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DIPLOMOVÁ PRÁCE

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Algebraické aspekty fuzzy logiky

Katedra aplikované matematiky
Vedoucí diplomové práce: Prof. RNDr. Petr Hájek, DrSc.
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Na tomto místě bych chtěl poděkovat panu prof. Petru Hájkovi za čas, který mi věnoval a za vedení semináře aplikované matematické logiky. Můj velký dík patří také konzultantovi Petru Cintulovi za množství připomínek k tomuto textu a za podporu, kterou mi poskytoval.

Prohlašuji, že jsem svou diplomovou práci napsal samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

V Praze dne Martin Víta

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Abstrakt: V této práci studujeme filtry na algebrách fuzzy logik jejich možná využití. Zobecňujeme pojmy implikativního/pozitivního implikativního/ fantastického filtru na BL-algebrách zavedením pojmu R-S-filtru na algebrách implikativních logik. Zformulujeme a dokážeme některé vlastnosti R-S-filtrů a následně ukážeme souvislost charakterizace R-S-filtru s alternativní axiomatizací dané logiky. Dále popíšeme způsoby, jak lze pomocí R-S-filtrů ekvivalentně charakterizovat konkrétní algebry implikativních logik. Ukážeme, že výsledky publikované v článcích [1] a [3] jsou jednoduché důsledky těchto zobecnění. Dalším tématem této práce jsou uniformní prostory a uniformní topologie nad algebrami implikativních logik. Zde filtry slouží k vytvoření tzv. Leibnitzovských kongruencí. Množina těchto kongruencí na dané algebře je (sub)bází uniformity, kterou následně zkoumáme. V této části ukazujeme, že výsledky uvedené v článcích [2] a [4] se dají snadno zobecnit pro libovolné implikativní logiky.
Klíčová slova: BL-algebra, filtry, abstraktní algebraická logika

Title: Algebraical Aspects of Fuzzy Logic
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Abstract: In this thesis we study filters on algebras of fuzzy logics and their possible applications. We are going to generalize the notion of an implicative/ positive implicative/fantastic filter on BL-algebras by introducing a notion of R-S-filter. We state and prove some properties of R-S-filters and then we show the connection between characterization of R-S-filter and alternative axiomatization of a given logic. We are going to describe the way how to characterize given algebra of implicative logic via R-S-filters. We show that the results published in [1] and [3] are simple consequences of our theory. Next topics of this work are uniform spaces and uniform topologies. Filters are there used to set up so-called Leibnitz congruences. A set of this congruences on the algebra is a (sub)base of a uniformity that we are going to study. We are going to show that results published in [2] and [4] can be easily generalized for any implicative logics.
Keywords: BL-algebra, filters, abstract algebraic logic

Introduction

In the last years many fuzzy logics have been presented and developed. Some important many-valued logics (such as infinite-valued Lukasiewicz logic, Gödel logic) were "transfered" into the framework of fuzzy logic and now they can be viewed as extensions of Hájek's Basic Fuzzy Logic (the logic BL in short). This development was accompanied by studying their algebraic counterparts – an overview is given by Jipsen, [11]. At the present time there is a big amount of particular results for particular logics and corresponding algebras. Hence some efforts leads towards generalizations of those results for classes of fuzzy logics (see Cintula [8]).

This thesis is focused on algebraic counterparts of fuzzy logics – the central notion of this thesis is the notion of (logical) filter.

Filters play an important role not only in algebraic point of view, but also in logic when defining matrix semantics of given logic. In the last years there were published several articles about filters on BL-algebras: Haveshki et al. [1], Kondo and Dudek [3], annother one by Haveshki et al. [2] (with a parallel about filters on Hilbert algebras in Saeid et al. [4]).

The main goal of this work is to explore the possibilites of a generalization of results in mentioned articles, correct mistakes and finding more general connection between [1] and [3].

The framework of abstract algebraic logic (AAL) was chosen due to two main reasons: it primarily allows us dealing with logics and their algebraic counterparts in a unified way. The second one is that the core theory of AAL provides deep results which we can be easily employed for our purposes.

In the name of the thesis there appears a word "fuzzy". The algebraic counterparts of fuzzy logic we are going to study as algebraic counterparts of implicational logics: main fuzzy logics – including the logic BL, Łukasiewicz, etc. – are covered by the class implicative logics.

The thesis is divided into three major parts. First chapter, Preliminaries contains mainly a summary of results published in cited articles that we are going to generalize in the next chapter – some of them are accompanied by our comments. In the last paragraphs of Preliminaries we mention some topological notions, which we will use further in the text.

The second chapter named "Filters on Algebras of Implicative Logics versus Filters in Abstract Algebraic Logic" connects the results of Haveshki et al. [1] and Kondo, Dudek [3] and put them in wider context. We show that this topic is strongly linked with alternative axiomatization of considered logics (axiomatizations of Łukasiewicz logic and Gödel logic – relative to the logic BL). Our results can by applied on an arbitrary implicative logic. The begining of this chapter contains an introduction to the framework of abstract algebraic logic – the environment we are going to work in. Then we present the axiomatic system of implicative logics and we continue with our core theory.

The third chapter is devoted to study the uniform topology based on conguences that arise from families of filters on a given algebra (of implicative logic). In the text we follow the work of Haveshki et al. [2] and we prove corresponding results for any implicative logic. At some theorems we point out possible weakenings of the assumptions of the original paper.

Chapter 1

Preliminaries

1.1 Notational Conventions

The main purpose of this short auxiliary section is to set up notational and typographical conventions we will use throughout the whole text. It corresponds appropriately to A Survey of Abstract Algebraic Logic [13].

For a function f with domain A and any $X \subseteq A$ we denote $f[X] = \{f(x) \mid x \in X\}$. The same convention we use for relations. If $R \subseteq A \times B$ and any $X \subseteq A$, $R[X] = \{y \mid \exists x \in X(x, y) \in R\}$.

We use the symbol $\Delta(X)$ for the indentity relation on a set X.

Algebraic structures will be denoted by boldface letters, e.g. $\mathbf{A}$, $\mathbf{Fm}$ and their universes by corresponding normal (italic) letters A, Fm.

The set of congruences of $\mathbf{A}$ is denoted by $\mathrm{Co}\mathbf{A}$.

If we deal with products of algebras, we use π_i as a symbol for projections as usual.

1.2 Some Important Algebras of Fuzzy Logics

In this section we are going to present the notion of BL-algebra and other algebras connected with important fuzzy logics. We will show the results about characterizing subclasses of BL-algebras via special types of filters (Haveshki et al., [1]). Then we continue by summarizing results of Kondo and Dudek [3], which are closely related to the previously cited work. These results will be generalized further in the second chapter of this thesis.

BL-algebras were introduced by Hájek [12] in order to provide a completeness theorem of Basic Logic.

Definition 1.1 A BL-algebra is an algebra $\mathbf{A} = (A, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying following conditions:

(i) $(A, \wedge, \vee, 0, 1)$ is a lattice with the largest element 1 and the least element 0 – with respect to lattice ordering $\leq$,

(ii) $(A, *, 1)$ is a commutative semigroup with the unit element 1,

(iii) $*$ and $\rightarrow$ form an adjoint pair, i. e. $z \leq (x \rightarrow y)$ iff $x * z \leq y$ for all x, y, $z \in A$,

(iv) $x \wedge y = x * (x \rightarrow y)$ for all x, $y \in A$,

(v) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ for all x, $y \in A$.

The last condition is often called *prelinearity* and the one but last is called *divisibility*. Unary operation of negation $\neg$ is defined $\neg x = x \rightarrow 0$. It can be shown that BL-algebras form a variety. Now we can continue with defining some subvarieties of the variety of BL-algebras.

Definition 1.2 A BL-algebra $\mathbf{A}$ is called Gödel algebra if $x * x = x$ for all $x \in A$ and $\mathbf{A}$ is called MV algebra if $\neg\neg x = x$ for all $x \in A$ or equivalently $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ for all x, $y \in A$.

Notice that in Gödel algebras the operations $\wedge$ and $*$ coincide. A BL-algebra $\mathbf{A}$ where $x \vee \neg x = 1$ for all $x \in A$ is termwise equivalent with a Boolean algebra $\mathbf{A}' = (A, \wedge, \vee, \neg, 0, 1)$ (by setting $* = \wedge$ and $x \rightarrow y = \neg x \vee y$).

The notion of filter on BL-algebra is also given in Hájek [12]:

Definition 1.3 Let $\mathbf{A}$ be a BL-algebra. A filter on $\mathbf{A}$ is a non-empty set $F \subseteq A$ such that for each x, $y \in A$:

(i) $x \in F$ and $y \in F$ implies $x * y \in F$,

(ii) $x \in F$ and $y \in A$, $x \leq y$ implies $y \in F$.

Definitions of all following types of filters were published by Haveshki et al. in [1]. Suppose in the rest of this section $\mathbf{A} = (A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra.

Definition 1.4 A non-empty subset $F \subseteq A$ is called an implicative/positive implicative/fantastic filter on $\mathbf{A}$ if $1 \in F$ and if it satisfies a following condition:

(i) in case of an implicative filter: $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$,

(ii) in case of a positive implicative filter: $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$,

(iii) in case of a fantastic filter: $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$,

for all x, y, $z \in A$.

After introduction of these definitions authors prove that every implicative/positive implicative/fanstastic filter is a filter. It is the opinion of the author of this text that this approach is a bit curious. We prefer slightly different (equivalent) definition: implicative/positive implicative/fantastic filter on $\mathbf{A}$ is a filter $F \subseteq A$ (hence it contains 1) satisfying a corresponding condition given in the list above. It is terminologically more adequate: we usually expect that a special type of filter is just a filter with some condition added.

Following theorems characterizing Gödel, Boolean and MV algebras match the same pattern. They all were presented in Haveshki [1].

Theorem 1.5 *In any* BL*-algebra* $\mathbf{A}$ *the following conditions are equivalent:*

1. $\{1\}$ *is an implicative filter.*
2. *Every filter on* $\mathbf{A}$ *is an implicative filter.*
3. $\mathbf{A}$ *is Gödel algebra.*

Theorem 1.6 *In any* BL*-algebra* $\mathbf{A}$ *the following conditions are equivalent:*

1. $\{1\}$ *is a positive implicative filter.*
2. *Every filter on* $\mathbf{A}$ *is a positive implicative filter.*
3. $\mathbf{A}$ *is Boolean algebra.*

Theorem 1.7 *In any* BL*-algebra* $\mathbf{A}$ *the following conditions are equivalent:*

1. $\{1\}$ *is a fantastic filter.*
2. *Every filter on* $\mathbf{A}$ *is a fantastic filter.*
3. $\mathbf{A}$ *is* MV *algebra.*

Theorem 1.8 *Let F be a filter on a* BL*-algebra* $\mathbf{A}$*,* $U_F = \{(x, y) \in A \times A \mid x \to y \in F \text{ and } y \to x \in F\}$ *is congruence on* $\mathbf{A}$ *and the corresponding quotient algebra* $\mathbf{A}/U_F$ *is a* BL*-algebra (Hájek [12]).*

In the rest of this section we will use $\mathbf{A}/F$ instead of $\mathbf{A}/U_F$ for the quotient algebra based on congruence U_F.

Propositions in the next two theorems summarize results contained in Haveshki et al. [1].

Theorem 1.9 *Let F be a filter on* $\mathbf{A}$*. Then*

(i) F is an implicative filter if and only if every filter on the quotient algebra $\mathbf{A}/F$ *is an implicative filter.*

(ii) F is a positive implicative filter if and only if every filter on the quotient algebra $\mathbf{A}/F$ *is a positive implicative filter.*

(iii) F is a fantastic filter if and only if every filter on the quotient algebra $\mathbf{A}/F$ *is a fantastic filter.*

Theorem 1.10 *Let $F, G \subseteq A$ be filters on* $\mathbf{A}$*, F be an implicative/positive implicative/fantastic filter and $F \subseteq G$. Then G is also an implicative/positive implicative/fantastic filter.*

Alternative characterizations of implicative/positive implicative/fantastic filters are given in Kondo and Dudek [3].

Theorem 1.11 *For any filter of a* BL*-algebra* $\mathbf{A}$*, the following conditions are equivalent:*

1. *F is an implicative filter,*
2. *$x \to x * x \in F$ for every $x \in A$.*

Theorem 1.12 *For any filter of a* BL*-algebra* $\mathbf{A}$*, the following conditions are equivalent:*

1. *F is a positive implicative filter,*
2. *$((x \to 0) \to x) \to x \in F$ for all $x \in A$.*

Theorem 1.13 *For any filter of a* BL*-algebra* $\mathbf{A}$*, the following conditions are equivalent:*

1. *F is a fantastic filter,*

2. $((x \to 0) \to 0) \to x \in F$ *for all* $x \in A$.

Now let us turn to syntactical part of the basic logic BL. We are going to present its axiomatic system and describe several extensions of this logic.

Definition 1.14 The following formulas are axioms of the basic logic BL.

1. $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
2. $(\varphi * \psi) \to \varphi$
3. $(\varphi * \psi) \to (\psi * \varphi)$
4. $(\varphi * (\varphi \to \psi)) \to (\psi * (\psi \to \varphi))$
5. $(\varphi \to (\psi \to \chi)) \to ((\varphi * \psi) \to \chi)$
6. $((\varphi * \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
7. $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
8. $\bar{0} \to \varphi$

The deduction rule of BL is modus ponens.

The following table shows us what logic we obtain when adjoining an axiom to the axiomatic system BL.

Logic	axiom
Gödel	$\varphi \to \varphi * \varphi$
Łukasiewicz	$\neg\neg\varphi \to \varphi$
Boolean	$\varphi \vee \neg\varphi$

All of logics mentioned above are sound and complete with respect to algebraic semantics:

Theorem 1.15 *Let* L *be a logic* BL*/Gödel/standard Łukasiewicz/Boolean,* T *a set of formulas and* φ *a formula. Then* $T \vdash_{\mathrm{L}} \varphi$ *if and only if* $T \models_{\mathbf{A}} \varphi$ *for each (linearly ordered)* $\mathcal{L}$*-algebra* $\mathbf{A}$.

1.3 Topological Background

Now we provide a short summary of topological background we will use in the next chapter.

Definition 1.16 A topology on a set X is a family $\mathcal{T}$ of subsets of X, called open sets, such that:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. $U \cap V \in \mathcal{T}$ for any U, $V \in \mathcal{T}$,
3. $\mathcal{T}$ is closed under arbitrary intersections.

The ordered pair $\langle X, \mathcal{T} \rangle$, where $\mathcal{T}$ is a topology on X is called a topological space. Subset $A \subseteq X$ is called closed in $\langle X, \mathcal{T} \rangle$ if $X \setminus A$ belongs to $\mathcal{T}$. The topological space $\langle X, \mathcal{T} \rangle$ is discrete if every subset $A \subseteq X$ belongs to $\mathcal{T}$.

Definition 1.17 $A \subseteq X$ is a clopen set in a topological space $\langle X, \mathcal{T} \rangle$ if $A \in \mathcal{T}$ and $X \setminus A \in \mathcal{T}$.

A topology on X can be equivalently defined in other ways: e.g. by defining neighborhoods (or neighborhood bases) for each point of X. See more in [6].

An open covering of topological space $\langle X, \mathcal{T} \rangle$ is a system $\mathcal{C} \subseteq \mathcal{T}$ such that $\bigcup \mathcal{C} = X$.

Definition 1.18 Topological space $\langle X, \mathcal{T} \rangle$ is compact if every open covering of $\langle X, \mathcal{T} \rangle$ admits a finite subcovering.

Definition 1.19 A subset $A \subseteq X$ is a compact set in $\langle X, \mathcal{T} \rangle$ if a subspace $\langle A, \mathcal{T}|A \rangle$ is compact.

Definition 1.20 Let $\langle X, \mathcal{T} \rangle$ and $\langle Y, \mathcal{V} \rangle$ be topological spaces, $f : X \to Y$ a function. Function f is called continuous (with respect to $\mathcal{T}$, $\mathcal{V}$) if for every open set U in $\langle Y, \mathcal{V} \rangle$ $f^{-1}[U]$ is an open set in $\langle X, \mathcal{T} \rangle$.

Definition 1.21 Topological space $\langle X, \mathcal{T} \rangle$ is completely regular (fulfils the axiom $T_{3\frac{1}{2}}$) if for every $x \in X$ and any closed subset $A \subseteq X$ such that $x \notin A$ exist continuous function $\Phi : X \to \langle 0, 1 \rangle$ satisfying $\Phi(x) = 0$ and $\Phi[A] \subseteq \{1\}$.

Chapter 2

Filters on Algebras of Implicative Logics versus Filters in Abstract Algebraic Logic

The main aim of this chapter is to develop a generalization of results published by Haveshki et al. [1] – the summarization is given in Preliminaries. This chapter provides equivalent characterizations of several algebras of implicative logics via families of the so called R-S-filters and shows the way how to obtain an alternative axiomatization of certain logics. Our further results show that characterization of positive, positive implicative and fantastic filters given by Kondo and Dudek in [3] can also be generalized. The last paragraphs of this chapter are devoted to quotient algebras.

This chapter is based on the framework of abstract algebraic logic – the "tool" we are going to present in the following section.

2.1 Introduction to Abstract algebraic logic

This section summarizes basic notions and results of abstract algebraic logic (AAL for short) we are going to use in the following text. It includes basic definitions and also establishes notational conventions. For comprehensive introduction of AAL see Czelakowski [9].

We assume that the notion of *propositional language* $\mathcal{L}$ is defined as usual and furthermore we assume that the propositional connectives have a finite arity. We denote by symbol $\mathbf{Fm}_{\mathcal{L}}$ the free term algebra over a denumerable set of propositional variables of $\mathcal{L}$. The universe of this free term algebra will be denoted by $Fm_{\mathcal{L}}$ (and the same convention we will use in the whole text: if $\mathbf{A}$ is an algebra, then A is its universe). Elements of $Fm_{\mathcal{L}}$ are called formulas. Somewhere in the text we omit the symbol of the language when it is clear from the context.

The endomorphisms of the algebra $\mathbf{Fm}_{\mathcal{L}}$ are called $\mathcal{L}$-substitutions and play an important role when defining a notion of logic.

Definition 2.1 A $\mathcal{L}$-consecution is a pair $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$. If Γ is finite, then we say that consecution $\Gamma \rhd \varphi$ is finitary.

Clearly, a set of consecutions can be viewed as a relation between sets of formulas and a single formula. We will see the connection between provability relation and consecutions in further text.

Definition 2.2 A propositional logic L is an ordered pair $\langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ where $\mathcal{L}$ is a propositional language and $\vdash_{\mathrm{L}}$ a set of $\mathcal{L}$-consecutions such that:

(i) $\varphi \vdash_{\mathrm{L}} \varphi$,

(ii) if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$,

(iii) if $\Gamma \vdash_{\mathrm{L}} \varphi$ and for every $\psi \in \Delta$, $\Delta \vdash_{\mathrm{L}} \psi$, then $\Delta \vdash_{\mathrm{L}} \varphi$,

(iv) for every $\mathcal{L}$-consecution $\Gamma \rhd \varphi$ and $\mathcal{L}$-substitution σ, $\Gamma \vdash_{\mathrm{L}} \varphi$ implies $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$

for all $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$.

First three conditions imply that $\vdash_{\mathrm{L}}$ is a *consequence relation* in the sense of Tarski and the last condition is known as *structurality* or *substitution invariance*. When the propositional language is clear from the context we identify a logic with its consequence relation. Since the text concerns only propositional (fuzzy) logics, we everywhere omit the word *propositional*.

Definition 2.3 A theory of a logic L is a set T of formulas such that for every formula $\varphi \in Fm_{\mathcal{L}}$, $T \vdash_{\mathrm{L}} \varphi$ implies $\varphi \in T$.

Rougly said, a theory is a set of formulas closed on the consequence relation.

Definition 2.4 A logic L is finitary if for every $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ there is a finite $\Gamma_0 \subseteq \Gamma$ satisfying $\Gamma_0 \vdash_{\mathrm{L}} \varphi$.

Given logic can be syntactically presented by means of several kinds of proof systems. In this work we allways deal with Hilbert style calculi.

Definition 2.5 Let $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ be a finitary logic. The set $\mathcal{AX}$ of finitary $\mathcal{L}$-consecutions is called a *presentation* of L (L is axiomatized by $\mathcal{AX}$) if the relation $\vdash_{\mathrm{L}}$ coincides with a provability relation $\mathcal{AX}$ as a Hilbert style calculi: for every $\Gamma \cup \varphi \subseteq Fm_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ if and only if there is a sequence of formulas $\langle \psi_0, \psi_1, \ldots, \psi_n \rangle$ such that $\varphi = \psi_n$ and for every $i < n$, $\psi_i \in \Gamma$ or for some $\Delta \rhd \alpha \in \mathcal{AX}$ there is a substitution σ satisfying $\sigma(\alpha) = \psi_i$ and $\sigma[\Delta] \subseteq \{\psi_0, \ldots, \psi_{i-1}\}$.

Definition 2.6 Let L be a logic and R a set of consecutions. The logic $\mathrm{L}+R$ is the logic axiomatized by any of the presentations of L plus consecutions from R.

Definition 2.7 Let $\mathcal{L}$ be a propositional language, $\mathrm{L}_1 = \langle \mathcal{L}, \vdash_{\mathrm{L}_1} \rangle$ and $\mathrm{L}_2 = \langle \mathcal{L}, \vdash_{\mathrm{L}_2} \rangle$ logics. A logic L_2 is an extension of L_1 if $\vdash_{\mathrm{L}_1} \subseteq \vdash_{\mathrm{L}_2}$.

The extension L_2 is an axiomatic extension of L_1 if $\mathrm{L}_2 = \mathrm{L}_1 + A$ for some set of axioms A.

Matrix models of propositional logics

Now we introduce crucial notions we will use in many parts of the thesis: the notion of $\mathcal{L}$-matrix, semantical consequence, and a model of a logic L.

Definition 2.8 Let $\mathcal{L}$ be a propositional language. The $\mathcal{L}$-matrix is an ordered pair $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is an $\mathcal{L}$-algebra and F a subset of A.

In this context F is called a filter of $\langle \mathbf{A}, F \rangle$. Homomorphisms from $\mathbf{Fm}_{\mathcal{L}}$ to an $\mathcal{L}$-algebra $\mathbf{A}$ are called $\mathbf{A}$-evaluations. If clear from the context, we use simply "evaluations".

Definition 2.9 Let $\mathbb{K}$ be a class of $\mathcal{L}$-matrices. We say that φ is a semantical consequence of Γ with respect to the class $\mathbb{K}$, symbolically $\Gamma \models_{\mathbb{K}} \varphi$, if for each $\langle \mathbf{A}, F \rangle \in \mathbb{K}$ and each $\mathbf{A}$-evaluation e, $e[\Gamma] \subseteq F$ implies $e(\varphi) \in F$.

Definition 2.10 Let $\mathcal{L}$ be a propositional language, L be a logic, $\langle \mathbf{A}, F \rangle$ an $\mathcal{L}$-matrix. We say that $\langle \mathbf{A}, F \rangle$ is a model of L if $\vdash_{\mathrm{L}} \subseteq \models_{\langle \mathbf{A}, F \rangle}$.

By $\mathbf{MOD}(\mathrm{L})$ we denote the class of all models of L. Now we can start with some completeness theorems. For proofs see [9].

Theorem 2.11 *Let* $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ *be a logic. Then for every* $\Gamma \cup \varphi \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ *if and only if* $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$.

Definition 2.12 Let $\mathbf{A}$ be an $\mathcal{L}$-algebra, S a logic, $F \subseteq A$. F is called an S-filter if $\langle \mathbf{A}, F \rangle \in \mathbf{MOD}(\mathrm{S})$.

By the symbol $\mathcal{F}i_{\mathrm{S}}\mathbf{A}$ we denote the set of all S-filters on $\mathbf{A}$. It is easy to check that $\mathcal{F}i_{\mathrm{S}}\mathbf{A}$ is closed under arbitrary intersections.

Definition 2.13 Let $\langle \mathbf{A}, F\rangle$ be a $\mathcal{L}$-matrix. A binary relation $\Omega_{\mathbf{A}}(F) \subseteq A \times A$ defined by the following rule: $\langle a, b\rangle \in \Omega_{\mathbf{A}}(F)$ if and only if for every sequence of parameters $\bar{z}$ and every $\mathcal{L}$-formula $\varphi(x, \bar{z})$ and arbitrary sequence $\bar{c}$ it holds that $\varphi^{\mathbf{A}}(a, \bar{c}) \in F \Leftrightarrow \varphi^{\mathbf{A}}(b, \bar{c}) \in F$ is called Leibnitz congruence of the matrix.

We say that a congruence $\theta \in \mathrm{Co}\mathbf{A}$ is compatible with $F \subseteq A$ if for every $a, b \in A$ such that $a \in F$ and $\langle a, b\rangle \in \theta$ imply $b \in F$.

Theorem 2.14 *$\Omega_{\mathbf{A}}(F)$ is maximum congruence of $\mathcal{A}$ compatible with F.*

Proof can be find in Czelakowski [9] again.

Definition 2.15 A logical matrix $\langle \mathbf{A}, F\rangle$ is called reduced, if its Leibnitz congruence is the identity.

The class of reduced matrix models of a given logic S will be denoted by $\mathbf{MOD}^*(\mathrm{S})$. According to the previous definition to each matrix $\langle \mathbf{A}, F\rangle$ we can assign a reduced matrix $\langle \mathbf{A}/\Omega_{\mathbf{A}}(F), F/\Omega_{\mathbf{A}}(F)\rangle$ – this matrix is called a reduction of $\langle \mathbf{A}, F\rangle$. We can improve our completeness theorem.

Theorem 2.16 *Let $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}}\rangle$ be a logic. Then for every $\Gamma \cup \varphi \subseteq Fm_{\mathcal{L}}$, $\Gamma \vdash_{\mathrm{L}} \varphi$ if and only if $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi$.*

Definition 2.17 Let S be a logic. The class of algebras associated with logic S is the class of algebraic reducts of the reducted models of S and it is denoted by the symbol $\mathbf{Alg}^*(\mathrm{S})$.

$$\mathbf{Alg}^*(\mathrm{S}) = \{\mathbf{A} \mid \exists F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A} \text{ such that } \Omega_{\mathbf{A}}(F) \text{ is the identity}\}$$

On many places we will use the following theorem.

Theorem 2.18 *Let L, S be logics over the same language $\mathcal{L}$ such that L is an axiomatic extension of S, let $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$ Then $\mathcal{F}i_{\mathrm{S}}\mathbf{A} = \mathcal{F}i_{\mathrm{L}}\mathbf{A}$*

The proof can be find in Czelakowski [9] again.

2.2 Implicative Logics

All logics we consider in this text – prominent fuzzy logics as the logic BL, Łukasiewicz, Gödel logic are examples of implicative logics. In this short section we are going to present them and point out some important properties of implicative logics.

Implicative logics were introduced by Rasiowa [10].

Definition 2.19 A logic $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ over the language $\mathcal{L}$ is said to be implicative logic if $\mathcal{L}$ contains a binary connective $(\rightarrow, 2)$ such that:

- $\vdash_{\mathrm{L}} p \rightarrow p$,
- $p, p \rightarrow q \vdash_{\mathrm{L}} q$,
- $p \rightarrow q, q \rightarrow r \vdash_{\mathrm{L}} p \rightarrow r$,
- $p \vdash_{\mathrm{L}} q \rightarrow p$,
- $p \rightarrow q, q \rightarrow p \vdash_{\mathrm{L}} c(a_1, \dots a_{i-1}, p, a_n) \rightarrow c(a_1, \dots a_{i-1}, q, a_n)$ for every for every n-ary connective $c \in \mathcal{L}$ and $i \leq n$

Examples of implicative logics: BL, Łukasiewicz, Gödel, etc. $\mathbf{Alg}^*(\mathrm{BL})$ are BL-algebras, $\mathbf{Alg}^*(\mathrm{S})$ where S is Łukasiewicz logic are MV algebras (due to historical reasons), $\mathbf{Alg}^*(\mathrm{S})$ where S is Gödel logic are Gödel algebras.

The following paragraphs summarize some important properties of implicative logics we will often use in next chapters.

Proposition 2.20 *Let* S *be an implicative logic and* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$*. Then there is an element* $1 \in A$ *such that for each* $\langle \mathbf{A}, F \rangle \in \mathbf{MOD}(\mathrm{S})$ *and each* $x \in A$ *we have* $1 \in F$ *and* $x \rightarrow x = 1$*.*

Let us assume from now on that any language of any implicative logic contains a nullary connective 1 defined as $p \rightarrow p$. Clearly, this definition is sound as in any $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$ the value of 1 is constant.

Theorem 2.21 *Let* S *be an implicative logic and* $\langle \mathbf{A}, F \rangle \in \mathbf{MOD}(\mathrm{S})$*. Then the Leibnitz congruence* $\Omega_{\mathbf{A}}(F)$ *is defined as* $(a, b) \in \Omega_{\mathbf{A}}(F)$ *if and only if* $a \rightarrow b \in F$ *and* $b \rightarrow a \in F$*. Moreover, the operator* $\Omega_{\mathbf{A}}$ *(assigning to each filter a corresponding Leibnitz congruence) is an isomorphism between the lattice of filters on* $\mathbf{A}$ *and the lattice of congruences of* $\mathbf{A}$*.*

A convention we are going to use when dealing with Leibnitz congruences is denotation of a quotient algebra: instead of $\mathbf{A}/\Omega_{\mathbf{A}}(F)$ we use $\mathbf{A}/F$.

Theorem 2.22 *Let* S *be an implicative logic,* $\mathbf{A}$ *an* $\mathcal{L}$*-algebra,* $F \subseteq A$*. Then* $\langle \mathbf{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{S})$ *if and only if* $F = \{1\}$*.*

Observation 2.23 Consider the same assumptions as in the previous theorem. Then $\langle A, \{1\}\rangle \in \mathbf{MOD}(\mathrm{S})$ if and only if $\langle A, \{1\}\rangle \in \mathbf{MOD}^*(\mathrm{S})$ if and only if $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$.

Theorem 2.24 *Let* L*,* S *be implicative logics. Then* $\mathrm{L} = \mathrm{S}$ *if and only if* $\mathbf{Alg}^*(\mathrm{L}) = \mathbf{Alg}^*(\mathrm{S})$*.*

Proofs of all theorems can be found in [9] and [13].

Definition 2.25 Let S be an implicative logic and $\mathbf{A}$ be an $\mathcal{L}$-algebra. We say that a S-filter $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ is (finitely) meet-irreducible filter in the lattice $\mathcal{F}i_{\mathrm{S}}\mathbf{A}$ if F cannot be represented as a (finite) meet of non-empty set of S-filters different from F.

Definition 2.26 The $\mathcal{L}$-algebra $\mathbf{A}$ is said to be (finitely) subdirectly irreducible if for every subdirect representation α of $\mathbf{A}$ with a family $\{\mathbf{A}_i \mid i \in I\}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism.

The class of all (finitely) subdirectly irreducible algebras with respect to S is denoted by $\mathbf{Alg}^*_{\mathbf{SI}}(\mathrm{S})$ or $\mathbf{Alg}^*_{\mathbf{FSI}}(\mathrm{S})$ respectively.

Theorem 2.27 *Let* S *be an implicative logic. Then* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$ *is (finitely) subdirectly irreducible if and only if* $\{1\}$ *is (finitely) meet-irreducible filter in* $\mathcal{F}i_{\mathrm{S}}\mathbf{A}$*.*

Definition 2.28 Let S be an implicative logic and $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$. We define the partial order $\leq$ as $a \leq b$ iff $a \to b = 1$. The algebra $\mathbf{A}$ is called *linear* if $\leq$ is linear order and the class of all such algebras is denoted by $\mathbf{Alg}^\ell(\mathrm{S})$.

Deductive Filters on the Algebras of Implicative Logics

At this moment we are familiar with the definition of the (logical) filter which is well-established notion in AAL in the matrix semantics. In texts about BL-logic we often meet a notion of deductive filter or deductive system (for example Turunen [7]).

This short subsection makes clear the relationship between these two notions.

Definition 2.29 Let $\mathbf{A}$ be an algebra of an implicative logic. A subset $D \subseteq A$ is a deductive filter of $\mathbf{A}$, if two following conditions are satisfied:

(i) $1 \in D$,

(ii) if $x \to y \in D$ and $x \in D$, then $y \in D$ for all $x, y \in D$.

Theorem 2.30 *Let* S *be an implicative logic with a presentation where modus ponens is the only deductive rule. Let further* $A \in \mathbf{Alg}^*(\mathrm{S})$ *and* $F \subseteq A$*. Then* $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ *if and only if* F *is a deductive filter of* A*.*

Proof Suppose that $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$. As $\vdash_{\mathrm{S}} 1$ we have $\vDash_{\langle \mathbf{A},F\rangle} 1$, i.e., $1 \in F$. To prove the second condition observe that $\{p \to q, p\} \vdash_{\mathrm{S}} q$ and so $\{p \to q, p\} \vDash_{\langle \mathbf{A},F\rangle} q$. Thus for any $\mathbf{A}$-evaluation e, if $e[\{p \to q, p\}] \subseteq F$, then $e(q) \in F$. Let $x \to y \in F$ and $x \in F$. Now it is enough to set up an $\mathbf{A}$-evaluation e such that $e(p) = x$ and $e(q) = y$, so $y \in F$.

Conversely, suppose F is a deductive filter on $\mathbf{A}$, let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ and let e be an arbitrary $\mathbf{A}$-evaluation. Moreover let us assume that $e[\Gamma] \subseteq F$ and $\Gamma \vdash_{\mathrm{S}} \varphi$. We will use induction over the construction of the proof of φ – let consider the proof in a following form $\langle \psi_0, \ldots, \psi_n\rangle$. We claim, that if $e(\psi_j) \in F$ for all $j \leq i$ where $1 \leq i \leq n-1$, then $e(\psi_{i+1}) \in F$. Suppose it holds for some $i \leq n-1$ and we will prove it for $i+1$.

If $\psi_{i+1} \in \Gamma$, then $e(\psi_{i+1}) \in F$ by the assumption. If ψ_{i+1} is an substitutional instance of an axiom we know that $\vdash_{\mathrm{S}} \psi_{i+1}$. Thus $\vdash_{\mathrm{S}} \psi_{i+1} \to 1$ and $\vdash_{\mathrm{S}} 1 \to \psi_{i+1}$. As symmetrization of $\to$ defines the Leibnitz congruence on $\langle \mathbf{A}, \{1\}\rangle$ which is the identity we know that $e(\psi_{i+1}) = 1$ for each evaluation e thus $e(\psi_{i+1}) \in F$. Because in S there is no other deductive rule than modus ponens, to complete the proof it is sufficient to prove that $e[\{\psi_k \to \psi_l, \psi_k\}] \subseteq F$, implies $e(\psi_l) \in F$ for some $k, l \leq i$. Since e is an endomorphism, we obtain that $e[\{\psi_k \to \psi_l, \psi_k\}] = \{e(\psi_k) \to e(\psi_l), e(\psi_k)\}$. Suppose that $e(\psi_k) = x$ and $e(\psi_l) = y$. Now we can apply that F is a deductive filter: thus we obtain that $e(\psi_l) = y \in F$, so $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$.

Connection between Standard Notion of Filter on an Algebra and Deductive Filters

The notion of filter is primarily known in the theory of lattice-ordered sets as "an upper set closed under finite meets". In this chapter it remains to show the connections between these filters and deductive filters and logical filters we have introduced above.

We will show that (lattice) filters on a Boolean algebra $\mathbf{A}$ are just deductive filters. In Boolean algebras $x \to y$ is defined just an abbreviation for $\neg x \vee y$ for all $x, y \in A$.

Theorem 2.31 *Let* $\mathbf{A}$ *be a Boolean algebra. Then* $F \subseteq A$ *is a deductive filter of* A *if and only if* F *is a filter on* $\mathbf{A}$*.*

Analogous theorem holds of course for BL-algebras, firstly proved by Turunen [7].

Theorem 2.32 *Let* $\mathbf{A}$ *be a* BL*-algebra,* $F \subseteq A$*. Then* F *is a filter (in the sense of Hájek's definition) if and only if* F *is a deductive filter of* $\mathbf{A}$*.*

Proof Assume F is a deductive filter. Then F is non-empty, because it contains 1. Let $x, y \in F$. In BL-algebra we have $x \to (y \to (x * y)) = 1$ for all $x, y \in A$, so we use use the "modus ponens" property twice and we obtain $x * y \in F$. If $x \in F$ and $x \leq y$, then $x \to y = 1 \in F$, so we use the "modus ponens" again and we have $y \in F$, thus F is a filter.

Conversely, suppose that F is a filter. Since F is non-empty, $1 \in F$, because $x \leq 1$ for all $x \in A$. Assume $x, x \to y \in F$: $x * (x \to y) \in F$, since F is a filter. In BL-algebra it holds that $x * (x \to y) \leq y$, so $y \in F$.

In a general case we can say that each deductive filter is a lattice filter (see the proof above) but not vice-versa (take e.g. three-valued linearly ordered MV-algebra with the domain $\{0, \frac{1}{2}, 1\}$. Because $\frac{1}{2} * \frac{1}{2} = 0$ we obtain that $\{\frac{1}{2}, 1\}$ is not a deductive filter but it clearly is a lattice one).

2.3 Applications of R-S-filters

Let us fix an implicative logic $\mathrm{S} = \langle \mathcal{L}, \vdash_{\mathrm{S}} \rangle$ and a set of $\mathcal{L}$-consecutions R. We are going to start with an instrumental definition of R-S-filter. This definition unifies dealing with BL-filters, which are defined by the same pattern as implicative, positive implicative and fantastic filters.

Definition 2.33 Let $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$ be an algebra of an arbitrary implicative logic S and $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$. F is called R-S-filter if it satisfies the following condition:

- if $e[T_i] \subseteq F$, then $e(\varphi_i) \in F$, for every $\mathbf{A}$-evaluation e and every $T_i \rhd \varphi_i \in R$

Let us recall that $\mathrm{S} + R$ is an extension of the logic S by the consecutions R – see the previous section for the exact definitions.

Observation 2.34 Let $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$ and F be an S-filter on $\mathbf{A}$. Then F is an R-S-filter on $\mathbf{A}$ if and only if $F \in \mathcal{F}i_{\mathrm{S}+R}\mathbf{A}$.

Proof of this observation is obvious – the reasons for not stating the definition of R-S-filter this way are rather formal: it makes possible to present results in the same form as theorems in [1] and [3].

The following tables summarizes types of R-S-filters we going to work with. We assume x, y, z are arbitrary elements of A.

Type of S-filter F	Condition in traditional form
Implicative	if $x \to (y \to z) \in F$ and $x \to y \in F$, then $x \to z \in F$
Positive implicative	if $x \to ((y \to z) \to y) \in F$ and $x \in F$, then $y \in F$
Fantastic	if $z \to (y \to x) \in F$ and $z \in F$, then $((x \to y) \to y) \to x \in F$
Boolean	$x \vee \neg x \in F$

The next table shows us our newly introduced notation. Assume p_1, p_2, p_3 are propositional variables of L.

Type of S-filter F	Condition in R-form
Implicative	$\{\{p_1 \to (p_2 \to p_3), p_1 \to p_2\} \rhd p_1 \to p_3\}$
Positive implicative	$\{\{p_1 \to ((p_2 \to p_3) \to p_2), p_1\} \rhd p_2\}$
Fantastic	$\{\{p_3 \to (p_2 \to p_1), p_3\} \rhd ((p_1 \to p_2) \to p_2) \to p_1\}$
Boolean	$\{\rhd p_1 \vee \neg p_1\}$

Now we can formulate one of the central theorems of this chapter.

Theorem 2.35 *Let* $\mathrm{L}_1 \subseteq \mathrm{L}_2$ *be implicative logics in the language* $\mathcal{L}$ *such that* L_2 *is an axiomatic extension of* L_1 *satisfying* $\mathrm{L}_1 + R = \mathrm{L}_2$, $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_1)$, *and* R *a set of* $\mathcal{L}$*-consecutions. Then the following statements are equivalent:*

(i) F *is* R-L_1*-filter on* $\mathbf{A}$ *for all* $F \in \mathcal{F}i_{\mathrm{L}_1}\mathbf{A}$.

(ii) $\{1\}$ *is* R-L_1*-filter on* $\mathbf{A}$.

(iii) $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_2)$.

Proof The implication (i)⇒ (ii) is trivial.

(ii) ⇒ (iii): Suppose $\{1\}$ is R-L_1-filter on $\mathbf{A}$. From the previous observation we have $\langle \mathbf{A}, \{1\} \rangle \in \mathbf{MOD}(\mathrm{L}_1 + R)$, thus $\langle \mathbf{A}, \{1\} \rangle \in \mathbf{MOD}^*(\mathrm{L}_1 + R)$. As we suppose $\mathrm{L}_1 + R = \mathrm{L}_2$, we obtain $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_2)$.

(iii) ⇒ (i): Suppose $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_2)$ and consider an arbitrary $F \in \mathcal{F}i_{\mathrm{L}_1}\mathbf{A}$. Since L_2 is an axiomatic extension of L_1, we have $\mathcal{F}i_{\mathrm{L}_1}\mathbf{A} = \mathcal{F}i_{\mathrm{L}_2}\mathbf{A}$, so $F \in$

$\mathcal{F}i_{\mathrm{L}_2}\mathbf{A}$. Thus $\langle \mathbf{A}, F\rangle \in \mathbf{MOD}(\mathrm{L}_2)$. Obviously $T_i \vdash_{\mathrm{L}_2} \varphi_i$ for every $T_i \rhd \varphi_i \in R$, because $\mathrm{L}_1 + R = \mathrm{L}_2$, so $T_i \models_{\langle \mathbf{A},F\rangle} \varphi_i$. Hence for each $\mathbf{A}$-evaluation e we have a following implication: if $e[T_i] \subseteq F$, then $e(\varphi_i) \in F$ for every $T_i \rhd \varphi_i \in R$ – and this is just the condition in the definition of R-L_1-filter.

Now we are going to apply this general theorem in a special case to obtain results published in Haveshki et al. [1]. But first we prove one key lemma:

Lemma 2.36 *Let R be an implicative/positive implicative/fantastic condition. Then* $\mathrm{BL} + R$ *is Gödel/Boolean/Łukasiewicz logic.*

Proof We give formal proofs. Let R be the implicative condition. As BL proves $\varphi \to (\varphi \to \varphi * \varphi)$ and $\varphi \to \varphi$ we obtain that $\mathrm{BL} + R$ proves $\varphi \to \varphi * \varphi$. Conversely: from $\varphi \to (\psi \to \chi) \vdash_{\mathrm{BL}} \psi \to (\varphi \to \chi)$ we obtain $\varphi \to (\psi \to \chi), \varphi \to \psi \vdash_{\mathrm{BL}} \varphi \to (\varphi \to \chi)$, by transitivity, residuation plus the Gödel axiom $\varphi \to \varphi^2$ completes the proof.

Let R be the positive implicative condition. BL proves $\varphi \to \varphi \vee \neg\varphi$, $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$, and $\neg\varphi \to \varphi \vee \neg\varphi$, thus it also proves $\neg(\varphi \vee \neg\varphi) \to \neg\varphi \vee \varphi$. Thus $\mathrm{BL} + R$ proves $\varphi \vee \neg\varphi$ (we use the rule R for $p_1 = 1$, $p_2 = \varphi \vee \neg\varphi$ and $p_3 = 0$). To prove the converse direction we could use the truth-table method of classical logic.

Finally, let R be the fantastic condition. As BL proves $1 \to (0 \to \varphi)$ and 1 we obtain that $\mathrm{BL} + R$ proves $((\varphi \to 0) \to 0) \to \varphi$. Conversely: Łukasiewicz logic proves $((\chi \to \psi) \to \psi) \to ((\psi \to \chi) \to \chi)$ thus it also proves $(\psi \to \chi) \to (((\chi \to \psi) \to \psi) \to \chi)$. Thus clearly it proves $\varphi \to (\psi \to \chi), \varphi \vdash ((\chi \to \psi) \to \psi) \to \chi$ by modus ponens.

In the next corollary, we use just simply "filter" instead of "BL-filter".

Corollary 2.37 *Let* $\mathbf{A}$ *a* BL-*algebra. The following statements are equivalent.*

(i) Every filter on $\mathbf{A}$ *is an implicative/positive implicative/fantastic filter.*

(ii) $\{1\}$ *is an implicative/positive implicative/fantastic filter on* $\mathbf{A}$.

(iii) $\mathbf{A}$ *is Gödel/Boolean/*MV-*algebra.*

Proof We use Theorem 2.35 for L_1 being BL, R being implicative/positive implicative/fantastic condition, and L_2 being Gödel/Boolean/Łukasiewicz logic (the previous lemma tells us that the conditions of Theorem 2.35 are satisfied in this setting).

So, we put results published in Haveshki et al. [1] in a more general setting and we just have shown that they are consequences of Theorem 2.35. On the other side we are going to describe the relationship between the equivalence of statements in Theorem 2.35 and the alternative axiomatization of extension of given logic: this is the content of the next theorem.

Theorem 2.38 *Let* $\mathrm{L}_1 \subseteq \mathrm{L}_2$ *be implicative logics in the language* $\mathcal{L}$ *such that* L_2 *is an axiomatic extension of* L_1 *satisfying* $\mathrm{L}_1 + R = \mathrm{L}_2$, $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_1)$, *and* R *a set of* $\mathcal{L}$*-consecutions.*

If the equivalence ($\{1\}$ *is* R-L_1*-filter on* $\mathbf{A} \Leftrightarrow \mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_2)$) *holds for all* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_1)$, *then* $\mathrm{L}_1 + R = \mathrm{L}_2$.

Proof Let $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_1 + R)$, so $\langle \mathbf{A}, \{1\} \rangle \in \mathbf{MOD}{*}(\mathrm{L}_1 + R)$, thus $\{1\}$ is a R-L_1-filter. Thanks to the equivalence ($\{1\}$ is R-L_1-filter on $\mathbf{A} \Leftrightarrow \mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_2)$) we also have that $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_2)$, so $\mathbf{Alg}^*(\mathrm{L}_1 + R) \subseteq \mathbf{Alg}^*(\mathrm{L}_2)$. Proof of the converse inclusion is analogous and Theorem 2.24 completes the proof.

Roughly summarized, we have proved that from the alternative axiomatization of some logic we can get a system of equivalent statements about S-filters in the algebra and conversely, from the set of equivalences in this form we can easily obtain an alternative axiomatization.

Now we mention a special case of adding a single axiom: if we have some implicative logic S and we add an axiom φ – in our terminology: a rule in the form $\rhd\ \varphi$, then (by Theorem 2.35) we automatically have a triple of equivalent statements:

1. Every S-filter on A is a $\{\rhd\ \varphi\}$-S-filter.
2. $\{1\}$ is $\{\rhd\ \varphi\}$-S-filter.
3. $\mathbf{A} \in \mathbf{Alg}^*(S + \{\rhd\ \varphi\})$.

If we add a single axiom, we get an axiomatic extension of the logic S, so the assumptions of Theorem 2.35 are fulfiled.

Example 2.39 Let S be the logic BL again. If we enrich axiomatic system by the axiom $p \to p^2$, we will get an axiomatization of Gödel fuzzy logic accompanied with its algebraic counterpart, Gödel algebra. By the previous corollaries we know that on Gödel algebra every filter F is R-BL-filter, where R is implicative condition from the table – but now we see that F is also $\{\rhd\ p \to p^2\}$-BL-filter.

The R-S filters can have many equivalent characterizations – these paragraphs make clear the relationship among them. The next theorem describes a kind of exchange property and provides us "a method for generating" equivalent definitions of several types of S-filters.

Theorem 2.40 *Let* S *be an implicative logic and* R_1*,* R_2 *sets of consecutions. Then the following statements are equivalent:*

(i) $\mathrm{S} + R_1 = \mathrm{S} + R_2$.

(ii) F *is an* R_1-S*-filter on* $\mathbf{A}$ *if and only if* F *is* R_2-S*-filter for every* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$ *and for every* S*-filter on* $\mathbf{A}$*.*

(iii) $\{1\}$ *is a* R_1-S*-filter on* $\mathbf{A}$ *if and only if* $\{1\}$ *is a* R_2-S*-filter on* $\mathbf{A}$ *for every* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$*.*

Proof (i) $\Rightarrow$ (ii): We assume $\mathrm{S} + R_1 = \mathrm{S} + R_2$, so $\mathcal{F}i_{\mathrm{S}+R_1}\mathbf{A} = \mathcal{F}i_{\mathrm{S}+R_2}\mathbf{A}$. If F is a R_1-S-filter, then by Observation 2.34 $F \in \mathcal{F}i_{\mathrm{S}+R_1}\mathbf{A}$, so $F \in \mathcal{F}i_{\mathrm{S}+R_2}\mathbf{A}$. We use this observation again – in the converse direction, hence F is a R_2-S-filter. The converse implication is now obvious.

(ii) $\Rightarrow$ (iii) is trivial.

It remains to prove the implication (iii) $\Rightarrow$ (i). Let us suppose $\{1\}$ is a R_1-S-filter on $\mathbf{A}$ if and only if $\{1\}$ is a R_2-S-filter on $\mathbf{A}$ for every $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$. Let $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S} + R_1)$. So $\langle \mathbf{A}, \{1\}\rangle \in \mathbf{MOD}^*(\mathrm{S} + R_1)$, thus $\{1\}$ is a R_1-S-filter on $\mathbf{A}$ and as we suppose $\{1\}$ is also a R_2-S-filter on $\mathbf{A}$, so $\langle \mathbf{A}, \{1\}\rangle \in \mathbf{MOD}(\mathrm{S}+R_2)$, thus $\langle \mathbf{A}, \{1\}\rangle \in \mathbf{MOD}^*(\mathrm{S}+R_2)$ and hence $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S}+R_2)$. Converse direction is analogical, so $\mathbf{Alg}^*(\mathrm{S} + R_2) = \mathbf{Alg}^*(\mathrm{S} + R_2)$, thus – we are in implicative logics – $\mathrm{S} + R_1 = \mathrm{S} + R_2$.

As a consequence of this theorem we get results published by Kondo and Dudek [3]. We are going to sumarize them in following corollaries.

Corollary 2.41 *Let* $\mathbf{A}$ *be a* BL*-algebra. Then:*

1. F *is a fantastic* BL*-filter if and only if* $(x \to 0) \to 0) \to x \in F$ *for all* $x \in A$*,*

2. F *is a positive implicative* BL*-filter if and only if* $(x- \to x) \to x \in F$ *for all* $x \in A$*,*

3. F *is an implicative* BL*-filter if and only if* $x \to x^2 \in F$ *for all* $x \in A$*.*

Proof In all cases it is enough to consider that BL $+R_1 =$ BL $+R_2$, where R_1 is a fantastic/positive implicative/implicative condition in R-form, R_2 are conditions at the right sides of these corollaries in the R-form. This was shown in Lemma 2.36.

Another consequence of Theorem 2.40 concerns Boolean filters. In Haveshki et al. [1] authors proved that maximal positive implicative filter on a BL-algebra is Boolean filter and stated an open problem: under what suitable condition the converse of this proposition is true? This question was answered by Kondo and Dudek [3] – they proved a stronger result: the class of positive implicative filters on a given BL-algebra $\mathbf{A}$ coincides with the class of Boolean filters on $\mathbf{A}$.

In our perspective is now the solution easy: BL $+ R_1 =$ BL $+ R_2$, where R_1 is positive implicative condition in R-form, R_2 the Boolean condition in R-form. In both cases we have an alternative axiomatization of Boolean algebras.

Theorem 2.42 *Let* S *be an implicative logic in the language* $\mathcal{L}$*,* R *a set of* $\mathcal{L}$*-consecutions such that* $\mathrm{S} + R$ *is an axiomatic extension of* S*, and* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$*. Further assume that* F *is an* R-S*-filter on* $\mathbf{A}$ *and* G *be a* S*-filter on* $\mathbf{A}$ *such that* $F \subseteq G$*. Then* G *is also a* R-S*-filter on* $\mathbf{A}$*.*

Proof As we suppose $\mathrm{S} + R$ is an axiomatic extension of the logic S we can obtain the same logic by adding some axioms, i.e. rules in form $\rhd\ \varphi_i$ for $i \in I$. Let us denote the set of such rules R'. We have $\mathrm{S} + R = \mathrm{S} + R'$.

So F is also R'-S-filter by Theorem 2.40. By the definition of R'-S-filter it means that $e[\{\varphi_i \mid i \in I\}] \subseteq F$ for each $\mathbf{A}$-evaluation e. Thanks to the supposition $F \subseteq G$ we also have $e[\{\varphi_i \mid i \in I\}] \subseteq G$ (for all mentioned e), hence G is R'-S-filter. Now we use Theorem 2.40 again, so G is also R-S-filter on A.

As we seen in the proof of this theorem, we did not use particular definitions of concrete types of S-filters – we were interested only in the fact whether we have an axiomatic extension of the logic S.

This theorem covers three theorems in Haveshki et al. [1] saying that in BL-algebras every filter, which is a superset of an implicative/positive implicative/fantastic filter is also an implicative/positive implicative/fantastic filter.

Now we are going to consider two extensions of a given logic. The next theorem says something about the relationship of classes of R-S-filters.

Proposition 2.43 *Let* S *be an implicative logic,* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$*,* R_1*,* R_2 *sets of consecutions such* $\mathrm{S} + R_1 \subseteq \mathrm{S} + R_2$*. If* F *is a* R_2-S*-filter on* $\mathbf{A}$*, then* F *is also a* R_1-S*-filter on* $\mathbf{A}$*.*

Proof Let us assume F is a R_2-S-filter on $\mathbf{A}$ thus $\langle \mathbf{A}, F\rangle \in \mathbf{MOD}(\mathrm{S} + R_2)$. As $\mathrm{S} + R_1 \subseteq \mathrm{S} + R_2$ we also know that $\langle \mathbf{A}, F\rangle \in \mathbf{MOD}(\mathrm{S} + R_1)$ and so F is a R_1-S-filter on $\mathbf{A}$

Notice that in fact it was a part of a proof of Theorem 2.40.

If we are able to derive for example the fantastic-BL-filter condition from positive implicative-BL-filter condition (consecution) and presentation of the logic BL, it means that every positive implicative filter on BL-algebra is a fantastic filter. This is the content of Theorem 4.5 in [1].

In the rest of this section we focus on quotient algebras. We are going to prove general result and show that further results of Haveshki et al. [1] and Kondo, Dudek [3] are its simple consequences.

Lemma 2.44 *Let* L_1*,* L_2 *be implicative logics,* R *a set of consecutions such that* $\mathrm{L}_1 + R = \mathrm{L}_2$*, and let* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L}_1)$*. If* F *is a* R-L_1*-filter, then* $\mathbf{A}/F \in \mathbf{Alg}^*(\mathrm{L}_2)$*.*

Proof Let us assume that F is a R-L_1-filter. By Observation 2.34 we obtain $F \in \mathcal{F}i_{\mathrm{L}_1+R}\mathbf{A}$, so $F \in \mathcal{F}i_{\mathrm{L}_2}\mathbf{A}$. Thus $\langle \mathbf{A}, F\rangle \in \mathbf{MOD}(\mathrm{L}_2)$. Hence $\langle \mathbf{A}/F, F/F\rangle = \langle \mathbf{A}/F, \{1\}\rangle \in \mathbf{MOD}(\mathrm{L}_2)$ and also $\langle \mathbf{A}/F, \{1\}\rangle \in \mathbf{MOD}^*(\mathrm{L}_2)$. Now we can make a conclusion that $\mathbf{A}/F \in \mathbf{Alg}^*(\mathrm{L}_2)$.

Example 2.45 If we have a fantastic filter F on a BL-algebra $\mathbf{A}$, then $\mathbf{A}/F$ is a MV-algebra.

Theorem 2.46 *Let* L *be an implicative logic,* R *a set of consecutions such that* $\mathrm{L}+R$ *is an axiomatic extension of* L*,* $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{L})$*, and* F *an* L*-filter on* $\mathbf{A}$*. Then* F *is a* R-L*-filter on* $\mathbf{A}$ *if and only if every* L*-filter on the quotient algebra* $\mathbf{A}/F$ *is a* R-L*-filter.*

Proof Let F be a R-L-filter on $\mathbf{A}$. From the previous lemma we know that $\mathbf{A}/F \in \mathbf{Alg}^*(\mathrm{L} + R)$. At this moment we can apply Theorem 2.35 – the implication (iii) $\Rightarrow$ (i), so we get the following: G is R-L-filter on $\mathbf{A}/F$ for all $G \in \mathcal{F}i_L\mathbf{A}/F$.

Conversely, suppose that every L-filter on $\mathbf{A}/F$ is a R-L-filter. Thus $\{[1]\}$ is a R-L-filter. Consider a morphism $h\colon \mathbf{A} \to \mathbf{A}/F$ defined by the following rule: $h(x) = [x]_F$ and let $\bar{e} = (h \circ e)$ for a given $\mathbf{A}$-evaluation e. Obviously, $\bar{e}$ is an $\mathbf{A}/F$-evaluation. $\{[1]\}$ is a R-L-filter, so for every $\mathbf{A}/F$-evaluation v, if $v[T_i] \subseteq \{[1]_F\}$ then $v(\varphi_i) = [1]_F$ for all $T_i \rhd \varphi \in R$, thus we obtain an implication: if $\bar{e}[T_i] \subseteq \{[1]_F\}$ then $\bar{e}(\varphi_i) = [1]_F$.

Let us suppose that for an arbitrary $\mathbf{A}$-evaluation e, $e[T_i] \subseteq F$ for all $T_i \rhd \varphi_i \in R$. Then $\bar{e}[T_i] \subseteq \{[1]_F\}$ and also $\bar{e}(\varphi_i) = [1]_F$ (because $\{[1]\}$ is a R-L-filter). Hence $e(\varphi) \in F$, thus F is a R-L-filter.

Corollary 2.47 *Let* $\mathbf{A}$ *a* BL-*algebra,* F *a* BL-*filter. Then* F *is an implicative/positive implicative/fantastic filter on* $\mathbf{A}$ *if and only if every* BL-*filter on the quotient algebra* $\mathbf{A}/F$ *is an implicative/positive implicative/fantastic* BL-*filter.*

Proof The assumptions of Theorem 2.46 are obviously fulfiled, so it is enough to consider the logic BL as L and corresponding set of consecutions R.

Chapter 3

Uniform Topologies on Algebras of Implicative Logics

The construction of topologies induced by uniformities on BL-algebras was described by M. Haveshki, A. B. Saeid and E. Eslami in [2]. These uniformities are based on congruences defined by filters in BL-algebras. Similar construction is used in [4] on Hilbert algebras (which also belongs to implicative logics).

In this chapter we are going to state and prove several theorems about properties of uniform spaces and uniform topologies that arise from Leibnitz congruences based on families of filters. Unlike cited articles we work in a general setting. Thus these results are applicable in all algebras of any implicative logic.

In this chapter we suppose we have a propositional language $\mathcal{L}$ such that $(\rightarrow, 2) \in \mathcal{L}$, an implicative logic $\mathcal{S}$ in the language $\mathcal{L}$ and $\mathbf{A} \in \mathbf{Alg}^*(\mathcal{S})$. Many of this results are dependent on the fact, that there is an isomorphism between lattice of S-filters and Leibnitz congruences on a given algebra.

3.1 From Filters to Uniformity

Definition 3.1 [6] A uniformity on a set X is a non-empty collection $\mathcal{K}$ of subsets of $X \times X$, such that:

(i) $\Delta(X) \subseteq U$ for each $U \in \mathcal{K}$,

(ii) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,

(iii) if $U \in \mathcal{K}$, then there exists $V \in \mathcal{K}$ such that $V \circ V \subseteq U$,

(iv) if U, $V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,

(v) if $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{K}$.

The ordered pair $\langle X, \mathcal{K} \rangle$ is called a uniform space. Members of $\mathcal{K}$ are called entourages or surroundings.

Remark 3.2 Clearly, uniformity on X is a lattice filter in boolean algebra $\langle \mathcal{P}(X \times X), \cap, \cup, ^{\mathcal{C}}, \emptyset, X \times X \rangle$ due to conditions (iv) and (v). The set of all uniformities on a given set X will be denoted by $\mathcal{U}ni(X)$.

First recall the definition of the Leibnitz congruence based on a given a filter $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$, in this chapter we denote it simply as U_F. Let us also recall that in implicative logics we have $U_F = \{(x, y) \mid x \to y \in F \text{ and } y \to x \in F\}$.

Theorem 3.3 *Let $\Gamma \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ be an arbitrary non-empty family of* S*-filters closed under finite intersections. Let*

$$\mathcal{K}_\Gamma = \{U \subseteq A \times A \mid (\exists F \in \Gamma)(U_F \subseteq U)\}.$$

Then $\mathcal{K}_\Gamma$ forms a uniformity on A.

Proof We are going to check conditions in the definition of uniformity.

(i) For any $F \in \Gamma$, $x \in A$ we have $x \to x \in F$, since F is a S-filter. Thus $\Delta(A) \subseteq U_F$ and hence $\Delta(A) \subseteq U$ for all $U \in \mathcal{K}_\Gamma$.

(ii) If $U \in \mathcal{K}_\Gamma$, then $U_F \subseteq U$ for some $F \in \Gamma$. Thus $U_F^{-1} \subseteq U^{-1}$. Clearly $U_F = U_F^{-1} \in \mathcal{K}_\Gamma$. According to the definiton of $\mathcal{K}_\Gamma$, we get $U_F^{-1} \subseteq U^{-1} \in \mathcal{K}_\Gamma$.

(iii) If $U \in \mathcal{K}_\Gamma$, then there exists $F \in \Gamma$ such that $U_F \subseteq U$. The transitivity of U_F implies that $U_F \circ U_F \subseteq U_F$, so $U_F \circ U_F \subseteq U$.

(iv) Let U, $V \in \mathcal{K}_\Gamma$. Then there exist $F, G \in \Gamma$ satisfying $U_F \subseteq U$, $U_G \subseteq V$. We claim that $U_F \cap U_G = U_{F \cap G}$: if $(x, y) \in U_F$ and $(x, y) \in U_G$, then $x \to y \in F$ and $x \to y \in G$, so $x \to y \in F \cap G$, the same argument we use for $y \to x$, hence $(x, y) \in U_{F \cap G}$.

Conversely, if $(x, y) \in U_{F \cap G}$, then $x \to y \in F \cap G$ and $y \to x \in F \cap G$. Thus $(x, y) \in U_F \cap U_G$. Γ is closed under finite intersections, so $F \cap G \in \Gamma$ and clearly $U_{F \cap G} \in \mathcal{K}_\Gamma$. Now it is sufficient to consider that $U_{F \cap G} \subseteq U \cap V$ and therefore $U \cap V \in \mathcal{K}_\Gamma$.

(v) Let us assume $U \subseteq V \subseteq A \times A$, $U \in \mathcal{K}_\Gamma$. There exists $F \in \Gamma$ such that $U_F \subseteq U$. Thus $U_F \subseteq V$, so $V \in \mathcal{K}_\Gamma$.

This proof is an analogy of the proof published by Haveshki et al. in [2] about BL-algebras. In fact, there is no reason to suppose **A** is a BL-algebra – as we have seen above, it is sufficient to have an algebra of an arbitrary implicative logic.

More generally, the proof does not depend on the way how the congruence U_F on the algebra **A** was constructed from the S-filters.

If we introduce a well known notion of a base and a subbase of a uniformity $\mathcal{U}$, we can deal with this uniformities more easily then in the articles [1] and [2]. A base of a uniformity $\mathcal{U}$ is a subset $\mathcal{B} \subseteq \mathcal{U}$ from which $\mathcal{U}$ can be recovered by taking supersets of elements of the base to fulfil the codition (v). So a base $\mathcal{B}$ of a uniformity $\mathcal{U}$ on X is a non-empty collection of subsets of $X \times X$ satisfying the following conditions:

(i) $\Delta(X) \subseteq U$ for each $U \in \mathcal{B}$,

(ii) if $U \in \mathcal{B}$, then there exists $V \in \mathcal{B}$ such that $V \circ V \subseteq U$,

(iii) if $U \in \mathcal{B}$, then there exists $V \in \mathcal{B}$ such that $V \subseteq U^{-1}$,

(iv) if U, $V \in \mathcal{B}$, then $W \subseteq U \cap V$ for some $W \in \mathcal{B}$.

A subbase of uniformity $\mathcal{U}$ is a subset of $\mathcal{D} \subseteq \mathcal{U}$ such that all finite intersection of elements of $\mathcal{D}$ form a base of $\mathcal{U}$. Notice that any set of equivalence relations (closed under finite intersections) is a subbase (base) of some uniformity. As we expect, the uniformity is uniquely determined by its base (subbase), for details see Page, [6].

If we take a non-empty family $\Lambda \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ of S-filters, then the system $\{U_F \mid F \in \Lambda\}$ is a subbase for a certain uniformity on A (it is enough to realize that U_F is an equivalence for every $F \in \Lambda$) and if we require that Λ is closed under finite intersections, we obtain just a uniformity base (direct consequence of the fact that there is an isomorphism between a lattice of S-filters and a lattice of congruences generated by these S-filters - fact noticed in the previous chapter).

In the following text we always denote the uniformity constructed from the non-empty family of S-filters $\Gamma \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ closed under finite intersections by the symbol $\mathcal{K}_\Gamma$. Let us make another convention: if we do not assume that Γ is closed under finite intersections, we denote the uniformity with a subbase $\{U_F \mid F \in \Gamma\}$ by the symbol $\mathcal{U}_\Gamma$

Definition 3.4 The uniformity $\mathcal{K}$ on the set X is discrete if every superset of $\Delta(X)$ belongs to $\mathcal{K}$.

Theorem 3.5

1. *Let* Γ *be a non-empty family of* S*-filters closed under finite intersections. Then* $\mathcal{K}_\Gamma$ *is the discrete uniformity if and only if* $\{1\} \in \Gamma$*.*

2. *Let* $\mathbf{A} \in \mathbf{Alg}^*_{\mathbf{FSI}}(\mathrm{S})$*,* Λ *a non-empty family of* S*-filters. Then* $\mathcal{U}_\Lambda$ *is the discrete uniformity if and only if* $\{1\} \in \Lambda$*.*

Proof

1. Assume $\{1\} \in \Gamma$. Since $U_{\{1\}} = \Delta(X)$, $\mathcal{K}_\Gamma$ contains every superset of $\Delta(X)$.

   Conversely, let $\mathcal{K}_\Gamma$ be the discrete uniformity, so $\Delta(X) \in \mathcal{K}_\Gamma$. Thus Γ contains a S-filter F such that $U_F \subseteq \Delta(X)$ and this S-filter F is preciously $\{1\}$.

2. If $\{1\} \in \Lambda$, then $\mathcal{U}_\Lambda$ is obviously the discrete uniformity (same argument as in the previous case).

   Let us suppose $\mathbf{A} \in \mathbf{Alg}^*_{\mathbf{FSI}}(\mathrm{S})$. Then $\{1\}$ is finitely meet-irreducible (Theorem 2.27). Let $\mathcal{U}_\Lambda$ be the discrete uniformity, so $\Delta(X) \in \mathcal{U}_\Lambda$. Thus Λ contains a finite subset of S-filters $F_1, \dots F_n$ such that $\bigcap\{U_{F_i} \mid i = 1, \dots, n\} \subseteq \Delta(X)$. Since $\Delta(X) = U_{\{1\}}$, we have $\{1\} = \bigcap\{F_i \mid i = 1 \dots, n\}$. Thus – because of $\{1\}$ is finitely meet-irreducible – $F_i = \{1\}$ for some $i = 1, \dots, n$. So $\{1\}$ is contained in Λ.

The second statement of this theorem doesn't have a parallel in [2] it is based on some special property of the algebra and clearly does not hold in general – it is enough to consider a simple example of four-valued Boolean algebra and set of filters $\Lambda = \{\{a, 1\}, \{\neg a, 1\}\}$. Then clearly $\mathcal{U}_\Lambda$ is discrete but $\{1\} \notin \Gamma$.

Definition 3.6 The uniform space $\langle X, \mathcal{K}\rangle$ is totally bounded (precompact) if for each $U \in \mathcal{K}$ there exists a finite subset $S \subseteq X$ such that $U[S] = X$.

Definition 3.7 Let $\langle X, \mathcal{K}\rangle$ be a uniform space, $U \in \mathcal{K}$. Subset $M \subseteq X$ is called U-small, if $M \times M \subseteq U$.

Remark 3.8 It is easy to see that the uniform space $\langle X, \mathcal{K}\rangle$ is totally bounded if and only if X can be covered by a finite number of U-small subsets for each $U \in \mathcal{K}$.

Lemma 3.9 *Let $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$. Then $\mathbf{A}$ can be covered by a finite number of U_F-small subsets of A if and only if there exists a finite set*

$$Q = \{q_1, \dots, q_n\} \subseteq A$$

satisfying the following condition:
for all $x \in A$ exists $q \in Q$ such that $x \to q \in F$ and $q \to x \in F$. In other words, $A = \bigcup\{U_F[q_i] \mid i = 1, \dots, n\}$.

Proof Let us assume a finite covering consisting of system of U_F-small sets $M_1, \dots, M_n$. If $x \in A$, then $x \in M_i$ for some $1 \leq i \leq n$. M_i is U_F-small, so $M_i \times M_i \subseteq U_F$. Now we can choose an arbitrary element q_i from M_i (if M_i is not finite, in this point we use axiom of choice). According to definition of U_F we have $x \to q_i \in F$ and $q_i \to x \in F$.

Conversely, suppose we have a subset $Q \subseteq A$ satisfying the condition in the theorem for a given S-filter F. Let M_i be a subset of $x \in A$ such that $x \to q_i \in F$ and $q_i \to x \in F$. Clearly, $M_i \times M_i \subseteq U_F$. The number of such M_i is finite, because Q is also finite.

This lemma refers to Theorem 5.7 in [2]. The theorem can be formulated in a more more natural way: A can be covered by a finite number of U_F-small subsets of A if and only if A/F is finite.

S-filters and the Refinement Relation

Uniformities on a given set can be naturally ordered by the refinement relation. In the following paragraphs we are going to focus on some basic connections between S-filters (ordered by inclusion) and the refinement relation on corresponding uniformities.

Definition 3.10 ([5]) A refinement of the uniformity $\mathcal{K}$ on the set X is a uniformity $\mathcal{R}$ on the same set such that each entourage $K \in \mathcal{K}$ belongs to $\mathcal{R}$. In this situation we say that $\mathcal{R}$ refines $\mathcal{K}$ or that $\mathcal{K}$ coarsens $\mathcal{R}$, symbolically $\mathcal{R} \leq \mathcal{K}$.

In fact, the refinement relation on the class of uniformities is just the "reversed" inclusion, similarly as in topologies on a given set.

Theorem 3.11 *Let Γ, Θ be non-empty subsets of $\mathcal{F}i_{\mathrm{S}}\mathbf{A}$ closed under finite intersections satisfying the following condition: for all $F \in \Gamma$ there is $G \in \Theta$ such that $G \subseteq F$. Then $\mathcal{K}_\Theta \leq \mathcal{K}_\Gamma$.*

Proof Consider $K \in \mathcal{K}_\Gamma$. By definition of $\mathcal{K}_\Gamma$ there is $F \in \Gamma$ such that $U_F \subseteq K$ and from the assumption of the theorem we have $G \in \Theta$ satistfying $G \subseteq F$. Thus $U_G \subseteq U_F \subseteq K$. Hence $K \in \mathcal{K}_\Theta$.

Remark 3.12 This theorem holds also for families of filters Γ, Θ which are not closed intersections: if the condition in the theorem is satisfied, then $\mathcal{U}_\Theta \leq \mathcal{U}_\Gamma$ – proof is analogous.

Theorem 3.13 *Let F, G be* S*-filters in* $\mathbf{A}$*. If $\mathcal{K}_{\{F\}} \leq \mathcal{K}_{\{G\}}$, then $F \subseteq G$.*

Proof Let $F \nsubseteq G$. Then $U_F \nsubseteq U_G$. By the definition of $\mathcal{K}_{\{G\}}$, $U_G \in \mathcal{K}_{\{G\}}$. Moreover, $K \in \mathcal{K}_{\{F\}}$ holds if and only if $U_F \subseteq K$. We can conclude that U_G does not belong to $\mathcal{K}_{\{F\}}$. Thus $\mathcal{K}_{\{F\}} \nleq \mathcal{K}_{\{G\}}$.

Corollary 3.14 *(i) Let F, G be* S*-filters in* $\mathbf{A}$*, $F \subseteq G$. Then $\mathcal{K}_{\{F\}} \leq \mathcal{K}_{\{G\}}$.*

(ii) A mapping $f\colon \mathcal{F}i_\mathrm{S}\mathbf{A} \to \mathcal{U}ni(A)$ defined by the following rule: $f(F) = \mathcal{K}_{\{F\}}$ is an embedding from $\langle \mathcal{F}i_\mathrm{S}\mathbf{A}, \subseteq\rangle$ to $\langle \mathcal{U}ni(A), \leq\rangle$.

(iii) Let $\mathbf{A} \in \mathbf{Alg}^\ell(\mathrm{S})$, M be an maximal S*-filter. Then for every non-empty $\Gamma \subseteq \mathcal{F}i_\mathrm{S}\mathbf{A}$ such that $\Lambda \neq \{A\}$ holds $\mathcal{U}_\Lambda \leq \mathcal{U}_{\{M\}}$.*

(iv) Let $\mathbf{A} \in \mathbf{Alg}^*_\mathbf{FSI}(\mathrm{S})$, Then for every non-empty $\Lambda \subseteq \mathcal{F}i_\mathrm{S}\mathbf{A}$ satisfying $\{1\} \notin \Lambda$ holds $\mathcal{K}_{\mathcal{F}i_\mathrm{S}\mathbf{A}\setminus\{1\}} \leq \mathcal{U}_\Lambda$.

(v) Let $\mathbf{A} \in \mathbf{Alg}^\ell(\mathrm{S})$, let Γ, Θ be non-empty subsets of $\mathcal{F}i_\mathrm{S}\mathbf{A}$ such that $\bigcap\Gamma \subset \bigcap\Theta$. Then $\mathcal{U}_\Gamma \leq \mathcal{U}_\Theta$.

Proof Item (i)is a direct consequences of Theorem 3.11, (ii) is a consequence of (i) and Theorem 3.13.

(iii): If $\mathbf{A}$ is linearly ordered, every filter (except the trivial one A) is a subset a maximal S-filter M. The rest is consequence of Remark 3.12.

(iv): The assumption $\mathbf{A} \in \mathbf{Alg}^*_\mathbf{FSI}(\mathrm{S})$ guarantee that finite intersection of any elements in Γ does not equal to $\{1\}$. Now it is enough to consider that every finite intersection of elements in Γ belongs to $\mathcal{F}i_\mathrm{S}\mathbf{A} \setminus \{1\}$.

(v): Assume that $\bigcap\Gamma \subset \bigcap\Theta$. Since $\mathbf{A}$ is linearly ordered, $\mathcal{F}i_\mathrm{S}$ is also linearly ordered by inclusion. Thus we have for all $F \in \Theta$ there exists $G \in \Gamma$ such that $G \subseteq F$. Hence we can use the Remark 3.12

The statements (iii), (iv) and (v) do not again have any counterpart in [2].

Theorem 3.15 *Let Λ be a non-empty family of* S*-filters in* $\mathbf{A}$ *closed under arbitrary intersections, $J = \bigcap\{F \mid F \in \Lambda\}$. Then $\mathcal{K}_{\{J\}} = \mathcal{K}_\Lambda$.*

Proof At first we are going to prove $\mathcal{K}_\Lambda \subseteq \mathcal{K}_{\{J\}}$. Let $K \in \mathcal{K}_\Lambda$. So $U_F \subseteq K$ for some $F \in \Lambda$. Clearly, $J \subseteq F$, thus $U_J \subseteq U_F \subseteq K$. Hence $K \in \mathcal{K}_{\{J\}}$.

Conversely, let K be an entourage in $\mathcal{K}_{\{J\}}$. So $U_J \subseteq K$. Since Λ is closed under arbitrary intersection, $J \in \Lambda$ and also $U_J \in \mathcal{K}_\Lambda$. This proves $K \in \mathcal{K}_{\{J\}}$, so $\mathcal{K}_{\{J\}} \subseteq \mathcal{K}_\Lambda$.

Remark on Uniform Continuity

One of the important motivations for studying uniform spaces is the notion of uniform continuity. Uniform spaces are in some certain sense a generalization of metrics spaces, where uniform continuity is a well known notion. In this paragraph we are going to present the notion of uniform continuity on a uniform space and give some examples of uniformly continuous functions based on common connectives of some implicative logics.

Definition 3.16 [6] Let $\langle X, \mathcal{K}\rangle$, $\langle Y, \mathcal{P}\rangle$ be uniform spaces, $f : X \to Y$ a function. Then f is uniformly continuous with respect to $\mathcal{K}$, $\mathcal{P}$ if for all $P \in \mathcal{P}$ exists $K \in \mathcal{K}$ such that $(f \times f)[K] \subseteq P$.

The condition in the definition above says that for every $P \in \mathcal{P}$ there is $K \in \mathcal{K}$ such that $(f(x), f(y)) \in P$ for all $(x, y) \in K$.

The following theorem provides us a sufficient condition for uniform continuity of a function $f : A \to A$ with respect to $\mathcal{K}_\Gamma$.

Theorem 3.17 *Let $\langle A, \mathcal{K}_\Gamma\rangle$ be a uniform space, where $\Gamma \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ is an arbitrary non-empty set of* S*-filters closed under finite intersections. Then every function $f : A \to A$ which satisfies $(f \times f)[U_F] \subseteq U_F$ for all $F \in \Gamma$, is uniformly continuous with respect to $\mathcal{K}_\Gamma$.*

Proof Let L be an arbitrary entourage of $\mathcal{K}_\Gamma$. Then there is $F \in \Gamma$ such that $U_F \subseteq L$. Our aim is to show an entourage $K \in \mathcal{K}_\Gamma$ such that $(f \times f)[K] \subseteq L$. By the assumption $(f \times f)[U_F] \subseteq U_F$ we have $(f \times f)[U_F] \subseteq L$, thus U_F is the requested entourage in $\mathcal{K}_\Gamma$.

The condition in Theorem 3.17 can be equivalently stated as follows:
for all $F \in \Gamma$, for all x, $y \in A$, if $x \to y \in F$ and $y \to x \in F$, then $f(x) \to f(y) \in F$ and $f(y) \to f(x) \in F$.

The uniform continuity is often defined with respect to bases of given uniformities. This theorem is in that sense a special case.

Now, thanks to previous results we can easily check the uniform continuity of several common functions.

Corollary 3.18 *The following functions from A to A are uniformly continuous with respect to $\mathcal{K}_\Gamma$: $\varphi^{\mathbf{A}}(y, \bar{a})$, where $\varphi(y, \bar{x})$ is an arbitrary formula in the language $\mathcal{L}$ with $n+1$ variables and $\bar{a} \in A^n$.*

Proof It is enough to consider, that these all these functions are congruent with the symmetrization of the operation $\rightarrow$ (see the definition of the implicative logics).

3.2 Uniform topologies

Theorem 3.19 *Let $\Gamma \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ be a family of filters in $\mathbf{A}$ closed under finite intersections, $\mathcal{K}_\Gamma$ associated uniform space on A. Then*

$$\mathcal{T}_\Gamma = \{Q \subseteq A \mid (\forall x \in Q)(\exists U \in \mathcal{K}_\Gamma)(U[x] \subseteq Q)\}$$

is a topology on A.

Proof Obviously, $\emptyset$ and A belongs to $\mathcal{T}_\Gamma$. It is also clear that $\mathcal{T}_\Gamma$ is closed under arbitrary unions.

Now we are going to prove that $\mathcal{T}_\Gamma$ is closed under finite intersections. Let Q_1, $Q_2 \in \mathcal{T}_\Gamma$ and consider some $x \in Q_1 \cap Q_2$. By the definition of $\mathcal{T}_\Gamma$ we have U_1, $U_2 \in \mathcal{K}_\Gamma$ such that $U_1[x] \subseteq Q_1$ and $U_2[x] \subseteq Q_2$. $\mathcal{K}_\Gamma$ is uniformity, so $U_1 \cap U_2 \in \mathcal{K}_\Gamma$. Clearly $U_1[x] \cap U_2[x] \subseteq Q_1 \cap Q_2$. Hence $Q_1 \cap Q_2 \in \mathcal{T}_\Gamma$ and $\mathcal{T}_\Gamma$ constitutes a topology on A.

This theorem holds generaly, not only for congruences that arise from the set of filters.

This topology is usually called a uniform topology on A. Similarly as by the uniformities, in the following text we always denote the uniform topology based on the uniformity $\mathcal{K}_\Gamma$ by the symbol $\mathcal{T}_\Gamma$. (We consider a family $_\Gamma$ of filters which is closed under finite intersections. If this condition is not fulfiled, we simply add these finite intersections to Γ.)

Remark 3.20 This uniform topology can be viewed equivalently as a topology in which a neighborhood base at point $x \in A$ is formed by the family of sets $U[x]$, where U runs through all entourages in $\mathcal{K}_\Gamma$ (James, [5]).

Theorem 3.21 *The topological space $\langle A, \mathcal{T}_\Gamma\rangle$ is completely regular. (Page, [6])*

It is easy to see that for all $x \in A$ and any $U \in \mathcal{K}_\Gamma$, $U[x]$ is an open set in the uniform topology $\mathcal{T}_\Gamma$.

Another easy observation is that if the uniformity $\mathcal{K}_\Gamma$ is discrete, then the topology is also discrete $\mathcal{T}_\Gamma$.

Theorem 3.22 *Let $\Lambda \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ be a non-empty family of* S*-filters closed under arbitrary intersections and $J = \bigcap\{F \mid F \in \Lambda\}$. Then $\mathcal{T}_{\{J\}} = \mathcal{T}_\Lambda$.*

Proof From the Theorem 3.15 we have $\mathcal{K}_{\{J\}} = \mathcal{K}_\Lambda$. Topologies induced by equal uniformities are equal, so $\mathcal{T}_\Lambda = \mathcal{T}_{\{J\}}$.

Remark 3.23 The assumption that Λ is closed under arbitrary intersection is essential – it is not sufficient to assume that Λ is closed under finite intersection: in the case that all S-filters are not trivial, but the intersection $\bigcap \Lambda = \{1\}$, $\mathcal{T}_{\{J\}}$ is discrete topology, but $\mathcal{T}_\Lambda$ coarser. In Theorem 5.1 in [2] there is this assumption omited.

Similarly as by the uniformities, we have a refinement relation on the class of all topologies on a given set. A refinement of the topology $\mathcal{T}$ is a topology $\mathcal{V}$ on the same set X such that each open set of $\mathcal{T}$ is also an open set of $\mathcal{V}$. Obviously, "finer uniformity induces finer topology", so it is possible to state analogies of Theorem 3.11 and its corollaries, e.g.:
Let Γ, Θ be a non-empty subsets of $\mathcal{F}i_{\mathrm{S}}\mathbf{A}$ closed under finite intersections satisfying the following condition: for all $F \in \Gamma$ there is $G \in \Theta$ such that $G \subseteq F$. Then $\mathcal{T}_\Theta$ a refinement of $\mathcal{T}_\Gamma$.
In Haveshki et al. [2] they consider a uniformity with a subbase $\{U_F \mid F \text{ is a maximal filter of } \mathbf{A}\}$ and topology Max induced by this uniformity and they claim that for every $\Lambda \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ closed under intersections the topology $\mathcal{T}_\Lambda$ is finer than Max. This statement is not true: let Λ be a family consisting of one maximal non-trivial filter and consider an algebra $\mathbf{A} \in \mathbf{Alg}^*(\mathrm{S})$ such that the intersection of some pair of maximal filters is a trivial filter. In that case $\mathcal{T}_\Lambda$ is coarser than Max.

Some Important Clopen and Compact Sets in $\mathcal{T}_\Gamma$

In this subsection we are going to state and prove some properties of S-filters and associated congruences in terms of clopen and compact subsets in $\mathcal{T}_\Gamma$.

Let us start with a "technical" lemma which we will use also in the further subsection about continuity. At first we are going to define one instrumental notation.

Definition 3.24 Let X, Y be subsets of A.

$$X \to Y := \{x \to y \mid x \in X, y \in Y\}.$$

Lemma 3.25 *Let F be a* S*-filter in* $\mathbf{A}$*, x, $y \in A$. Then*

$$(U_F[x] \to U_F[y]) \subseteq U_F[x \to y]$$

Proof Let $a \in U_F[x]$ and $b \in U_F[y]$. Thus $a \to x \in F$ and $x \to a \in F$, analogiously $b \to y \in F$ and $y \to b \in F$. Therefore $(a \to b, x \to y) \in U_F$, so $a \to b \in U_F[x \to y]$, hence $U_F[x] \to U_F[y] \subseteq U_F[x \to y]$

Theorem 3.26 *Let* $\Gamma \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ *be a non-empty family of* S*-filters in* $\mathbf{A}$ *closed under finite intersections. Then every* S*-filter in* Γ *is a clopen subset in the topological space* $\langle A, \mathcal{T}_\Gamma \rangle$*.*

Proof Consider a S-filter $F \in \Gamma$. First we are going to show that F is a union of $\{U_F[q] \mid q \in F\}$ in $\mathcal{T}_\Gamma$. Let $q \in F$: clearly $q \in U_F[q]$, so we have $F \subseteq \bigcup\{U_F[q] \mid q \in F\}$. The second inclusion we are going to prove by contradiction. Suppose $\bigcup\{U_F[q] \mid q \in F\} \nsubseteq F$. Then there is $a \in \bigcup\{U_F[q] \mid q \in F\} \setminus F$. Thus there is some $r \in F$ such that $a \in U_F[r]$, so $a \to r \in F$ and $r \to a \in F$. Since F is S-filter, $a \in F$ and this is contradiction. Thus F is a union of a certain system of open sets.

Now we are going to show that F^c is open (F^c is an abbreviation for $A \setminus F$). Proof is similar – let $q \in F^c$, since $q \in U_F[q]$ we have $F^c \subseteq \bigcup\{U_F[q] \mid q \in F^c\}$. Second inclusion: for the contradiction let us consider $a \in \bigcup\{U_F[q] \mid q \in F^c\} \setminus F^c$, so $a \in F$. Then there is some $r \in F^c$ such that $a \to r \in F$ and $r \to a \in F$. Sice F is S-filter, r also belongs to F, which leads to a contradiction.

Thus F is clopen in $\mathcal{T}_\Gamma$.

Theorem 3.27 *Let* $\Gamma \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ *be a non-empty family of* S*-filters in* $\mathbf{A}$ *closed under finite intersections. Then for all* $x \in A$ *and* $F \in \Gamma$*,* $U_F[x]$ *is a clopen subset in the topological space* $\langle A, \mathcal{T}_\Gamma \rangle$*.*

Proof As noticed above, $U_F[x]$ is open in $\mathcal{T}_\Gamma$ for all $x \in A$. Our aim is to prove that $(U_F[x])^c$ is also open in $\mathcal{T}_\Gamma$.

Consider $y \in (U_F[x])^c$. So, at least one of the following conditions is satisfied: $x \to y \in F^c$, $y \to x \in F^c$. Without loose of generality let us suppose $y \to x \in F^c$.

By Lemma 3.25 we have $(U_F[y] \to U_F[x]) \subseteq U_F[y \to x]$.

We claim $U_F[y \to x] \subseteq F^c$: if some $a \in U_F[y \to x] \setminus F^c$, then $a \in F$ and also $a \in U_F[y \to x]$, so $a \to (y \to x) \in F$. Since F is S-filter, then $y \to x \in F$, so we have a contradiction (we have supposed $y \to x \in F^c$).

We are going to show that $U_F[y] \subseteq (U_F[x])^c$.

Suppose $z \in U_F[y]$. Clearly $z \to y \in F$ and $y \to z \in F$. Since $z \in U_F[y]$ and $x \in U_F[x]$ and by Lemma 3.25 we have

$$z \to x \in (U_F[y] \to U_F[x]) \subseteq U_F[y \to x] \subseteq F^c,$$

hence $z \to x \in F^c$, so $z \in (U_F[x])^c$. Thus for any $y \in (U_F[x])^c$, $U_F[y] \subseteq (U_F[x])^c$, hence $(U_F[x])^c$ is open, so $U_F[x]$ is clopen.

Lemma 3.28 *Let $F_0 \subseteq F_1$ be a pair of* S*-filters in* $\mathbf{A}$*. Then each F_1 is a clopen set in the topological space $\langle A, \mathcal{T}_{F_0}\rangle$.*

Proof Consider $\Lambda = \{F_i \mid i \in \mathbf{N}\}$. Clearly $F_0 = \bigcap \Lambda$. By Theorem 3.22 we have $\mathcal{T}_\Lambda = \mathcal{T}_{\{F_0\}}$ and by Theorem 3.26 F_i $(i \in \mathbf{N})$ is a clopen set in the topological space $\langle A, \mathcal{T}_{F_0}\rangle$.

Note that since F is a clopen set in $\mathcal{T}_\Gamma$, the topological space $\langle A, \mathcal{T}_\Gamma\rangle$ is not connected, if $F \neq A$ (i.e. F is a proper filter in $\mathbf{A}$).

Theorem 3.29 *Let $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ and $A \setminus F$ is a finite set, then the topological space $\langle A, \mathcal{T}_{\{F\}}\rangle$ is compact.*

Proof Let $\{O_\alpha \mid \alpha \in I\}$ be an open covering of A, $A \setminus F = \{x_1, x_2, \ldots, x_n\}$. Then we have α_0, $\alpha_1, \ldots,$ $\alpha_n \in I$, such that $1 \in O_{\alpha_0}$, $x_i \in O_{\alpha_i}$ $(i \in \{1, 2, \ldots, n\})$. According to the definition of $\mathcal{T}_{\{F\}}$, $O_{\alpha_0} \in \mathcal{T}_{\{F\}}$ implies that $U_F[1] \subseteq O_{\alpha_0}$. Since $U_F[1] = F$, we can cover F by a single open set in $\mathcal{T}_{\{F\}}$ and the subset $A \setminus F$ is covered by finite number of O_{α_i}, hence the topological space is compact.

Theorem 3.30 *Let $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$. Then F is a compact set in the topological space $\langle A, \mathcal{T}_{\{F\}}\rangle$.*

Proof Let $\{O_\alpha \mid \alpha \in I\}$ be a system of open sets in $\mathcal{T}_{\{F\}}$ covering F. F is a filter, so $1 \in F$, thus there is some $\gamma \in I$ such that $1 \in O_\gamma$. By the same argument as in the previous proof, $U_F[1] \subseteq O_\gamma$. Since $U_F[1] = F$, F can be covered by a single open set in $\mathcal{T}_{\{F\}}$.

Theorem 3.31 *Let $F \in \mathcal{F}i_{\mathrm{S}}\mathbf{A}$, $x \in A$. Then $U_F[x]$ is a compact set in the topological space $\langle A, \mathcal{T}_{\{F\}}\rangle$.*

Proof Consider an open covering $\{O_\alpha \mid \alpha \in I\}$ of $U_F[x]$. Clearly $x \in U_F[x]$, so there is some $\alpha \in I$ such that $x \in O_\alpha$. Recall the definition of $\mathcal{T}_{\{F\}}$: $O_\alpha \in \mathcal{T}_{\{F\}}$, thus $U_F[x] \subseteq O_\alpha$, so $U_F[x]$ can be covered by single a open set in $\mathcal{T}_{\{F\}}$.

Lemma 3.32 *Let $F_0 \supseteq F_1$ be* S*-filters in* $\mathbf{A}$*. Then both* S*-filters are compact sets in the topological space $\langle A, \mathcal{T}_{F_0}\rangle$.*

Proof By Theorem 3.30 F_0 is a compact set in $\langle A, \mathcal{T}_{F_0}\rangle$. Obviously, any subset of a compact set is compact.

Continuity of Implication and Other Functions in $\mathcal{T}_\Gamma$

Roughly said, topological spaces are sets equipped by an additional structure. In various branches of mathematics an important role is played by "structure-preserving" mappings. In topology are these mappings continuous functions. In this subsection we will prove the continuity (with respect to $\mathcal{T}_\Gamma$) of all (binary) operations congruent with the symmetrization of the operation $\to$, in particular the operation $\to$.

Theorem 3.33 *Let $\Gamma \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ a non-empty family of* S*-filters in* $\mathbf{A}$ *closed under finite intersections, c an arbitrary binary operation on* $\mathbf{A}$ *congruent with symmetrization of the operation $\to$. Then the operation c is continuous with respect to $\mathcal{T}_\Gamma$.*

Proof In this proof we will use the following notation:

$$c(X, Y) := \{c(x, y) \mid x \in X, y \in Y\}.$$

To prove the continuity of c is enough to show that following statement holds:
Let O be an open set in $\mathcal{T}_\Gamma$, x, $y \in A$ such that $c(x, y) \in O$. Then there exist open sets O_1 and O_2 such that $x \in O_1$, $y \in O_2$ and $c(O_1, O_2) \subseteq O$.

Consider x, $y \in A$ such that $c(x, y) \in O$, where $O \in \mathcal{T}_\Gamma$. There exist $U[c(x, y)] \subseteq O$ and $F \in \Gamma$ such that $U_F \subseteq U$.

Assume $a \in U_F[x]$, $b \in U_F[y]$, so $a \to x \in F$ and $x \to a \in F$, analogiously $b \to y \in F$ and $y \to b \in F$. Since c is congruent with $\to$, it holds that $(c(x, y), c(a, b)) \in U_F$, thus $c(a, b) \in U_F[c(x, y)]$. As a conclusion we have $c(U_F[x], U_F[y]) \subseteq U_F[c(x, y)]$. The choice $U_F[x]$ as O_1 and $U_F[y]$ as O_2 completes the proof.

Corollary 3.34 *Let $\Gamma \subseteq \mathcal{F}i_{\mathrm{S}}\mathbf{A}$ a non-empty family of filters in* $\mathbf{A}$ *closed under finite intersections. Then the operation $\to$ is continuous with respect to $\mathcal{T}_\Gamma$.*

Proof It is a direct consequence of the previous theorem, it enough to consider that $\to$ is congruent with the symetrization of the implication.

In [2] Haveshki et al. presented the notion of topological BL-algebra. We are going to recall the definition and show that their result is a corollary of Theorems 3.34 and 3.33.

Definition 3.35 Let $\mathbf{A}$ be a BL-algebra, $\mathcal{T}$ a topology on A. We say that a pair $\langle A, \mathcal{T} \rangle$ is a topological BL-algebra if the operations $\to$ and $*$ of $\mathbf{A}$ are continuous with respect to $\mathcal{T}$.

Theorem 3.36 *Let* $\mathbf{A}$ *be a* BL*-algebra,* Γ *a family of filters in* $\mathbf{A}$ *closed under finite intersections. Then* $\langle A, \mathcal{T}_\Gamma \rangle$ *is a topological* BL*-algebra.*

Proof The theorem is an easy corollary of Theorems 3.34 and 3.33. It is enough to consider that $*$ is congruent with symetrization of the operation $\rightarrow$.

Similar notion of topological Hilbert algebra was defined by Saeid in [4]. Hilbert algebra is an algebraic counterpart of the implicative fragment of intuicionistic logic which belongs to the class of implicative logics, so Theorem 3.34 is there also applicable.

Roughly said, we have proved that if we have an arbitrary implicative logic with connectives congruent with symetrization of the implication together with the associated algebra $\mathbf{A}$, the realizations of these connectives will be continuous in described topology on A.

Chapter 4

Conclusion

4.1 Summary of Our Results

The thesis is devoted to study the filters on algebras of the implicative (fuzzy) logics.

In the second chapter we proved general results about connections of the alternative axiomatization of a given logic and the properties of all filters on corresponding algebras. Most of work of Haveshki et al. [1] and Kondo, Dudek [3] are just consequences of our propositions. Thanks to chosen framework (AAL) all proofs became relatively short. Given examples concern BL-algebras and Gödel/Boolean/MV-albegras and we mentioned the link between equivalent definitions of implictive/positive implicative/fantastic filters.

In the third chapter we have shown that most of the theorems published in Haveshki et al. [2] hold not only in BL-algebras, but in any algebra of implicative logic. Their results are particular: presented statements do not depend on special properties of BL-algebras, their results are based on the fact that they work with a congruence on the algebra. In some cases we point out possible weakenings of the original assumptions (e.g. when the algebra of implicative logic is linearly ordered or when the algebra is finitely meet-irreducible).

4.2 Fields for Further Study

At the end we would like to mention the possible extensions of this work:

- Relax the assumtions about implicative logic and make steps towards analogical theory for weakly implicative logics and/or equvalential logics.

- Apply the results presented in the third chapter on the other types of filters defined by the same pattern as implicative/positive implicative/fantastic filters and cover more results published in other articles.

- In chapter three we started with the notion of the filter and then we used families of filters to construct a Leibnitz congruence on the algebra. These congruences were taken as (sub)bases of the uniformities. Another approach could start with the notion of congruence (more natural in the AAL environment) and study the properties of uniformities/uniform topologies in case of Leibnitz congruence, Fregean congruence, etc.

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