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# BAKALÁŘSKÁ PRÁCE

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# Kosmologické modely a jejich perturbace

Ústav teoretické fyziky

Vedoucí práce: prof. RNDr. Jiří Bičák, DrSc., dr.h.c.
Studijní program: Fyzika, obecná fyzika

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Ráda bych poděkovala mému vedoucímu prof. Jiřímu Bičákovi za to, že jsem u něho vůbec mohla psát tuto práci. Děkuji za jeho vedení a čas, rovněž za jeho podněty, rady a připomínky a také za zajímavé články a knihy, kterými jsem byla zásobena.

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**BACHELOR THESIS**

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# Cosmological Models and Their Perturbations

Institute of Theoretical Physics

Supervisor: prof. RNDr. Jiří Bičák, DrSc., dr.h.c.
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Abstrakt: V práci podáváme přehled hlavních rysů newtonovské kosmologie. Popisujeme kosmologické modely a rozebíráme omezení kladené na newtonovskou kosmologii a problémy spojené s gravitačním potenciálem v případě rovnoměrného rozdělení hmoty. Ukazujeme možný vztah mezi kosmologickou konstantou a celkovou hmotou v konečném vesmíru a zavádíme tlak do rovnic popisujících newtonovské kosmologické modely. Poté studujeme růst malých lineárních perturbací hustoty a tlaku v jinak izotropním vesmíru. Ukazujeme, že původní odvození Jeansova vztahu pro gravitační nestabilitu ve statickém vesmíru není správné. V dynamickém vesmíru odvodíme vztah velmi podobný.

Klíčová slova: Newtonovská gravitace, kosmologické modely, gravitační nestabilita.

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Abstract: We give the review of the main features of the Newtonian cosmology. We describe the cosmological models and discuss the limits of the Newtonian cosmology and problems connected with the gravitational potential in case of the uniform matter distribution. We show the possible relation between the cosmological constant and the total mass of a finite universe and introduce pressure into the equation describing Newtonian cosmological models. We study the growth of small linear perturbations in the density and pressure in an otherwise isotropic universe. We show that the original derivation of Jeans' formula for gravitational instability in a static universe is incorrect, however, in a dynamical universe a very similar formula appears.

Keywords: Newton's gravity, cosmological models, gravitational instability.

# Chapter 1

# Introduction

Cosmology belongs to the main areas of fundamental physics. As particle physics describes interactions between basic constituents of the universe, the main goal of cosmology is to explain the structure and evolution of the universe on the large scales. The interaction governing primarily the macroscopic matter in the universe is gravitation. Although electromagnetic interaction is stronger than gravitation and also has long range, there exist both negative and positive charges and most of macroscopic bodies are neutral. While magnetic fields can be important in astrophysics, in cosmology their role appears to be much less significant.

The best present theory of gravitation is Einstein's general theory of relativity. However, similarly to the situation in astrophysics where it is possible to use classical Newton's theory of gravitation in most of the cases of interest, it is possible to employ Newtonian gravitation in the analysis of the behaviour of cosmological models. The review of the main features of the Newtonian cosmology is the subject of this thesis.

In Chapter 2 we analyse the basic principles of cosmology – the Copernican principle and the cosmological principle. In Chapter 3 we describe Olbers' paradox of the dark night sky and its resolution. Next we deal with the Newtonian potential for the uniform distribution of matter and illustrate the problems connected with this potential. We also add the cosmological constant into Poisson's field equation as the weak-field limit of Einstein's equations with the non-vanishing cosmological term requires. Then cosmological models of homogenous and isotropic universes are derived and the limits of the applicability of the Newtonian cosmology are discussed. Finally we are concerned with the newest aspects of the Newtonian cosmology. We show the possible relation between the cosmological term and the total mass of a finite universe. Then we introduce pressure into the equations governing the dynamics of the universe to enable us to treat the cosmological term as a perfect fluid in the Newtonian framework.

In Chapter 4 we focus our attention on the growth of small linear perturbations in the density and pressure in an otherwise homogenous and isotropic universe. Jeans' formula for gravitational instability in a static universe is discussed and it is shown that its proof is incorrect. However, further we derive that in a dynamical universe a very similar formula for gravitational instability is valid. In Chapter 5 we give a brief summary.

Since our attention is primarily concentrated on the Newtonian models we relegated a brief review of relativistic cosmology to Appendix. At the end of the Appendix we explicitly demonstrate that the equations for the dynamical evolution of the homogenous and isotropic universes within relativistic cosmology are identical to those of the Newtonian cosmology. Our discussion of the behaviour of the Newtonian models in Chapter 3 thus goes over to relativistic cosmology without change.

# Chapter 2

# The Cosmological Principle

## 2.1 The Homogeneity and Isotropy of the Universe

One of the main postulates on which the standard cosmology is based, is the Copernican principle: "The Earth is not in a special location in the Universe." Copernicus originally proposed that we are not in a central position, indeed, but in fact he only replaced the geocentric picture with the heliocentric one and set the Sun at the centre of the cosmos. Nevertheless nowadays the Copernican principle is understood in the sense that humans are not privileged observers of the universe. If we accept this principle and contrast it with astronomical observational data which reveal the universe to be isotropic on a large-scale, we may assume that an arbitrary place enables existence of observers also perceiving the space around them isotropic. This argument leads to the conclusion that the universe is globally isotropic and hence also homogenous, since global isotropy implies homogeneity. This is called the cosmological principle and is the foundation of most of the cosmological theories.

Evidence for the large-scale isotropy lies in the distribution of the galaxies, reflected in the distribution of their apparent magnitudes and redshifts, and in the distribution of the radio sources. And, of course, there is the remarkable isotropy of the low-temperature cosmic microwave background (CMB) radiation – the most convincing source of evidence. Precise measurements show that the CMB has a small "dipole" anisotropy. It is slightly warmer in one direction of the sky than in the opposite direction. As the

Earth moves within our solar system, the Sun moves in the Galaxy, and the Galaxy moves with respect to the local group, the intensity of the radiation is modified by the Doppler effect. Hence, the standard explanation of this anisotropy of the CMB is that our local group of galaxies moves relatively to the "fundamental" reference frame in which the CMB is isotropic. From the CMB data, the relative velocity of this motion appears to be about 630 km/s.

The cosmological principle is basically unprovable. We should be aware that its postulating is not quite self-evident. It is a bit like explorers put at a fixed place on a strange planet who on the basis of their observation conclude that the planet is covered with desert altough there can be deep jungle on the other hemisphere. As an example of inhomogeneous universe we can consider an island of stars in a flat infinite universe. Franz Selety [12] proved that such an island would remain stable if its density vanishes as $1/r^2$.

We shall now treat the concepts of homogeneity and isotropy further. Homogeneity of the universe means that, at a given moment of a preferred time, all places and, of course, the physical laws are the same everywhere, i.e., that the physical conditions are identical. Thus we cannot distinguish one point from the others by any physical measurement at a given moment of time. Similarly, isotropy of the universe means that any observer who is moving with the cosmological substratum cannot distinguish one direction from the others by any physical measurement. There is no ambiguity about a given moment of time in Newton's theory (time is determined up to an additive constant). However, in general relativity the problem how to understand this intuitive phrase arises. But if we divide the entire spacetime into preferred spacelike hypersurfaces marked with the time parameter, the concept "at a given moment of time", then translated into "on a given spacelike hypersurface", is made precise. A homogenous universe can also be anisotropic but if one place is isotropic, the universe is globally isotropic. In an inhomogenous universe there might be one place from which the space seems isotropic. We perceive a state of isotropy. Provided we could observe the universe homogeneous, the cosmological principle would become valid automatically. However, when we look out into the universe we also look back into the past and thus the cosmological principle remains an useful heuristic hypothesis from which various consequences follow. Their validity can then be checked by confronting them with observation.

# Chapter 3

# Idealized Newtonian Cosmology

## 3.1 Olbers' Paradox

Olbers' paradox, sometimes also referred to as the dark night sky paradox, shows contradiction between the observed darkness of the night sky and an assumption of a uniform infinite eternal universe[1] (which was often believed to be the universe we live in). The paradox can be easily derived from the following consideration.

Assume that we live in an infinite universe with an infinite number of uniformly distributed luminous stars which all are for convenience assumed to be identical "average" stars. Every line of sight should then eventually terminate on the surface of a star. At this point, due to the finite speed of light, we also need to suppose that the uniform distribution did not change rapidly in the course of time. The mean free path of the line of sight (i.e., the mean distance of background stars) can be calculated as

$$\lambda = \frac{V}{\pi R_S^2}, \tag{3.1}$$

where $V$ is the volume that contains on the average one star, $R_S$ is the radius of the star and $\pi R_S^2$ is the star's cross-sectional area. Consider an observer on the Earth who observes two stars at the distances $r_1, r_2$ at their respective solid angles $\Omega_1, \Omega_2$. Since the stars have the same magnitude

[1] In the relativistic framework it is also possible to have the paradox in a finite unbounded universe.

we have $\Omega_1 r_1^2 = \Omega_2 r_2^2$. The observed brightness or the apparent luminosity decreases with distance $r$ as $L/4\pi r^2$ where $L$ is the absolute luminosity. Hence, the apparent luminosity of observed stars related to the solid angle is the same

$$\frac{L}{4\pi r_1^2 \Omega_1} = \frac{L}{4\pi r_2^2 \Omega_2}. \tag{3.2}$$

The amount of light is thus equal at every solid angle, independent of the distance which the light travels from. The night sky should then be entirely covered by stars and it should appear like the Sun's disk. Now the question arises: Why the sky is dark at night?

Despite its name, Olbers' paradox was first revealed in 1610 by Johannes Kepler who preferred a finite bounded universe. He used the paradox as an argument against the infinite universe proposed by Thomas Digess in 1576 [5]. Since then, many scientist paid attention to the problem of the dark night sky. In 1823 Heinrich Wilhelm Olbers proposed as the explanation the interstellar absorption. However, from thermodynamics it follows that the absorbing medium would soon heat up and the emission of radiation would equal the absorption.

American poet and writer Edgar Allan Poe was the first who came with the right idea how to solve the paradox. In his essay Eureka: A Prose Poem (1848) he suggested that the distance of the background (i.e., (3.1)) is so immense that the light from it has not yet reached the Earth. Hence, the paradox can be resolved on the assumption that our universe lasted for a finite time only and is also young enough so that the size of the visible universe is much less than the background distance. The first person who quantitatively derived the finite-age solution was William Thomson Kelvin (in 1901).

However, strictly speaking the finite age of the universe itself is not sufficient since the theory describing the creation of the universe must also account for the paradox. Nowadays the mainstream theory of the universe really assumes that our universe has a finite age and its creation is described by the Big Bang theory. According to this theory the sky was much brighter in the past, especially in the first few seconds of the universe. However, the Big Bang theory also involves subsequent expansion which redshiftened this radiation to microwave wavelenghts and so formed the invisible CMB.

Since observation reveals that our universe expands the redshift also affects the light from stars. Nevertheless, despite a common belief this diminution caused by the expansion is not great enough to resolve the paradox completely without the finite-age assumption as it is shown for example in [6].

## 3.2 The Newtonian Gravitational Potential

Gravitation in the classical conception is considered to be a force with long range of action which cannot be shielded by any known means. For our purposes we can modify the formulation of Newton's first law under the presence of gravitation to take the following form [14]:

i) There exist free falls - a class of motions of bodies acted upon only by a gravitational force.

ii) There exists a global Cartesian coordinate system $\{x^i\}, i = 1, 2, 3$, and a real-valued function $\Phi$, called the gravitational potential, such that free falls are characterized by the equation of motion

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\frac{\partial \Phi(\vec{x}, t)}{\partial x^i}, \tag{3.3}$$

where $t$ is an absolute time. This equation is invariant under the transformations which consist of a rigid rotation and a transformation of the form

$$x^i \to x'^i = x^i + f^i(t) \tag{3.4}$$

which can be understood as an arbitrarily accelerated motion of the rigid frame $\{x^i\}$. This must be accompanied by the transformation of the potential

$$\Phi(\vec{x}, t) \to \Phi'(\vec{x}\,', t) = \Phi(\vec{x}, t) - \sum_{i=1}^{3} \frac{\mathrm{d}^2 f^i(t)}{\mathrm{d}t^2} x^i + g(t). \tag{3.5}$$

Here the real-valued functions $f^i(t)$ and $g(t)$ are arbitrary.

Newton's law of attraction between two point masses can be derived from Poisson's equation which playes a role of a gravitational "field equation" in Newton's theory:

$$\Delta \Phi = 4\pi G \rho, \tag{3.6}$$

where $G$ is the gravitational constant and $\rho$ is the mass density (which is the delta function for a point mass); the potential is assumed to be bounded for $|\vec{x}| \to \infty$. Note that under this condition the solutions of (3.6) differ only by a constant and are therefore a special case of (3.4) and (3.5) leaving

(3.3) invariant. If we require the potential to tend to zero for $|\vec{x}| \to \infty$ the general solution of (3.6) is then given by

$$\Phi(\vec{x}) = -G \int_{\Re^3} \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|} \mathrm{d}^3 x' \tag{3.7}$$

provided the integral exists. This can be guaranteed by assuming $\rho$ to decrease sufficiently rapidly.

In accordance with the cosmological principle we will now consider the uniform matter distribution and global isotropy. Evidently, (3.7) diverges for $\rho =$ const. but we can obtain an unbounded solution directly from (3.6). Hereafter we look only for a spherically symmetric potential because an observer at the origin (we shall consider only observers moving with the cosmological fluid) must also see the universe isotropic. On that condition the solution takes the form

$$\Phi(r) = \frac{2\pi G\rho}{3} r^2 + \text{const.}, \tag{3.8}$$

where $r = |\vec{x}|$. We will get back to this solution later.

In order to avoid an unbounded potential Hugo von Seeliger [13] (in 1895) and Carl Neumann [10] (in 1896) proposed to replace Poisson's equation by

$$\Delta\Phi - \Lambda\Phi = 4\pi G\rho, \quad \Lambda = \text{const.} \tag{3.9}$$

which has for constant $\rho$ the solution

$$\Phi = -\frac{4\pi G\rho}{\Lambda}. \tag{3.10}$$

Obviously this potential appears alike to all observers. Furthermore we can see that the gravitational forces vanish, hence such a universe remains static. Alternatively, we may say there exists a particular inertial frame in which the matter is at rest. The solution of (3.9) for a unit point mass located at the origin is

$$\Phi(r) = -\frac{G}{r} e^{-\sqrt{\Lambda} r}. \tag{3.11}$$

However, Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{3.12}$$

(where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ the metric tensor, $R$ the scalar curvature, $\Lambda$ the cosmological constant, $c$ the speed of light, and $T_{\mu\nu}$ the stress-energy tensor) in the weak-field limit do not go over into (3.9) but into the equation

$$\Delta\Phi + \Lambda c^2 = 4\pi G\rho. \tag{3.13}$$

Then (3.6) is a special case of (3.13). The solution for constant $\rho$ takes the form

$$\Phi = \frac{4\pi G\rho - \Lambda c^2}{6} r^2 + \text{const.} \tag{3.14}$$

Negative $\Lambda$ contributes to density whereas positive $\Lambda$ causes "anti-gravity". At first sight this potential appears to violate the cosmological principle since it seems that an observer at the origin is the only one who can see the space isotropic. However, if we make a translation with $a^i$ arbitrary

$$\begin{aligned} x^i &\to x'^i = x^i - a^i, \\ \Phi(\vec{x},t) &\to \Phi'(\vec{x}\,',t) = \Phi(\vec{x},t) \end{aligned} \tag{3.15}$$

leaving (3.3) invariant, the potential remains the same and we have actually moved "the origin" to the point which was in the old coordinates marked as $[a^1, a^2, a^3]$. Since $a^i$ are arbitrary, all points of the universe are equivalent and the potential appears radial to all observers. In addition, due to the generally nonvanishing force the matter cannot be at rest in any frame of reference. So we are forced to deal with a non-static universe.

Nonetheless, potential (3.14) faces up some difficulties. In any observer's system the gravitational force is radial and so there is no preferred direction of the force at a particular point. Moreover, since the force vanishes at the origin every observer's system may be regarded as an inertial frame although different observers can be mutually accelerated. We shall discuss these conceptual issues connected with the Newtonian cosmology in Section 3.4.

For unit point mass $M$ equation (3.13) gives

$$\Phi = -\frac{MG}{r} - \frac{\Lambda c^2}{6} r^2 + \text{const.} \tag{3.16}$$

By differentiating this potential we obtain modified Newton's law of gravitation for a unit point mass attracted to point mass $M$ under the presence of the cosmological constant:

$$\vec{F} = \left(-\frac{MG}{r^2} + \frac{\Lambda c^2 r}{3}\right)\frac{\vec{x}}{r}. \tag{3.17}$$

$F \propto r^{-2}$ and $F \propto r$ are the only two cases which allow stable planetary orbits. Moreover in both cases a spherically symmetric mass can be treated as a point mass located at the centre. In addition the force inside a spherical shell is zero in the first case whereas in the other case it remains constant.

This fact enables us to derive (3.14) alternatively. Consider the uniform matter distribution with density $\rho$. Choose a system of coordinates and an arbitrary point at arbitrary distance $r$ from the origin. The point lies on the sphere of radius $r$ with the origin as its centre. Hence a unit mass at this point acts upon by the force

$$\vec{F} = \left(-\frac{MG}{r^2} + \frac{\Lambda c^2 r}{3}\right)\frac{\vec{x}}{r} = \left(-\frac{4\pi\rho G}{3} + \frac{\Lambda c^2}{3}\right)\vec{x}. \tag{3.18}$$

By integration we get the potential (3.14). Obviously the arbitrariness in a selection of a sphere, on which the point under consideration lies, makes the direction and the magnitude of the force fully indeterminate. The potential, however, due to the allowed transformations, remains independent of this choice.

## 3.3 Cosmological Models

In this section we fully accept the cosmological principle as the essential assumption. Now the purpose is to describe the motion of the substratum of the universe ("fundamental particles", or "typical galaxies") idealized as a perfect fluid. For this purpose consider two observers $O$ and $O'$ moving with the substratum. They set up a system of coordinates, each of them considering themselves being at their respective origins, and observe at time $t$ the motion of the substratum which they characterize by the velocity fields $\vec{v}(\vec{r},t)$ and $\vec{v}\,'(\vec{r}\,',t)$ where $\vec{r}$ and $\vec{r}\,'$ are the position vectors of fluid particles relative to $O$, $O'$ respectively. From the cosmological principle we deduce that the functions $\vec{v}$, $\vec{v}\,'$ must be the same, for otherwise $O$ and $O'$ would have different pictures of the universe.

Suppose that they measure the velocity of a general point $P$ of the liquid and find the values $\vec{v}(\vec{r},t)$ and $\vec{v}\,'(\vec{r}\,',t) = \vec{v}(\vec{r}\,',t)$. Let the vector $OO'$ be $\vec{a}$. Then the velocity of $O'$ relative to $O$ is $\vec{v}(\vec{a},t)$. Since the observers look at the same particle we get $\vec{r}\,' = \vec{r} - \vec{a}$ and consequently

$$\vec{v}(\vec{r},t) - \vec{v}(\vec{a},t) = \vec{v}\,'(\vec{r}\,',t) = \vec{v}(\vec{r}\,',t) = \vec{v}(\vec{r}-\vec{a},t). \tag{3.19}$$

Thus $\vec{v}(\vec{r}, t)$ is a linear function of $\vec{r}$ and we may write

$$v^i = A^i{}_j(t) r^j, \tag{3.20}$$

where $A^i{}_j$ is a tensor and $i, j = 1, 2, 3$. Since in general a tensor distinguishes directions, $A^i{}_j$ must be a multiple of the unit tensor. Then (3.20) becomes

$$\vec{v} = H(t)\vec{r}, \tag{3.21}$$

where we have introduced Hubble's parameter $H(t)$. If we put

$$H(t) = \frac{1}{R(t)} \frac{\mathrm{d}R(t)}{\mathrm{d}t}, \tag{3.22}$$

where $R(t)$ is a non-negative funtion called the scale factor, (3.21) takes the form

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{1}{R} \frac{\mathrm{d}R}{\mathrm{d}t} \vec{r}, \tag{3.23}$$

and may be integrated into

$$\vec{r} = R(t)\vec{r}_0, \quad \vec{r}_0 = \vec{r}(t_0), \quad R(t_0) = 1. \tag{3.24}$$

This result shows that the requirement of global isotropy and homogeneity implies that each observer occupies the origin of a unique frame of reference in which they observe a uniform expansion or contraction due to a time-dependent scale factor.

From the cosmological principle we also infer that the density and pressure must be the same throughout the whole space, i.e.,

$$\rho = \rho(t), \tag{3.25}$$

$$p = p(t). \tag{3.26}$$

To complete the analysis we impose the conservation laws in the form of the continuity equation and Euler's equation of motion for a perfect fluid and supplement them by Poisson's equation (3.13). If we substitute (3.21), (3.22) and (3.25) into the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{3.27}$$

we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\rho}{R} \frac{\mathrm{d}R}{\mathrm{d}t} \nabla \cdot \vec{r} = \frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{3\rho}{R} \frac{\mathrm{d}R}{\mathrm{d}t} = 0, \tag{3.28}$$

and, consequently,

$$\rho(t) = \frac{\rho(t_0)}{R^3(t)}. \tag{3.29}$$

This is also a natural result of (3.24).

Since every observer's system is inertial we can impose Euler's equation

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0 \tag{3.30}$$

without contradictions. We have already shown that in case of uniform density Poisson's equation (3.13) has the solution (3.14). By substituting (3.14), (3.21), (3.22), (3.24) and (3.26) into Euler's equation we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}\vec{r}\right) + \frac{4\pi G\rho - \Lambda c^2}{3}\vec{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\vec{r}_0\right) + \left(\frac{4\pi G\rho}{3} - \frac{\Lambda c^2}{3}\right)R\vec{r}_0 =$$

$$= \left(\frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{4\pi G\rho}{3}R - \frac{\Lambda c^2}{3}R\right)\vec{r}_0 = 0. \tag{3.31}$$

Or we may write

$$\frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{4\pi G\rho}{3} - \frac{\Lambda c^2}{3} = 0. \tag{3.32}$$

Finally, by integration and by virtue of (3.29) we obtain

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8\pi G\rho}{3}R^2 - \frac{\Lambda c^2}{3}R^2 + K = 0, \tag{3.33}$$

where $K$ is a constant of integration. It is remarkable that this equation has precisely the same form as the relativistic Einstein-Friedmann equation for the universe filled with (pressure-free) dust (see Appendix A for a brief description of relativistic cosmology).

Integration of (3.33) leads to elliptic functions but we shall settle for the qualitative integration. Our discussion is similar to that in [1], for example. There is a number of various solutions which are of interest. For convenience we put $8\pi G\rho(t_0) = 3C$, rewrite the equation as

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{C}{R} + \frac{\Lambda c^2}{3}R^2 - K, \tag{3.34}$$

and denote the function on the right-hand side by $F(R)$. First of all note that the equation is invariant under the time inversion. In order to eliminate the solutions which can be obtained from the others by reversing the time direction we will choose the direction of time, where possible, such that $R(t)$ is increasing. The origin of time is also matter of arbitrariness and so we will choose that one to get $R(0) = 0$, if possible. Then different solutions can be distinguished on the basis of the signs of $\Lambda$ and $K$.

I) $K < 0$

(i) $\Lambda > 0$. In this case $F(R)$ is a positive function of $R$. For small $t$ we have $R \approx (9Ct^2/4)^{\frac{1}{3}}$. The expansion slows down until $R_m = (3C/2\Lambda c^2)^{\frac{1}{3}}$, which is a minimum of $F(R)$, then it accelerates and for large $t$ we get $R \approx \exp\left[t(\frac{1}{3}\Lambda c^2)^{\frac{1}{2}}\right]$. This exponential expansion is typical for the model called de Sitter universe.

(ii) $\Lambda = 0$. $F(R)$ is now a positive decreasing function, hence the rate of the expansion decreases continuously. For small $t$ we again have $R \approx (9Ct^2/4)^{\frac{1}{3}}$, and for large $t$, $R \approx (-K)^{\frac{1}{2}}t$.

(iii) $\Lambda < 0$. $F(R)$ is a decreasing function. It takes positive values in $0 < R < R_c$, where $F(R_c) = 0$, and negative values when $R > R_c$. For small $t$ we again have $R \approx (9Ct^2/4)^{\frac{1}{3}}$. However, the expansion does not go on forever as in the previous cases. It slows down until $R$ reaches its maximum value $R_c$, after that, the contraction begins. The system runs through its previous phase but in the opposite sense, until $R = 0$. Then the cycle starts again and we get an oscillating model. From physical point of view, of course, fundamental problems arise with passing through singularities at $R = 0$.

II) $K = 0$

(i) $\Lambda > 0$. $F(R)$ is positive, with a minimum at $R_m = (3C/2\Lambda c^2)^{\frac{1}{3}}$. Thus the behaviour is very similar to the case where $K < 0$ and $\Lambda > 0$. But now we have the explicit solution available

$$R^3 = \frac{3C}{2\Lambda c^2}\left\{\cosh\left[t(3\Lambda c^2)^{\frac{1}{2}}\right] - 1\right\}. \tag{3.35}$$

(ii) $\Lambda = 0$. Here we can immediately write down the simple solution

$$R = \left(\frac{9Ct^2}{4}\right)^{\frac{1}{3}}. \tag{3.36}$$

(iii) $\Lambda < 0$. Now $F(R)$ is a decreasing function. As for $K < 0$ and $\Lambda < 0$ we get the oscillating model with similar properties. The explicit solution takes the form

$$R^3 = \frac{3C}{2(-\Lambda)c^2}\left\{1 - \cos\left[t\left(-3\Lambda c^2\right)^{\frac{1}{2}}\right]\right\}. \tag{3.37}$$

III) $K > 0$

If $\Lambda > 0$, $F(R)$ has the minimum

$$F(R_m) = \left(\frac{9C^2\Lambda c^2}{4}\right)^{\frac{1}{3}} - K. \tag{3.38}$$

Evidently the behaviour of $R(t)$ depends, among others, on the sign of $F(R)$. We shall define a critical value $\Lambda_c$ for which $F(R_m) = 0$,

$$\Lambda_c = \frac{4K^3}{9C^2c^2}. \tag{3.39}$$

(i) $\Lambda > \Lambda_c$. $F(R)$ is now positive. The behaviour of $R$ is similar to the previous cases where $\Lambda > 0$.

(ii) $\Lambda = \Lambda_c$. In this case $F(R)$ is positive except the point $R_m$ at which $F = 0$. This allows three different solutions:

(ii *a*) Obviously there is a static solution $R = R_m$. Since $R(t_0) = 1$, it follows that $R_m = 1$. From that we infer

$$\Lambda = \Lambda_c = \frac{4\pi G\rho}{c^2}. \tag{3.40}$$

Hence, the potential (3.14) is really constant consistently with our expectations. This model corresponds to the Einstein static universe. It was the first model of the universe, constructed by Einstein in 1917, but it is not stable.

(ii *b*) Next we start with expansion, which slows down and $R$ approaches $R_m$ asymptotically from below. For small $t$, as usual, $R \approx (9Ct^2/4)^{\frac{1}{3}}$.

(ii *c*) Last solution approaches $R_m$ asymptotically from above for $t \to -\infty$. Expansion continuously increases and $R \approx \exp\left[t(\frac{1}{3}\Lambda c^2)^{\frac{1}{2}}\right]$ for large $t$. This is so-called Lemaître model.

Eddington in 1930 proposed another model. His universe initially exists for an infinite period of time as a static Einstein universe and then, as a result of disturbance, it begins to expand.

(iii) $0 < \Lambda < \Lambda_c$. Now the equation $F(R) = 0$ has two different roots $R_1 < R_2$. Hence $F(R)$ is positive except the interval $R_1 \leq R \leq R_2$ and we have two solutions:

(iii *a*) $0 \leq R \leq R_1$. We again get an oscillating universe which behaves similarly to the preceding oscillating models.

(iii *b*) $R_2 \leq R$. As $t \to -\infty$, $R \approx \exp\left[-t(\frac{1}{3}\Lambda c^2)^{\frac{1}{2}}\right]$. The rate of contraction decreases until $R$ reaches its minimum value $R_2$. Then expansion sets in and continuously increases. The system runs through the previous phase, but in the opposite sense. For large $t$, $R \approx \exp\left[t(\frac{1}{3}\Lambda c^2)^{\frac{1}{2}}\right]$. Again the same expansion as in de Sitter universe.

(iv) $\Lambda \leq 0$. In this case we have again an oscillating model similar to the previous ones.

Now we have described all possible models of the homogeneous and isotropic universe filled with incoherent matter (dust) within Newtonian cosmology. To sum up, we may divide the models into six classes: the static universe, expanding models starting at a definite time from $R = 0$, expanding models starting with a finite value $R$ at $t = -\infty$, expanding models starting at a definite time from $R = 0$ and asymptotically approaching a finite value $R$, oscillating models, and, finally, the model which contracts from infinity to a finite value $R$ and then expands again to infinity.

The qualitative behaviour of these models can be derived by using the method of the effective potential. Equation (3.34) can be rewritten in the form

$$H^2 = \frac{\Lambda c^2}{3} - V(R), \tag{3.41}$$

where

$$V(R) = \frac{K^2}{R^2} - \frac{C}{R^3} \tag{3.42}$$

is the effective potential. In Figure 3.1 we have displayed the plot of $V(R)$ versus $R$. The diagram is only schematic, not quantitative. The models of possible universes can be derived from the simple consideration that the difference between $\Lambda c^2/3$ and $V(R)$ must be non-negative to have Hubble's parameter real. The regions where this difference is negative are inaccesible.

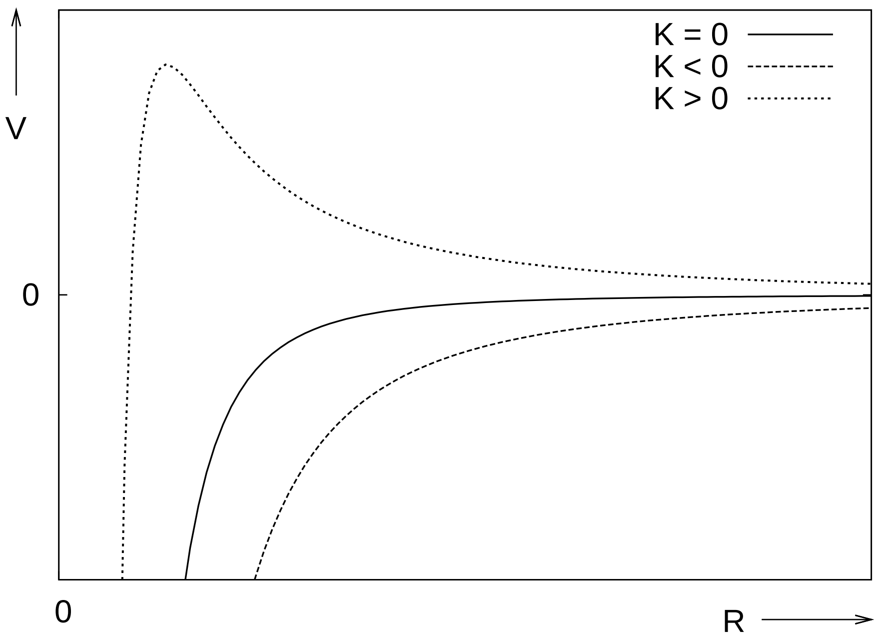


Figure 3.1: Schematic plot of the effective potential $V(R)$ versus the scale factor $R$. See the text for details.

## 3.4 Limits of the Newtonian Cosmology

We have now treated the largest possible system - the universe. With respect to the limitations of Newtonian mechanics we can hardly expect that our results are aplicable to that whole system. Classical mechanics is adequate if the velocity of particles is small compared with the speed of light $c$ and if gravity is weak, i.e., the gravitational potential $\Phi$ is small in comparison with $c^2$: $\frac{\Phi}{c^2} << 1$. Regarding (3.21) and (3.14) we see that the results can be use only for sufficiently small regions.

We have already noticed that Newtonian cosmology assumes that each of two mutually accelerated observers possesses an inertial system. This is of course not Newtonian since in the classical Newtonian theory inertial frames are global and move with constant relative velocities. Moreover, the uniform matter distribution defines no preffered direction of gravitational forces which contradicts the general statement that the gravitational force at a given point is determined completely by the instantaneous distribution of matter in the universe.

Consider a sphere of radius $R(t)$ imbedded in an empty space. On condition that the density inside the sphere remains uniform and the motion is radial at every point, the dynamics of such a sphere is the same as if it was a part of a uniform isotropic universe. So we may investigate a spherical region of the universe as a sphere in an empty space. This approach also means that the direction and magnitude of the gravitational force are fully determined and, in general, the only observer (moving with the fluid) possesing an inertial system is the one at the centre of the sphere. Hence, we see that Newtonian mechanics is valid for sufficiently small spherical volumes.

## 3.5 The Cosmological Constant and Dark Energy

There are two possible interpretations of cosmological constant $\Lambda$. First, it can be considered as a term contributing to Newton's gravitational force. We shall treat this option in detail.

Consider a linear force $\vec{F}_L$ by which point mass $m$, with position vector $\vec{r}_m$, acts on a unit point mass, with position vector $\vec{r}$,

$$\vec{F}_L(\vec{r}) = Cm(\vec{r} - \vec{r}_m), \tag{3.43}$$

where $C$ is an arbitrary constant. By integration we find corresponding potential $\Phi_L$:

$$\Phi_L = \frac{Cm}{2}(\vec{r} - \vec{r}_m)^2 + \text{const.} \tag{3.44}$$

The potential can also be derived from Poisson's equation

$$\Delta\Phi_L = 3Cm, \tag{3.45}$$

on condition that we assume spherical symmetry around the mass $m$.

In case of general mass distribution and provided that the integrals below exist, there are expressions for $\vec{F}_L$ and $\Phi_L$ in the form of convolution:

$$\vec{F}_L(\vec{r}) = C\int_V \rho(\vec{r}\,')(\vec{r} - \vec{r}\,')\mathrm{d}V = CM(\vec{r} - \vec{r}_T), \tag{3.46}$$

$$\Phi_L(\vec{r}) = \frac{C}{2}\int_V \rho(\vec{r}\,')(\vec{r} - \vec{r}\,')^2\mathrm{d}V + \text{const.} = \frac{CM}{2}(\vec{r} - \vec{r}_T)^2 + \text{const.} \tag{3.47}$$

Here $M$ is overall mass and $\vec{r}_T$ is a position vector of the centre of mass. Hence, we see that the linear force and its potential do not depend on the mass distribution whatsoever, but acts as if the total mass is concentrated at its centre of mass. Considering the symmetry around the centre of mass, equations (3.46), (3.47) can also be derived from Poisson's equation (3.45), where point mass $m$ is replaced with overall mass $M$.

The problem, of course, arises with the uniform mass distribution. Since the space in Newtonian mechanics is an infinite Euclidean space, the mass corresponding to the uniform matter distribution takes the infinite value. However, in general relativity there may exist finite cosmological models. Newtonian cosmology gives a good approximation in sufficiently small volumes without relativistic objects such as black holes etc.; so let us now suppose that the space is finite, described on "small" scales with sufficient accuracy by the Newtonian cosmology and the total mass, which the universe consists of, is $M$. Since in case of uniform matter distribution, there is no mass center we shall suppose that the force $F_L$ and potential $\Phi_L$ is radially symmetric in a selected coordinate system - as we have already assumed previously and also discussed the difficulties connected with such arbitrariness.

Adding the linear force to Newton's gravitational force, in case of uniform matter distribution, we get

$$\vec{F} = \left( -\frac{4\pi\rho G}{3} + CM \right) \vec{r}. \tag{3.48}$$

By identifying the corresponding constants in (3.18) and (3.48) we find $CM = \Lambda c^2/3$, i.e.,

$$\Lambda \propto M. \tag{3.49}$$

Hence, except the interpretation of the cosmological term as a fundamental constant, we could possibly relate $\Lambda$ to universe's total mass. This useful insight which appears to be given for the first time very recently [3], is easily given by the Newtonian cosmology.

In relativistic treatment, the positive cosmological constant can also be represented as a perfect fluid with constant energy density $\rho_\Lambda c^2 = \Lambda c^4/8\pi G$ and negative pressure $p_\Lambda = -\rho_\Lambda c^2$. However, in Newtonian mechanics a perfect fluid with constant matter density $\rho_\Lambda$ does not satisfy the continuity equation (3.28) for a dynamical universe (when $R$ depends on time). Regarding $\Lambda$ as a fluid also in the Newtonian cosmology is useful from practical reasons; therefore our next goal will be to modify the continuity equation and consequently equations determining the scale factor [11].

Consider a volume $V(t) = V(t_0)R^3(t)$ adiabatically expanding with the universe. By substituting the relativistic expression of the energy in the volume

$$E = V(t_0)R^3\rho c^2, \tag{3.50}$$

into the first law of thermodynamics

$$\mathrm{d}E + p\mathrm{d}V = 0, \tag{3.51}$$

we get the continuity equation in an expanding universe

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H\left(\rho + \frac{p}{c^2}\right) = 0. \tag{3.52}$$

As it is possible in relativistic physics, we shall suppose that $\Lambda$ contributes to the total density with the term $\rho_\Lambda = \Lambda c^2/8\pi G$. To fit (3.52), the corresponding pressure $p_\Lambda$ must be equal to $-\rho_\Lambda c^2$. We see that on the right-hand side of equation (3.33) then appears only the density $\rho + \rho_\Lambda$, no pressure. Hence, we let that equation unchanged. By differentiating (3.33) and using (3.52) we obtain the corrected form of (3.32):

$$\frac{1}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}t^2} = -\frac{4\pi G}{3}\left(\rho_t + 3\frac{p_t}{c^2}\right) = -\frac{4\pi G}{3}\left(\rho_b + 3\frac{p_b}{c^2}\right) + \frac{\Lambda c^2}{3}, \tag{3.53}$$

where $\rho_t$ and $p_t$ are the total density and pressure respectively, and $\rho_b = \rho_t - \rho_\Lambda$, $p_b = p_t - p_\Lambda$. Since equation (3.32) is derived from

$$\frac{1}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}t^2}\vec{r} = -\nabla\Phi, \tag{3.54}$$

(3.53) is also consistent with the fact that if we do not neglect the pressure, Einstein's equations (3.12), in the weak-field limit, go over into

$$\Delta\Phi = 4\pi G\left(\rho_t + 3\frac{p_t}{c^2}\right). \tag{3.55}$$

Now the equations describing evolution of the universe, have the same form as Friedmann's equations. Equation (3.55) nicely exhibits the feature of general-relativistic physics: the source of gravity is not just $\rho$ but $\rho + 3\frac{p}{c^2}$;the pressure also contributes to gravity.

As it follows from (3.53), the negative pressure contributes to acceleration whereas the positive pressure deccelerates. This seems to be in contradiction with our intuition that compressed substance expands with greater effect. However, this intuition is related to the pressure *gradient* which is zero in idealized cosmology. Strictly speaking, the pressure should not appear in the Newtonian idealized cosmology. If we introduce pressure into the Newtonian continuity equation and field equation, we could also introduce other non-Newtonian terms which also can be sources of gravity as, for example, viscosity, electromagnetic fields etc. And if the pressure cannot be neglected we do not have a weak field any more. Nevertheless, the pressure corrections can help us to understand some interesting features of cosmology within the relatively simpler Newtonian framework.

In order to solve the evolution equations, we need to know an equation of state. In case of barotropic fluid, the equation is given by $w = p_t/\rho_t c^2$. For instance, dust, radiation and $\Lambda$ correspond to $w = 0$, $1/3$ and $-1$ respectively. On condition that the universe is filled with a perfect fluid with the equation of state $w = p_t/\rho_t c^2 = \text{const.}$, we find from (3.52)

$$\rho_t \propto R^{-3(1+w)}, \tag{3.56}$$

and from (3.33) for large $R$ or in case of $K = 0$ for any $R$

$$R(t) \propto t^{\frac{2}{3(1+w)}}, \quad \omega > -1. \tag{3.57}$$

$$R(t) \propto e^{\sqrt{\frac{\Lambda}{3}}ct}, \quad w = -1. \tag{3.58}$$

Recent observational data reveal that our universe expands at an increasing rate (”accelerating universe”). Therefore, we infer from (3.53)

$$p_t < -\frac{\rho_t c^2}{3}, \tag{3.59}$$

i.e., we need an exotic ”fluid”, called the dark energy, to fit the observations. The simplest candidate is provided by the cosmological constant. It appears that ordinary matter (planets, stars, intergalactic gas etc.) makes up only 4% of the total mass of the universe. Non-baryonic dark matter[2] makes up about 21% and the remaining 75% - the dark energy - has not been satisfactorily explained.

There are other theories explaining the accelerating expansion. In the quintessence model of the dark energy, the observed acceleration is caused by the potential energy of a dynamical scalar field, referred to as a quintessence field. In contrast to the cosmological constant, quintessence can vary in space and time. A special case of quintessence is the phantom energy - a fluid which has an equation of state $w < -1$. In a phantom energy-dominated universe, the solution of (3.33) for large $R$ or for $K = 0$ for any $R$ takes the form

$$R(t) \propto (t_s - t)^{\frac{2}{3(1+w)}}, \tag{3.60}$$

$$H(t) \propto \frac{1}{t_s - t}, \tag{3.61}$$

where $t_s$ is an integration constant. A phantom-dominated universe will end itself in a singularity, known as the Big Rip. As $t \to t_s$, i.e., at a finite future time, the scale factor, Hubble's parameter and the density diverge.

[2]Mostly cold dark matter hypothesis is considered. The particles which the cold dark matter should consist of are hypotethical WIMPs (weakly interacting massive particles), e.g. neutralino.

# Chapter 4

# Perturbed Newtonian Cosmology

## 4.1 Jeans' Formula for Gravitational Instability

In the previous chapter we have worked with a perfectly isotropic and homogenous universe. Clearly, such idealized models cannot represent precisely the real universe with stars, galaxies, galactic clusters and other types of inhomogenities. Therefore, we now focus our attention on small linear perturbations of velocity and density in an otherwise uniform universe. These played a role in the structure formation in the early homogenous universe. An attempt was made by James H. Jeans [7] already who derived a formula which gives the condition for gravitational (in)stability of a static cloud of gas. We shall first derive the formula in its original form and see that the proof is questionable.

Let us suppose that the mass of gas under consideration is in equilibrium characterized by density $\rho_0$ and pressure $p_0$ and that it stays at rest in a certain inertial frame of reference. The only force which the gas is acted upon is the gravitational force $\vec{F}_0$. General hydrodynamical Euler's equation and the continuity equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} + \frac{1}{\rho}\nabla p - \vec{F} = 0, \tag{4.1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{4.2}$$

then become

$$\frac{1}{\rho_0} \nabla p_0 - \vec{F}_0 = 0, \tag{4.3}$$

$$\rho_0 = \rho_0(x). \tag{4.4}$$

The force $\vec{F}_0 = -\nabla \Phi_0$ satisfies the equation

$$\nabla \cdot \vec{F}_0 = -4\pi G \rho_0 \tag{4.5}$$

as a consequence of Poisson's equation (3.6). Further, assume that the pressure is a function of the density only ("barotropic fluid"), hence $p = p(\rho)$.

Now we shall consider small perturbations such that the gas can move with a small velocity $\vec{v}$ and the density and force can vary like

$$\rho(\vec{x}, t) = \rho_0(\vec{x}) + \rho_1(\vec{x}, t), \tag{4.6}$$

$$\vec{F} = \vec{F}_0 + \vec{F}_1. \tag{4.7}$$

We suppose that the changes of the density are due to $\vec{F}_1$, so

$$\nabla \cdot \vec{F}_1 = -4\pi G \rho_1. \tag{4.8}$$

Squares and products of small deviations $\vec{v}$, $\rho_1$, $\vec{F}_1$, and of their derivatives will be neglected. On that condition equations (4.1), (4.2) for perturbed gas, by virtue of (4.3) and (4.4), give

$$\frac{\partial \vec{v}}{\partial t} - \vec{F}_0 - \vec{F}_1 + \frac{1}{\rho_0} \nabla p_0 + \nabla \left( \frac{\rho_1}{\rho_0} \frac{\mathrm{d}p}{\mathrm{d}\rho} \right) = \frac{\partial \vec{v}}{\partial t} - \vec{F}_1 + \nabla \left( \frac{\rho_1}{\rho_0} \frac{\mathrm{d}p}{\mathrm{d}\rho} \right) = 0, \tag{4.9}$$

$$\frac{\partial}{\partial t}(\rho_0 + \rho_1) + \nabla \cdot (\rho_0 \vec{v}) = \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \vec{v}) = 0, \tag{4.10}$$

where $\mathrm{d}p/\mathrm{d}\rho$ is understood to be taken at $\rho = \rho_0$. To derive (4.9) we have used the chain of identities

$$\frac{1}{\rho_0 + \rho_1} \nabla \left[ p(\rho_0 + \rho_1) \right] = \left( \frac{1}{\rho_0} - \frac{\rho_1}{\rho_0^2} \right) \cdot \nabla \left( p_0 + \frac{\mathrm{d}p}{\mathrm{d}\rho} \rho_1 \right) =$$

$$= \frac{1}{\rho_0}\nabla p_0 + \frac{1}{\rho_0}\nabla\left(\rho_1\frac{\mathrm{d}p}{\mathrm{d}\rho}\right) - \frac{\rho_1}{\rho_0^2}\nabla p_0 = \frac{1}{\rho_0}\nabla p_0 + \frac{1}{\rho_0}\nabla\left(\rho_1\frac{\mathrm{d}p}{\mathrm{d}\rho}\right) - \frac{\rho_1}{\rho_0^2}\frac{\mathrm{d}p}{\mathrm{d}\rho}\nabla\rho_0 =$$

$$= \frac{1}{\rho_0}\nabla p_0 + \frac{1}{\rho_0}\nabla\left(\rho_1\frac{\mathrm{d}p}{\mathrm{d}\rho}\right) + \rho_1\frac{\mathrm{d}p}{\mathrm{d}\rho}\nabla\left(\frac{1}{\rho_0}\right) = \frac{1}{\rho_0}\nabla p_0 + \nabla\left(\frac{\rho_1}{\rho_0}\frac{\mathrm{d}p}{\mathrm{d}\rho}\right).$$

By applying the divergence on (4.9), $\partial/\partial t$ on (4.10) and by virtue of (4.4) and (4.5), we get

$$\frac{\partial}{\partial t}(\nabla\cdot\vec{v}) + 4\pi G\rho_1 + \Delta\left(s\frac{\mathrm{d}p}{\mathrm{d}\rho}\right) = 0, \tag{4.11}$$

$$\frac{\partial^2 s}{\partial t^2} + \frac{\partial}{\partial t}(\nabla\cdot\vec{v}) + \frac{(\nabla\rho_0)}{\rho_0}\frac{\partial\vec{v}}{\partial t} = 0, \tag{4.12}$$

where we have put $s = \rho_1/\rho_0$. These equations reduce to

$$\frac{\partial^2 s}{\partial t^2} + \frac{(\nabla\rho_0)}{\rho_0}\frac{\partial\vec{v}}{\partial t} = 4\pi G\rho_1 + \Delta\left(s\frac{\mathrm{d}p}{\mathrm{d}\rho}\right). \tag{4.13}$$

Now to obtain the equation for $s$ given by Jeans [7]

$$\frac{\partial^2 s}{\partial t^2} = 4\pi G\rho_0 s + \Delta\left(s\frac{\mathrm{d}p}{\mathrm{d}\rho}\right), \tag{4.14}$$

we have to suppose $\rho_0 = \text{const}$. However, if $\rho_0$ (and therefore $p_0$) is constant, hydrodynamical equation (4.3) with Poisson's equation (4.5) have no solution! Jeans did not explicitly mention such an assumption but, in fact, in order to get (4.14) he, indeed, had to take $\rho_0$ constant.

Nevertheless, we shall proceed and deduce Jeans' condition for gravitational instability following his derivation. Consider a solution of (4.14) in the form of a wave propagated along axis $x$, for example, characterized by the wave length $\lambda$ and the function $h(t)$, describing how the amplitude changes with time,

$$s = h(t)\cos\frac{2\pi x}{\lambda}. \tag{4.15}$$

By substituting (4.15) into (4.14) we find

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = \left(4\pi G\rho_0 - \frac{4\pi^2}{\lambda^2}\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)h. \tag{4.16}$$

Hence, we see that the perturbation increases exponentially with time if

$$\lambda > \left(\frac{\pi}{G\rho_0}\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)^{\frac{1}{2}}. \tag{4.17}$$

Therefore, a clump of mass of gas with the size greater than this will be unstable. From (4.16) it is clearly seen that the gravitational part in the equation, $\propto G$, implies a smaller velocity of propagation. Nevertheless, for $\lambda$ small, the second term on the r.h.s. $\propto \frac{1}{\lambda^2}$ will be large and the gravitational term will represent just a small correction to this term.

## 4.2 Gravitational Stability in Newtonian Models

We have already shown that the solution of equations (4.1), (4.2) and (3.13) for a perfectly isotropic universe takes the form

$$\rho_0 = \frac{\rho_0(t_0)}{R(t)^3}, \tag{4.18}$$

$$p_0 = p_0(t), \tag{4.19}$$

$$\vec{v}_0 = \frac{\dot{R}(t)}{R(t)}\vec{r}, \tag{4.20}$$

$$\vec{F}_0 = \left(-\frac{4\pi\rho_0 G}{3} + \frac{\Lambda c^2}{3}\right)\vec{r}, \tag{4.21}$$

where $\dot{}$ denotes $\partial/\partial t$ and $R(t)$ satisfies the differential equation

$$\dot{R}^2 - \left(8\pi G\rho_0 + \frac{\Lambda c^2}{3}\right)R^2 + K = 0 \tag{4.22}$$

and the initial condition $R(t_0) = 1$. We add to that solution small perturbations $\rho_1, p_1, \vec{v}_1, \vec{F}_1$ and seek for a perturbed solution [2]. Euler's hydrodynamical equations (4.1), the continuity equation (4.2) and Poisson's equation (3.13) for perturbed quantities give

$$\dot{\rho}_1 + 3\frac{\dot{R}}{R}\rho_1 + \frac{\dot{R}}{R}(\vec{r}\cdot\nabla)\rho_1 + \rho_0\nabla\cdot\vec{v}_1 = 0, \tag{4.23}$$

$$\dot{\vec{v}}_1 + \frac{\dot{R}}{R}\vec{v}_1 + \frac{\dot{R}}{R}(\vec{r}\cdot\nabla)\vec{v}_1 = \vec{F}_1 - \frac{1}{\rho_0}\nabla p_1, \tag{4.24}$$

$$\nabla\cdot\vec{F}_1 = -4\pi G\rho_1. \tag{4.25}$$

Squares and products of perturbations and of their derivatives were again neglected.

As in the previous section, we shall suppose that there is an equation of state $p = p(\rho)$. On that condition the pressure perturbation can be written as

$$p_1 = \rho_1 \frac{\mathrm{d}p}{\mathrm{d}\rho}, \tag{4.26}$$

where $\mathrm{d}p/\mathrm{d}\rho$ is understood to be calculated at $\rho = \rho_0$.

We shall first assume spherically symmetric perturbation, i.e.,

$$\rho_1 = \rho_1(r), \tag{4.27}$$

$$\vec{v}_1 = \frac{\vec{r}}{r}v_1(r), \tag{4.28}$$

$$\vec{F}_1 = \frac{\vec{r}}{r}F_1(r). \tag{4.29}$$

The equations (4.23) – (4.25) then become

$$\dot{\rho}_1 + 3\frac{\dot{R}}{R}\rho_1 + r\frac{\dot{R}}{R}\frac{\partial\rho_1}{\partial r} + \rho_0\left(\frac{\partial v_1}{\partial r} + \frac{2v_1}{r}\right) = 0, \tag{4.30}$$

$$\dot{v}_1 + \frac{\dot{R}}{R}v_1 + r\frac{\dot{R}}{R}\frac{\partial v_1}{\partial r} = F_1 - \frac{1}{\rho_0}\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\partial\rho_1}{\partial r}, \tag{4.31}$$

$$\frac{\partial F_1}{\partial r} + \frac{2F_1}{r} = -4\pi G\rho_1. \tag{4.32}$$

For convenience we change to a new variable by the transformation

$$r = \chi R(t). \tag{4.33}$$

Hence, (4.30) – (4.32) read as follows

$$\dot{\rho}_1 + 3\frac{\dot{R}}{R}\rho_1 + \frac{\rho_0}{R}\left(\frac{\partial v_1}{\partial \chi} + \frac{2v_1}{\chi}\right) = 0, \tag{4.34}$$

$$\dot{v}_1 + \frac{\dot{R}}{R}v_1 = F_1 - \frac{1}{R\rho_0}\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\partial \rho_1}{\partial \chi}, \tag{4.35}$$

$$\frac{\partial F_1}{\partial \chi} + \frac{2F_1}{\chi} = -4\pi G\rho_1 R. \tag{4.36}$$

It is also convenient to express the changes in the density by using the fractional term $s = \frac{\rho_1}{\rho_0} = \frac{\rho_1 R^3(t)}{\rho_0(t_0)}$. The equations above change into

$$\frac{\partial v_1}{\partial \chi} + \frac{2v_1}{\chi} = -R\dot{s}, \tag{4.37}$$

$$\dot{v}_1 + \frac{\dot{R}}{R}v_1 = F_1 - \frac{1}{R}\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\partial s}{\partial \chi}, \tag{4.38}$$

$$\frac{\partial F_1}{\partial \chi} + \frac{2F_1}{\chi} = -4\pi G s\rho_0 R. \tag{4.39}$$

From (4.38) we calculate $\frac{\partial F_1}{\partial \chi} + \frac{2F_1}{\chi}$ and substitute into (4.39). We get

$$\frac{\partial \dot{v}_1}{\partial \chi} + \frac{2\dot{v}_1}{\chi} + \frac{\dot{R}}{R}\left(\frac{\partial v_1}{\partial \chi} + \frac{2v_1}{\chi}\right) + \frac{1}{R}\frac{\mathrm{d}p}{\mathrm{d}\rho}\left(\frac{\partial^2 s}{\partial \chi^2} + \frac{2}{\chi}\frac{\partial s}{\partial \chi}\right) = -4\pi G\rho_0 R s. \tag{4.40}$$

Expressing $\frac{\partial v_1}{\partial \chi} + \frac{2v_1}{\chi}$ from (4.37) and using the result in (4.40) gives

$$\frac{1}{R^2}\frac{\mathrm{d}p}{\mathrm{d}\rho}\left(\frac{\partial^2 s}{\partial \chi^2} + \frac{2}{\chi}\frac{\partial s}{\partial \chi}\right) = \ddot{s} + 2\frac{\dot{R}}{R}\dot{s} - 4\pi G\rho_0 s. \tag{4.41}$$

Finally, we consider a radial-wave solution

$$s = h(t)\frac{e^{ik\frac{r}{R(t)}}}{\frac{r}{R(t)}} = h(t)\frac{e^{ik\chi}}{\chi}, \tag{4.42}$$

where $k$ is a real number (no confusion with index of curvature $k$ should arise) and $h(t)$ is the function of interest which characterizes how the amplitude of the wave changes with time. The factor $1/R(t)$ means that the wavelength is stretched out by the expansion of the universe. Substituting (4.42) into (4.41) we find

$$\ddot{h} + 2\frac{\dot{R}}{R}\dot{h} + h\left(\frac{k^2}{R^2}\frac{\mathrm{d}p}{\mathrm{d}\rho} - 4\pi G\rho_0\right) = 0. \tag{4.43}$$

There is an obvious similarity between (4.43) and Jeans' equation (4.16). If we replace $k$ by the wavelength of the disturbance

$$\lambda = \frac{2\pi R}{k}, \tag{4.44}$$

the only term, in which the equations differ, is the second term on the r.h.s., i.e., $2\dot{h}\dot{R}/R$ due to the expansion (or contraction). In Einstein's static universe ($R$ = const. $= 1, \Lambda = 4\pi G\rho_0(t_0)$) (4.43) goes exactly over into Jeans' relation. Before we investigate (4.43) in more detail we shall show that assuming a plane-wave solution, equation (4.43) for corresponding function $h(t)$ remains the same as for the radial-wave solution [15].

Indeed, suppose the perturbations take the form

$$\rho_1 = {\rho_1}'(t)e^{\frac{i\vec{r}\cdot\vec{q}}{R(t)}}, \tag{4.45}$$

$$\vec{v}_1 = {\vec{v}_1}'(t)e^{\frac{i\vec{r}\cdot\vec{q}}{R(t)}}, \tag{4.46}$$

$$\vec{F}_1 = {\vec{F}_1}'(t)e^{\frac{i\vec{r}\cdot\vec{q}}{R(t)}}. \tag{4.47}$$

Equations for perturbed quantities (4.23)–(4.25) then become

$$\frac{\dot{\rho}_1{}'}{\rho_0} + 3\frac{\dot{R}}{R}\frac{{\rho_1}'}{\rho_0} + \frac{i}{R}\vec{q}\cdot\vec{v}_1 = 0, \tag{4.48}$$

$$\dot{\vec{v}}_1{}' + \frac{\dot{R}}{R}{\vec{v}_1}' = {\vec{F}_1}' - \vec{q}\frac{i}{R}\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{{\rho_1}'}{\rho_0}, \tag{4.49}$$

$$i\vec{q}\cdot{\vec{F}_1}' = -4\pi G\rho_1 R. \tag{4.50}$$

The latter equation has the solution

$$\vec{F}_1{}' = \frac{4\pi i G \rho_1{}' R \vec{q}}{q^2}, \tag{4.51}$$

where $q = |\vec{q}|$.

In the next step we decompose $\vec{v}_1{}'$ into two parts – perpendicular and paralel to $\vec{q}$:

$$\vec{v}_1{}'(t) = \vec{v}_\perp(t) + \frac{\vec{q}\,(\vec{q}\cdot\vec{v}_1{}'(t))}{q^2} = \vec{v}_\perp(t) + \vec{q}\epsilon(t), \tag{4.52}$$

$$\vec{q}\cdot\vec{v}_\perp = 0. \tag{4.53}$$

Equations (4.48) and (4.49) then split into the part determining $\vec{v}_\perp$,

$$\dot{\vec{v}}_\perp + \frac{\dot{R}}{R}\vec{v}_\perp = 0, \tag{4.54}$$

and into the part which describes the perturbation of the velocity parallel to $\vec{q}$:

$$\dot{h} = -\frac{i}{R}q^2\epsilon, \tag{4.55}$$

$$\dot{\epsilon} + \frac{\dot{R}}{R}\epsilon = \left(\frac{4\pi G\rho_0 R}{q^2} - \frac{1}{R}\frac{\mathrm{d}p}{\mathrm{d}\rho}\right) ih, \tag{4.56}$$

where we have introduced the function $h(t) = \rho_1'/\rho_0$.

Using (4.55) to eliminate $\epsilon$ in (4.56) implies

$$\ddot{h} + 2\frac{\dot{R}}{R}\dot{h} + h\left(\frac{q^2}{R^2}\frac{\mathrm{d}p}{\mathrm{d}\rho} - 4\pi G\rho_0\right) = 0. \tag{4.57}$$

This is the same differential equation as we have derived for the radial wave. To be correct, we should also investigate equation (4.54) which has a simple solution

$$\vec{v}_\perp(t) \propto \frac{1}{R(t)}. \tag{4.58}$$

Nevertheless these modes are not connected with changes in the density.

Equation (4.57) is the fundamental differential equation that determines the gravitational condensation in Newtonian dynamical cosmology. Assume a perturbation at time $t_1$ such that

$$h(t_1) > 0, \quad \dot{h}(t_1) > 0. \tag{4.59}$$

Obviously $h(t)$ increases after $t_1$ and will stop only if it reaches a maximum, say at time $t_2$, i.e.,

$$\dot{h}(t_2) = 0, \quad \ddot{h}(t_2) < 0. \tag{4.60}$$

It follows from (4.57) that if $\dot{h}(t_2) = 0$ and also

$$\frac{q^2}{R^2}\frac{\mathrm{d}p}{\mathrm{d}\rho} - 4\pi G\rho_0 < 0 \quad \text{at} \quad t = t_2, \tag{4.61}$$

$\ddot{h}(t_2)$ must be positive. Thus, $h(t)$ does not have the maximum at $t_2$ and continues increasing. This analysis shows that if (4.61) is satisfied at time $t_1$, the condensation will surely proceed until such time $t'$ when (4.61) ceases to be valid. By using the definition (4.44) – with $q$ instead of $k$ – (4.61) becomes exactly the formula (4.17) given by Jeans. In the present case of a dynamical model, however, physical quantities are time-dependent.

# Chapter 5

# Conclusion

Although Einstein's general theory of relativity provides the best so far known description of gravitation, the dynamics of the universe on sufficiently small scales can also be described by the Newtonian cosmology. In Chapter 3 we showed that the Newtonian cosmology can exhibit the evolution of the perfectly isotropic and homogenous universe. In Section 3.2 the Newtonian gravitational potential was treated. We derived its form from modified Poisson's field equation in case of the uniform matter distribution and showed that the magnitude and direction of the corresponding gravitational force are undetermined. However, as we discussed in Section 3.4, if we restrict our investigation to a spherical volume filled with uniformly distributed matter embedded in empty space, the potential remains the same whereas the force becomes fully determined. In Section 3.3 we found the equations governing the dynamics of the universe. It is remarkable that they have the same form as the relativistic Friedmann equations for the universe filled with (pressure-free) dust.

In Section 3.5 we dealt with the interesting features of the Newtonian cosmology which were presented quite recently. In the Newtonian framework it was possible to show that the cosmological constant may be related to the finite universe's total mass. We also introduced pressure into the Newtonian continuity equation to enable us to treat the cosmological constant as a perfect fluid.

In Chapter 4 we focused our attention on small linear perturbations in the density and pressure in an otherwise uniform universe. First we derived Jeans' formula for gravitational instability of a clump of gas in a static universe. In order to get his formula we needed to assume that the background unperturbed density and pressure of the gas are constant. However, Euler's hydrodynamical equation, the continuity equation and Poisson's equation have no solution for constant density and pressure. In a dynamical universe, where the background density and pressure are time-dependent, we found a formula for gravitational instability very similar to the Jeans' one.

Newtonian cosmology still offers a number of interesting problems. For instance, David Langlois and Filippo Vernizzi [8] recently found some quantites which are conserved in the relativistic perturbation theory. It would be interesting to derive the Newtonian analogy of these conservation laws and describe them in more familiar Newtonian framework. Perhaps a rigorous formulation of the Newtonian limit within so-called frame theory [4] could be applied in such a problem.

# Appendix A

# Relativistic Cosmology

Throughout this appendix we set $c = G = 1$. Latin indices run over the three spatial coordinates (except the letter $t$ which denotes the time coordinate) whereas Greek indices run over the four spacetime coordinates (except $\phi, \theta$, and $\chi$ which are used for particular space coordinates). The subscripts $_{,\mu}$ and $_{;\mu}$ denotes the partial and covariant derivation respectively, with respect to $x^\mu$.

We will consider a perfectly isotropic and homogenous universe. This requirement of global homogeneity and isotropy places tight demands on the geometry of spacetime and on the motion of the cosmological fluid. Homogeneity of the universe means that the entire spacetime can be divided into a one-parameter family of spacelike hypersurfaces of homogeneity, i.e., physical conditions are identical at every event on such a hypersurface, in particular, the curvature of spacetime, density and pressure must be the same on the homogenous hypersurface.

Isotropy of the universe means that every observer who is moving with the cosmological fluid cannot distinguish one of their space directions from the others by any local physical measurements. Isotropy also implies that the cosmological substratum is at rest relative to a hypersurface of homogeneity, thus the world lines of the fluid are orthogonal to the family of such hypersurfaces. Otherwise, an observer moving with the cosmological fluid could measure his or her non-zero ordinary velocity relative to that hypersurface and so distinguish one space direction in his or her rest frame from all others.

This result enables us to set up so-called comoving coordinate systems. Consider world lines of the cosmological fluid and a one-parameter family of spacelike homogenous hypersurfaces orthogonal to them. Let the time coordinate $t'$ be the parameter which labels the hypersurfaces. Assign to every event on the given world line the space coordinates $(x^1, x^2, x^3)$. Since the world lines do not intersect, such an assignment is unique. Moreover, the fluid is then at rest relative to the space coordinates - the space coordinates are comoving.

In addition, as the time coordinate we can choose proper time $\tau$ measured along the world lines. This can be simply deduced from the following consideration: Two different observers moving with the fluid along different world lines are at time $t'$ on the same hypersurface $S_1$, i.e., perceiving the same physical conditions. They make observations after the same interval $\Delta\tau$ of their respective proper time. Since the Einstein equations are deterministic – identical initial conditions and identical lapses of proper time leads to identical final conditions – the observers must see the same physics again and, therefore, they must be on the same hypersurface of homogeneity $S_2$, now labeled by the time $t' + \Delta\tau$. The observers and $\Delta\tau$ were arbitrary, hence, we see that the hypersurfaces of homogeneity are also the hypersurfaces of constant proper time measured along the world lines. Thus we can replace the time coordinate $t'$ with more familiar cosmic time $t = \tau$.

Let us have the comoving spatial coordinates and the cosmic-time coordinate, $x^\mu = (t, x^1, x^2, x^3)$. The basis vector $\frac{\partial x^\mu}{\partial t}$ at any given event is tangent to the world line which passes through that event. And the basis vectors $\frac{\partial x^\mu}{\partial x^i}$ at any given event are tangent to the hypersurfaces of homogeneity which goes through that event. From the orthogonality of the world lines to the hypersurfaces we require the orthogonality of $\frac{\partial x^\mu}{\partial t}$ to $\frac{\partial x^\mu}{\partial x^i}$:

$$g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial x^i} = 0. \tag{A.1}$$

Hence, from (A.1) we infer $g_{ti} = 0$. The 4-velocity of the cosmological fluid reads

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\partial x^\mu}{\partial t} = (1, 0, 0, 0), \tag{A.2}$$

where $\tau$ is the proper time measured along the world lines of the fluid. The 4-velocity is normalized:

$$g_{\mu\nu}\frac{\partial x^\mu}{\partial t}\frac{\partial x^\nu}{\partial t} = -1. \tag{A.3}$$

This implies $g_{tt} = -1$.

Then the expression for the line element takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j = -\mathrm{d}t^2 + \mathrm{d}\sigma^2. \tag{A.4}$$

$\mathrm{d}\sigma^2$ describes the spatial (time-dependent) 3-dimensional geometry with constant curvature. There are three such geometries. One obvious possibility is 3-dimensional flat Euclidean space with the line element

$$\mathrm{d}\sigma^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2. \tag{A.5}$$

Another possibility is a spherical 3-dimensional hypersurface

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2, \quad a > 0, \tag{A.6}$$

imbedded in 4-dimensional Euclidean space with the line element

$$\mathrm{d}\sigma^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2. \tag{A.7}$$

And the last geometry is a 3-dimensional hyperboloid

$$(x^4)^2 - \left[(x^1)^2 + (x^2)^2 + (x^3)^2\right] = a^2, \quad a > 0, \tag{A.8}$$

imbedded in 4-dimensional Minkowski space with the line element

$$\mathrm{d}\sigma^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 - (\mathrm{d}x^4)^2. \tag{A.9}$$

We use the spherical polar coordinates:

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi, \\ x^2 &= r\sin\theta\sin\phi, \\ x^3 &= r\cos\theta, \end{aligned} \tag{A.10}$$

for which the equations for the 3-sphere and 3-hyperboloid become

$$(x^4)^2 \pm r^2 = a^2, \tag{A.11}$$

and the line elements of 4-dimensional Euclidean and Minkowski spaces take the form

$$\mathrm{d}\sigma^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \pm (\mathrm{d}x^4)^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \pm (\mathrm{d}x^4)^2. \tag{A.12}$$

The differential of equation (A.11) gives

$$x^4 \mathrm{d}x^4 = \mp r \mathrm{d}r. \tag{A.13}$$

By substituting (A.11) into (A.13) we can express $(\mathrm{d}x^4)^2$ as

$$(\mathrm{d}x^4)^2 = \frac{r^2 \mathrm{d}r^2}{a^2 \mp r^2}. \tag{A.14}$$

By virtue of (A.14) geometry (A.12) reads as follows

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 \mp \frac{a^2}{r^2}} + r^2 \mathrm{d}\Omega^2. \tag{A.15}$$

This can be extended for the case of flat 3-dimensional Euclidean space by writing it as

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1 - k\frac{a^2}{r^2}} + r^2 \mathrm{d}\Omega^2, \tag{A.16}$$

where $k$ is the index of curvature;

$$\begin{aligned} &k = +1 \quad \text{for sphere,} \\ &k = -1 \quad \text{for hyperboloid,} \\ &k = 0 \quad \text{for Euclidean space.} \end{aligned} \tag{A.17}$$

For convenience we change to a new coordinate by the transformation

$$r = a\Sigma, \tag{A.18}$$

$$\begin{aligned} &\Sigma = \sin\chi, \quad \chi \in (0, \pi), \quad \text{if } k = +1, \\ &\Sigma = \sinh\chi, \quad \chi \in (0, +\infty), \quad \text{if } k = -1, \\ &\Sigma = \chi, \quad \chi \in (0, +\infty), \quad \text{if } k = 0. \end{aligned} \tag{A.19}$$

Geometry (A.16) then becomes

$$\mathrm{d}\sigma^2 = a^2(\mathrm{d}\chi^2 + \Sigma^2 \mathrm{d}\Omega^2). \tag{A.20}$$

By substituting (A.20) into (A.4) we finally get the spacetime geometry

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(\mathrm{d}\chi^2 + \Sigma^2 \mathrm{d}\Omega^2). \tag{A.21}$$

This is a commonly used form of so-called FLRW (Friedmann - Lemaître - Robertson - Walker) metric. Since the geometry of 3-dimensional hypersurfaces may depend on time, the term $a$ is, in fact, time dependent; $a = a(t)$. All the dynamics is then imprinted in this term.

From (A.21) it is seen that the metric tensor has only diagonal elements: $g_{tt} = -1$, $g_{\chi\chi} = a^2(t)$, $g_{\theta\theta} = a^2(t)\Sigma^2$ and $g_{\phi\phi} = a^2(t)\Sigma^2 \sin^2\theta$.

The volume of the above universes can be calculate as

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\chi_{\max}} \sqrt{g_{\chi\chi}g_{\theta\theta}g_{\phi\phi}}\mathrm{d}\chi\mathrm{d}\theta\mathrm{d}\phi = 4\pi a^3 \int_0^{\chi_{\max}} \Sigma^2 \mathrm{d}\chi. \tag{A.22}$$

The universe for $k = +1$ is finite with volume $2\pi^2 a^3$ and like any spherical surface it has no boundary. The spaces for $k = -1$ and $k = 0$ are infinite, however, they may be constructed finite by imposing suitable periodical conditions. There is an infinite number of possible conditions of periodicity for $k = -1$ and $k = 0$ as well as for $k = +1$. But we will not consider these models which arise from changing the topology of 3-spaces by suitable identifications. In this way some global symmetries are lost.

Before we use Einstein's equations (3.12) to determine $a(t)$, we need to calculate the Ricci tensor $R_{\mu\nu}$. Thus let us first calculate the affine connection

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}). \tag{A.23}$$

The individual components read as follows:

$$\Gamma^{t}_{tt} = -\frac{1}{2}(g_{tt,t} + g_{tt,t} - g_{tt,t}) = 0, \tag{A.24}$$

$$\Gamma^{t}_{ti} = -\frac{1}{2}(g_{tt,i} + g_{ti,t} - g_{ti,t}) = 0, \tag{A.25}$$

$$\Gamma^{i}_{tt} = \frac{1}{2}g^{ij}(g_{jt,t} + g_{jt,t} - g_{tt,j}) = 0, \tag{A.26}$$

$$\Gamma^{t}_{ij} = -\frac{1}{2}(g_{ti,j} + g_{tj,i} - g_{ij,t}) = \frac{1}{2}g_{ij,t} = \frac{\dot{a}}{a}g_{ij}, \tag{A.27}$$

$$\Gamma^{i}_{tj} = \frac{1}{2}g^{ik}(g_{kt,j} + g_{kj,t} - g_{tj,k}) = \frac{1}{2}g^{ik}g_{kj,t} = \frac{\dot{a}}{a}g^{ik}g_{kj} = \frac{\dot{a}}{a}\delta^{i}_{j}, \tag{A.28}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l}). \tag{A.29}$$

The only non-vanishing purely spatial components of the affine connection are $\Gamma^\theta_{\theta\chi}, \Gamma^\phi_{\phi\chi}, \Gamma^\phi_{\phi\theta}, \Gamma^\chi_{\theta\theta}, \Gamma^\chi_{\phi\phi}$, and $\Gamma^\theta_{\phi\phi}$.

Following identities will be useful

$$\begin{aligned} \Gamma^i_{it,t} = 3\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right), \quad \Gamma^t_{ij,t} = \frac{\partial(a\dot{a})}{\partial t}\frac{g_{ij}}{a^2}, \\ \Gamma^i_{tj}\Gamma^j_{ti} = 3\frac{\dot{a}^2}{a^2}, \quad \Gamma^t_{ik}\Gamma^k_{jt} = \frac{\dot{a}^2}{a^2}g_{ij}, \quad \Gamma^t_{ij}\Gamma^l_{tl} = 3\frac{\dot{a}^2}{a^2}g_{ij}. \end{aligned} \tag{A.30}$$

The expression for the Ricci tensor in terms of the affine connection takes the form

$$R_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\lambda\mu,\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}. \tag{A.31}$$

By using (A.24)–(A.30) we have

$$R_{tt} = -\Gamma^\lambda_{\lambda t,t} - \Gamma^\lambda_{t\sigma}\Gamma^\sigma_{t\lambda} = -\Gamma^i_{it,t} - \Gamma^i_{tj}\Gamma^j_{ti} = -3\frac{\ddot{a}}{a}, \tag{A.32}$$

$$\begin{aligned} R_{ti} =& \Gamma^\lambda_{ti,\lambda} - \Gamma^\lambda_{\lambda t,i} + \Gamma^\lambda_{ti}\Gamma^\sigma_{\lambda\sigma} - \Gamma^\lambda_{t\sigma}\Gamma^\sigma_{i\lambda} = \\ =& \Gamma^j_{ti}\Gamma^k_{jk} - \Gamma^j_{tk}\Gamma^k_{ij} = 0, \end{aligned} \tag{A.33}$$

$$\begin{aligned} R_{ij} = \left[\Gamma^k_{ij,k} + \Gamma^t_{ij,t}\right] - \Gamma^k_{ki,j} + \left[\Gamma^k_{ij}\Gamma^l_{kl} + \Gamma^t_{ij}\Gamma^l_{tl}\right] - \\ - \left[\Gamma^t_{ik}\Gamma^k_{jt} + \Gamma^k_{it}\Gamma^t_{jk} + \Gamma^l_{ik}\Gamma^k_{jl}\right] = \tilde{R}_{ij} + \left(2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) g_{ij}, \end{aligned} \tag{A.34}$$

where $\tilde{R}_{ij}$ is the purely spatial Ricci tensor

$$\tilde{R}_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ki,j} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}. \tag{A.35}$$

It can be shown ([9]) that

$$\tilde{R}_{ij} = 2\frac{k}{a^2}g_{ij}. \tag{A.36}$$

Hence,

$$R_{ij} = \left(2\frac{k}{a^2} + 2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) g_{ij}. \tag{A.37}$$

Except the Ricci tensor, we also need the values of the stress-energy tensor $T_{\mu\nu}$ for the cosmological fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{A.38}$$

$p = p(t)$ and $\rho = \rho(t)$ are the pressure and energy density of the cosmological fluid and $u^\mu = -u_\mu = (1, 0, 0, 0)$ is its 4-velocity.

$$T_{tt} = \rho, \quad T_{ij} = p g_{ij}, \quad T_{it} = 0 \tag{A.39}$$

$$T^\mu{}_\mu = (\rho + p)u^\mu u_\mu + p\delta^\mu_\mu = -\rho + 3p. \tag{A.40}$$

The momentum conservation law, $T^{i\mu}{}_{;\mu} = 0$, is automatically satisfied whereas the energy conservation law gives

$$\begin{aligned} 0 = T^{t\mu}{}_{;\mu} =& T^{t\mu}{}_{,\mu} + \Gamma^t_{\mu\nu}T^{\nu\mu} + \Gamma^\mu_{\mu\nu}T^{t\nu} = T^{tt}{}_{,t} + \Gamma^t_{ij}T^{ij} + \Gamma^i_{it}T^{tt} = \\ =& \dot\rho + \frac{\dot a}{a} g_{ij}T^{ij} + \frac{\dot a}{a}\rho\delta^i_i = \dot\rho + 3\frac{\dot a}{a}(\rho + p), \end{aligned} \tag{A.41}$$

so that

$$\dot\rho + 3\frac{\dot a}{a}(\rho + p) = 0. \tag{A.42}$$

We have already solved (A.42) in case of barotropic fluid characterized by the equation of state $w = p/\rho$, see (3.56).

We shall use the Einstein equations in the convenient form

$$R_{\mu\nu} = 8\pi T_{\mu\nu} - 4\pi T^\sigma{}_\sigma g_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{A.43}$$

From $tt$ component we get

$$3\frac{\ddot a}{a} = -4\pi(\rho + 3p) + \Lambda, \tag{A.44}$$

and from the diagonal spatial components we obtain

$$2\frac{\dot a^2}{a^2} + \frac{\ddot a}{a} = 4\pi(\rho - p) - \frac{k}{a^2} + \Lambda. \tag{A.45}$$

The other components are identically zero. By using (A.44) to eliminate $\ddot a$ in the equation above, (A.45) becomes

$$\frac{\dot a^2}{a^2} = \frac{8\pi\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{A.46}$$

(A.44) and (A.46) are the Friedmann equations which govern the dynamics of the universe. By differentiating (A.46) and substituting it into (A.44) one gets the energy conservation equation (A.42). This is not suprising since the conservation law is a consequence of the Einstein equations.

In case of $k = +1$, $a(t)$ can be called the radius of the universe. If $k = 0$, $a(t)$ may be fixed to any arbitrary positive value at a particular time. However, the form of the Friedmann equations is invariant under the transformation:

$$\begin{aligned} a(t) \rightarrow R(t) &= \frac{a(t)}{a(t_0)}, \\ k \rightarrow K &= \frac{k}{a^2(t_0)}. \end{aligned} \tag{A.47}$$

$R(t)$ is the scale factor and $K$ may be, in general, any number. Transformed equations (A.44) and (A.46) have then precisely the same form as the Newtonian equation (3.33), if considering $p = 0$, or as the corrected Newtonian equations (3.53) together with (3.33). Hence, cosmological models derived from (3.33) remain the same also in relativistic cosmology. Our discussion of their behaviour can thus be directly taken over to the models based on general relativity.

# Bibliography


[1] Bondi H.: *Cosmology*, 2d ed., Cambridge University Press, Cambridge, 1961, 75–125.

[2] Bonnor W. B.: *Jeans' Formula for Gravitational Instability*, Monthly Notices of the Royal Astronomical Society, **117**, 104–117 (1957).

[3] Calder L., Lahav O.: *Dark Energy: back to Newton?*, Astronomy and Geophysics, **49**, 1.13–1.18 (2008).

[4] Ehlers, J.: *Newtonian Limit of General Relativity* in *Encyclopedia of mathematical physics*, eds. Francoise J.P., Naber G.L., Tsou S.T., vol. 3, Elsevier, 2006, 503–509.

[5] Harrison E. R.: *Cosmology, the science of the universe*, Cambridge University Press, Cambridge, 1981, 249–265.

[6] Harrison E. R.: *Olbers' Paradox and the background radiation density in an isotropic homogeneous universe*, The Galactic and Extragalactic Background Radiation, proceedings of the 139th Symposium of the International Astronomical Union, held in Heidelberg, FRG, June 12-16, 1989. Eds. S. Bowyer, C. Leinert, Kluwer Academic Publishers, Dordrecht, Boston, 1990.

[7] Jeans J. H.: *Astronomy and Cosmogony*, Cambridge University Press, Cambridge, 1929, 345–348.
Originally Jeans discussed the problem of gravitational instability in Phil. Trans. **199** (1902).

[8] Langlois D., Vernizzi F.: *Conserved nonlinear quantities in cosmology*, Physical Review, **72**, 103501 (2005).

[9] Misner C. W., Thorne K. S., Wheeler J. A.: *Gravitation*, Freeman, San Francisco, 1973, 703–762.

[10] Neumann C.: *Allgemeine Untersuchungen über das Newtonsche Prinzip der Fernwirkungen*, Teubner, Leipzig, 1896.

[11] Sami M.: *A primer on problems and prospects of dark energy*, Invited Revue for Current Science, to be published.
Preprint see at http://arxiv.org/abs/0904.3445

[12] Selety F.: *Beiträge zum kosmologischen Problem*, Annalen der Physik, **373**, 281–334 (1922).

[13] Seeliger H.: *Über das Newtonsche Gravitationsgesetz*, Astronomische Nachrichten, **137**, 129–136 (1895).

[14] Trautman A.: *Foundations and Current Problems of General Relativity* in *Lectures on General Relativity*, eds. A. Trautman, F. A. E. Pirani and H. Bondi (Brandeis Summer Institute in Theoretical Physics, 1964), Prentice-Hall, New Jersey, 1965, 104–106, 228–238.

[15] Weinberg S.: *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, J. Wiley, New York, 1972, 571–573.