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ON GEOMETRICAL PROPERTIES OF THE r-NEIGHBORHOOD OF BROWNIAN MOTION AND RELATED RANDOM STRUCTURES

Doctoral thesis

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Preface

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Prague, 6.2.2007 Rostislav Černý

To Barborka.

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Chapter 1

Introduction

This work is mainly focused on the study of a particular random compact set called *Wiener sausage*. Heuristically, it is the trace of a moving spherical object of radius $r > 0$ which moves in d–dimensional Euclidean space along Brownian trajectories up to time $t \geq 0$.

Wiener sausage is used in physics and technology to model various phenomena. There exists only limited literature concerning Wiener sausage (see e.g. [32] and references inside) and its geometrical properties.

Basic settings that are needed later in this thesis are given in Chapter 2, Preliminaries. Standard apparatus concerning stochastic geometry and integral geometry in particular is described at the beginning. We define random compact set, its distribution, intrinsic volumes and techniques used to their estimation. Two sections devoted to Wiener process and Brownian motion follow. There is no new result, on the other hand, given connection between theory of Brownian motion and theory of potential is very interesting and its techniques are rarely explained in standard lectures on Brownian motion or potential theory. We focus on the first hitting time of a set A by the Brownian motion starting at $x \in \mathbb{R}^d$

$$\tau_A^x = \inf\{t : t > 0, B(t) \in A\}.$$

The crucial theorem by Hunt saying that the distribution of τ_A^x can be assessed as a solution to heat conduction problem is stated.

Chapter 3 starts with formal definition of the Wiener sausage. Its geometrical properties, namely its expected volume and surface area, are presented. These quantities have been previously published (see [1], [24]). Newly, we concentrate on leading terms of asymptotics when total time t tends to infinity or radius r tends to zero.

In Proposition 3.1 we show the leading term for mean volume both for $t \to \infty$ and $r \to 0$ in case when the dimension $d \geq 3$. The asymptotic behavior of Bessel functions is used:

$$\mathrm{E}V_d(S_{r,t}) \simeq \omega_d \frac{d(d-2)}{2} r^{d-2} t, \qquad \frac{t}{r^2} \to \infty.$$

Similarly, the mean surface area is treated in Proposition 3.2

$$\mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t}) \simeq d\omega_d r^{d-1} \frac{(d-2)^2}{2} \frac{t}{r^2}, \quad \frac{t}{r^2} \to \infty.$$

Existence and finiteness of other Minkowski functionals of the Wiener sausage is more or less open problem. Theorem 3.3 is a consequence of a recent result that in dimensions $d = 2, 3$ the Wiener sausage is almost surely a d–dimensional Lipschitz manifold.

A new finding is given in Subsection 3.1.4. In Theorem 3.4 we explain a possible approach how covariogram of the Wiener sausage can be assessed. Explicit formula is given for its derivative at zero ($d = 2, 3$)

$$\widetilde{C}'_{S_{r,t}}(0) = -\frac{\omega_{d-1}}{d\omega_d} \mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t}).$$

The Chapter is closed up with a self contained section on approximations of the Wiener sausage. These are closely related to approximations of the Brownian motion itself. Several possibilities with different types of convergence are presented. The main results (for $d = 2, 3$) of this section are stated in Theorem 3.7 and Theorem 3.8. We estimate the distribution of the approximation error and its asymptotical behavior. Those are consequences of generally formulated Proposition 3.4.

The approximation of the covariogram of the Wiener sausage is shown to be convergent in Proposition 3.5. This result is later used for numerical computations in Chapter 4.

Chapter 4 is devoted to the Boolean model of Wiener sausages. Its content is motivated by the recent joint work [4] and given results are more or less new. Main characteristics, like capacity functional, volume fraction and contact distributions are discussed. Numerical results for computation of the covariance function are presented.

The chapter is finished with a Theorem 4.2 with a new result on specific surface area of the Boolean model of Wiener sausages

$$S_\Xi = \lambda\, \mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t})\, e^{-\mathrm{E}V_d(S_{r,t})}.$$

In the last Chapter 5 we define a new special case of the Wiener sausage where the total time t is random. We differentiate two situations when t is independent and dependent on the trajectory of underlying Brownian motion.

In the first case the mean volume and mean surface area can be achieved easily by conditioning on the independent time t. The second case where Wiener sausage is terminated at the time the underlying Brownian motion reaches the boundary of a specific ball is much more complicated. We briefly discuss the possibilities of numeric computation of the mean volume and show that the mean surface area is almost surely finite.

Chapter 2

Preliminaries

Random sets and stochastic geometry

The major part of this work is devoted to the theory of random closed sets (RACS). Denote by $\mathcal{B}$, $\mathcal{F}$, $\mathcal{K}$ the space of Borel, closed and compact sets in $\mathbb{R}^d$ respectively. For $B \in \mathcal{B}$ we set

$$\begin{aligned} \mathcal{F}_B &= \{F \in \mathcal{F} : \; F \cap B \neq \emptyset\}, \\ \mathcal{K}_B &= \{K \in \mathcal{K} : \; K \cap B \neq \emptyset\}. \end{aligned}$$

Following Matheron [19], the random closed set in $\mathbb{R}^d$ is defined to be a measurable mapping from some probability space $(\Omega, \mathcal{A}, \mathrm{P})$ into $(\mathcal{F}, \mathfrak{F})$, where the σ–algebra $\mathfrak{F}$ is generated by the system of sets $\mathcal{F}_K$:

$$\mathfrak{F} := \sigma\{\mathcal{F}_K : \; K \in \mathcal{K}\}. \tag{2.1}$$

Let $\Xi : (\Omega, \mathcal{A}, \mathrm{P}) \to (\mathcal{F}, \mathfrak{F})$ be a random closed set. Its distribution Q_Ξ is given as an induced measure on the space $(\mathcal{F}, \mathfrak{F})$,

$$Q_\Xi = \mathrm{P} \circ \Xi^{-1}.$$

This is a standard definition of the distribution of any random object. The similar role as distribution function plays in the theory of random variables the capacity functional T_Ξ plays in the theory of random sets. It is defined for any compact $C \subset \mathbb{R}^d$ by

$$T_\Xi(C) = \mathrm{P}(\Xi \cap C \neq \emptyset). \tag{2.2}$$

It can be shown that the distribution Q_Ξ is determined uniquely by $T_\Xi(C)$, $C \subset \mathbb{R}^d$ compact.

A typical problem of the theory of random closed sets is the characterization by geometrical properties. A possible approach yields the estimation of means of intrinsic volumes. These are defined for convex sets by a well known theorem of convex geometry, the Steiner formula, characterizing the volume of a parallel set to a convex body by a polynomial in the dilation parameter.

Let $\Sigma^{d-1} := \{u \in \mathbb{R}^d : |u| = 1\}$ denote the $(d-1)$–dimensional unit sphere, $a \cdot b$ be the scalar product of two vectors $a, b \in \mathbb{R}^d$ and let $C \subseteq \mathbb{R}^d$ be convex.

The *support function* of C is defined as

$$h(C, z) := \sup_{x \in C} x \cdot z, \qquad z \in \mathbb{R}^d.$$

Subsequently, the *support hyperplane* of C in direction $u \in \Sigma^{d-1}$ is given by $\{y : y \cdot u = h(C, u)\}$.

The *width of C in direction* $u \in \Sigma^{d-1}$ is defined as

$$w(C, u) := h(C, u) + h(C, -u)$$

and the *mean breadth* $b(C)$ is its average over all directions

$$b(C) = \frac{1}{d\omega_d} \int\limits_{\Sigma^{d-1}} w(C, u) \, \mathcal{H}^{d-1}(du),$$

where $\omega_d = \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ is the volume of a unit ball in $\mathbb{R}^d$ (it is known, that $d\omega_d$ is then the surface content of Σ^{d-1}) and $\mathcal{H}^{d-1}$ is the $(d-1)$–dimensional Hausdorff measure (for definition of Hausdorff measures see e.g. [7]).

Let V_d denote the Lebesgue measure. The origin in $\mathbb{R}^d$ will be denoted by o, the closed ball centered at $a \in \mathbb{R}^d$ with radius $r > 0$ will be denoted by $B(a, r)$. Furthermore, $\oplus$ will stand for the Minkowski addition, i.e. a pointwise set addition. Let $\mathcal{C}$ denote the set of all convex and compact sets in $\mathbb{R}^d$.

Theorem 2.1 (Steiner formula) *There exists functionals $V_i : \mathcal{C} \to \mathbb{R}$, $i = 0, \dots, d$ such that for any $C \in \mathcal{C}$ and $\varrho \geq 0$:*

$$V_d(C_\varrho) = \sum_{m=0}^{d} \varrho^{d-m} \omega_{d-m} V_m(C), \tag{2.3}$$

where $C_\varrho = C \oplus B(o, \varrho)$ is the dilation of the set C.

Proof: See [28]. □

Remark: Intrinsic volumes are closely related to Minkowski functionals or quermassintegrals, see [33, §1.6].

In particular, we have the following identities which allow us to estimate easily all intrinsic volumes in two and three–dimensional Euclidian space

$$V_0(C) = 1, \tag{2.4}$$
$$V_1(C) = \frac{d\omega_d}{2\omega_{d-1}} b(C), \tag{2.5}$$
$$V_{d-1}(C) = \frac{1}{2}\mathcal{H}^{d-1}(\partial C), \tag{2.6}$$

for any $C \in \mathcal{C}$, where $b(C)$ is the mean breadth of C and $\mathcal{H}^{d-1}(\partial C)$ is the surface area of C. □

Moreover, the well known Hadwiger's representation theorem reflects the importance of intrinsic volumes. It can be shown that given a motion invariant additive and continuous function $\varphi : \mathcal{C} \to \mathbb{R}$ there exist numbers $a_0, \ldots, a_d$ such that

$$\varphi(C) = \sum_{i=0}^{d} a_i V_i(C), \quad \text{for all } C \in \mathcal{C}. \tag{2.7}$$

This means that V_i's are essentially the only functionals on convex bodies possessing motion invariance, additivity and continuity properties.

The intrinsic volumes can be extended to polyconvex sets (finite unions of compact convex sets) by inclusion–exclusion formula. They can further be represented in case of $\mathcal{C}^2$ (twice differentiable) smooth convex body as integrals of the symmetric functions of principal curvatures. This representation enables us to extend the notion to non–convex smooth bodies.

A (closed) subset X of $\mathbb{R}^d$ is said to have *positive reach* if there exists an $r_0 > 0$ such that any point with distance less than r_0 from X has its unique nearest neighbour in X.

The notion of intrinsic volumes can be extended to sets with positive reach (this was done by Federer [6]). The intrinsic volumes $V_j(X)$ are well defined by Steiner formula with $\rho < r$ whenever X is a set with positive reach $r > 0$ and compact boundary.

Curvature measures and intrinsic volumes have been extended consistently to certain full-dimensional Lipschitz manifolds of $\mathbb{R}^d$ in [27] (for definition of Lipschitz manifolds see the last section of this chapter). In particular, if X is a compact d-dimensional Lipschitz manifold such that its closure of complement, $\overline{\mathbb{R}^d \setminus X}$, has positive reach, then its Minkowski functionals are given by $V_j(X) = (-1)^{d-j-1} V_j(\overline{\mathbb{R}^d \setminus X})$, $j = 0, \ldots, d-1$, and they satisfy the Gauss-Bonnet formula ($V_0(X)$ equals the Euler-Poincaré characteristic of X) and the Principal Kinematic Formula (see [27, Theorem 4]).

Obviously, if a random closed set takes its realizations in some class of sets mentioned above it is then possible to define its intrinsic volumes (for any realization) and examine respective expected volumes.

For random polyconvex sets an algorithm called *method of moments* (see e.g. [29]) can be applied to estimate all intrinsic volumes via the estimation of local connectivity number (the Euler–Poincaré characteristic of the set intersected with moving ball).

For certain more complicated sets a method of estimation of the Euler number was proposed in [23]. The algorithm works recursively and the Euler number is estimated from the projections of thin slabs.

Having several methods how to estimate intrinsic volumes one can use the strong law of large numbers and apply it to independent realizations of a random closed set Ξ. Estimators of the mean intrinsic volumes $\mathrm{E}V_i(\Xi)$, $i = 0, \dots, d$ can be achieved (provided that $\mathrm{E}V_i(\Xi) < \infty$) and thus Ξ can be characterized by its geometrical properties.

Wiener process and Brownian bridge

A standard Wiener process $\{W(t),\, t \geq 0\}$ is defined to be a random element of the space $C_{[0,\infty)}$ of continuous functions on $[0, \infty)$ with the following properties:

- W(0) = 0 a. s.,
- for any $t \geq 0$, $W(t)$ is Gaussian distributed with mean 0 and variance t,
- for any $k \in \mathbb{N}$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k < \infty$, random variables $W(t_1) - W(t_0)$, $W(t_2) - W(t_1), \dots, W(t_k) - W(t_{k-1})$ are independent (and hence also Gaussian distributed),

while the space $C_{[0,\infty)}$ can be equipped by the following supremal metric

$$d_\infty(x, y) = \sum_{T=1}^{\infty} 2^{-T} \min\left(1, \sup_{0 \leq t \leq T} |x(t) - y(t)|\right).$$

In the sequel, by a Wiener process we shall always mean a standard Wiener process as defined above, unless stated otherwise.

For the existence and construction of the Wiener process we refer to [2, Chapter 2]. There exist several ways how to define a Wiener process. A constructive approach using so–called Haar–Schauder series will be described in Section 3.2.2.

Trajectory of the Wiener process has several useful properties. Next lemma is stated without proof, since the mentioned properties can be found in any book on Wiener process (e.g. [3], [13]).

Lemma 2.1 *Let* $\{W(t),\, t \geq 0\}$ *be a Wiener process. Then*

(1) $\{-W(t),\, t \geq 0\}$ *and* $\{\frac{1}{\sqrt{\alpha}} W(\alpha t),\, t \geq 0\}$, $\alpha > 0$ *are Wiener processes.*

(2) For any $t_0 \geq 0$, $\{W(t+t_0) - W(t_0),\, t \geq 0\}$ *is a Wiener process independent of the* σ*–algebra* $\sigma(\{W(t),\, t \leq t_0\})$.

(3) $\mathrm{P}(\max_{0 \leq t \leq b} W(t) \geq x) = 2\,\mathrm{P}(W(b) \geq x)$, *for* $x, b > 0$.

(4) $\mathrm{P}(\max_{0 \leq t \leq b} |W(t)| \geq x) \leq 2\,\mathrm{P}(|W(b)| \geq x)$, *for* $x, b > 0$.

(5) $\mathrm{P}(W(1) \geq \epsilon) \leq \exp(-\epsilon^2/2)$, $\epsilon > 1$.

Remark: The second part of (1) in Lemma 2.1 is often called the scaling invariance property. Inequality (4) is known as maximal inequality for Wiener process, inequality (5) is sometimes called Feller inequality.

Let $x, y \in \mathbb{R}$ and $l > 0$ be given. A continuous Gaussian process $\{X^{x,l,y},\, 0 \leq t \leq l\}$ with

$$\begin{aligned} \mathrm{E}X^{x,l,y}(t) &= x + (y-x)\frac{t}{l}, \\ \mathrm{cov}\left(X^{x,l,y}(t), X^{x,l,y}(s)\right) &= \min(s,t) - \frac{st}{l}, \end{aligned}$$

is called a *Brownian bridge* from x to y of length l.

The Brownian bridge can be also derived from a standard Wiener process $W(t)$ by

$$X^{x,l,x} := \left\{ x + W(t) - \frac{t}{l} W(l),\, 0 \leq t \leq l \right\}. \tag{2.8}$$

Furthermore, $X^{x,l,y}$ can be viewed as $X^{x,l,x}$ with a drift

$$X^{x,l,y}(t) = X^{x,l,x}(t) + (y-x)\frac{t}{l}, \quad 0 \leq t \leq l. \tag{2.9}$$

An important property which will be used later is that $X^{x,l,y}$ is a Wiener process started at x and conditioned to be at y at time l (see [3, IV.4.23]).

We have the following equality for the distribution of supremum of one–dimensional Brownian bridge (see [3, IV.4.26])

$$\mathrm{P}\left(\sup_{0 \leq t \leq l} X^{x,l,y}(t) < b \right) = 1 - \exp\left\{ -\frac{(y+x-2b)^2}{2l} + \frac{(y-x)^2}{2l} \right\}. \tag{2.10}$$

Brownian motion and the connection to potential theory

Let $\{W_1(t),\, t \geq 0\}, \ldots, \{W_d(t),\, t \geq 0\}$ be independent identically distributed Wiener processes. A d–dimensional Brownian motion is defined as

$$\{B(t),\, t \geq 0\} = \{(W_1(t), \ldots, W_d(t)),\, t \geq 0\}. \tag{2.11}$$

We recall here first some essential geometric properties of the Brownian motion. Almost surely, $\{B(t) : t \geq 0\}$ is a continuous nowhere differentiable curve and it has Hausdorff dimension equal to 2, nevertheless, its two-dimensional Hausdorff measure vanishes. The scaling invariance property is preserved in the multi–dimensional case as well, i.e.

$$\left\{ \frac{1}{\sqrt{\alpha}} B(\alpha t),\, t \geq 0 \right\}, \tag{2.12}$$

is again a Brownian motion for any $\alpha > 0$. The distribution of $B(t)$ is known to be invariant with respect to rotations. Hence, any projection of $\{B(t) : t \geq 0\}$ to a lower dimensional subspace of $\mathbb{R}^d$ is again a Brownian motion.

The theory of Brownian motion has a close connection to classical potential theory. Brownian motion is a Feller–type process (e.g. [13, Chapter 19]). It means that its transition operator (and hence transition kernel) can be recovered from a so–called generator of the process. Generator of the Brownian motion corresponds to Laplace operator $\triangle$:

$$\triangle f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}.$$

Potential theory, on the other hand, deals with harmonic functions. Let $U \subseteq \mathbb{R}^d$ be an open set. A function $h : U \to \mathbb{R}$ is said to be *harmonic* ($h \in \mathcal{H}(U)$), if it is of class C^2 on U and it satisfies the Laplace equation on U:

$$\triangle h = 0.$$

Due to this connection, many fundamental problems from potential theory can be solved by probabilistic approach. Vice–versa, various hitting distributions of the Brownian motion can be given by potential interpretation.

Assume A to be a compact subset of $\mathbb{R}^d$. We define *the first hitting time* of A by a Brownian motion $\{B^x(t),\, t \geq 0\}, B^x(0) = x$ starting at $x \in \mathbb{R}^d$ as

$$\tau_A^x = \inf\{t : t > 0,\, B(t) \in A\},$$

and we set $\inf \emptyset = \infty$. A point $x \in A$ is called *regular* if

$$\mathrm{P}[\tau_A^x = 0] = 1. \tag{2.13}$$

Note that all interior points of A are regular.

The regularity defined above has its potential counterpart. To go further some more basic settings from potential theory are needed. First, we shall define capacity as a set function to show later its connection to classical Dirichlet problem of potential theory.

For $t > 0$ we define

$$p(t) = \begin{cases} \frac{1}{d\omega_d} \log \frac{1}{t} & \text{when } d = 2, \\ \frac{1}{(d-2)d\omega_d} \frac{1}{t^{d-2}} & \text{when } d > 2. \end{cases}$$

For $x, y \in \mathbb{R}^d$ set $N(x,y) = p(|x-y|)$. For $d = 2$, the function N is known as *logarithmic kernel* and in higher dimensions N is called *Newtonian kernel*.

Let μ be a Radon measure (i.e. Borel measure such that $\mu(K) < \infty$ for any compact K). If the dimension $d = 2$, assume additionally that its support is compact. Define

$$N\mu : \, x \mapsto \int N(x,y)d\mu(y), \qquad x \in \mathbb{R}^d. \tag{2.14}$$

The function $N\mu$ is called *logarithmic potential of the measure* μ when $d = 2$, when $d > 2$ it is called *Newtonian potential of the measure* μ.

Let $\mathcal{R}(K)$ denote the set of all Radon measures μ such that $\mathrm{supp}(\mu) \subseteq K$. For $K \in \mathcal{K}$ define the capacity as

$$\mathrm{cap}(K) = \sup\{\mu(K) : \, \mu \in \mathcal{R}(K), N\mu \leq 1\}$$

and for $U \subseteq \mathbb{R}^d$ set

$$\mathrm{cap}(U) = \sup\{\mathrm{cap}(K) : \, K \subseteq U \text{ compact}\}.$$

The number $\mathrm{cap}(U)$ is called the *capacity* of the set U.

A classical problem of potential theory is the well known Dirichlet problem. Given a bounded open set $U \subseteq \mathbb{R}^d$ and real continuous function $f : \partial U \to \mathbb{R}$ the aim is to find a harmonic function h on U such that

$$h|_{\partial U} = f.$$

If the solution of the Dirichlet problem exists it is known to be unique. A set U is called *regular* if the Dirichlet problem has solution for any continuous boundary condition f. For example, any ball is regular set and the solution is given by Poisson integral (see e.g. [21]).

Classical Dirichlet problem can be generalized. It is known that there exists a unique linear nonnegative operator $H : C(\partial U) \to \mathcal{H}(U)$ such that it coincides with the solution to classical Dirichlet problem if it exists. This operator is called Keldysch operator. A point $z \in \partial U$ is called *regular* if for any $f \in C(\partial U)$ holds

$$Hf(x) \to f(z) \qquad x \to z.$$

Such a definition of a regular point coincides with the probabilistic one given in (2.13). Boundary points that are not regular are called irregular and the set consisting of all irregular points is denoted by $\partial_{irr}U$. It is known that

$$\mathrm{cap}(\partial_{irr}U) = 0$$

for any bounded open set $U \subseteq \mathbb{R}^d$.

The following Theorem was proved by Hunt, [12].

Theorem 2.2 (Hunt, 1956) *Let $A \subseteq \mathbb{R}^d$ be a set with positive capacity. Then*

$$u_A(t,x) = \mathrm{P}[\tau_A^x \le t] > 0, \qquad \text{for all } t > 0.$$

Moreover, $u_A(t,x)$ is the unique solution of the heat conduction problem

$$\frac{\partial u}{\partial t} = \frac{1}{2} \triangle u, \qquad t > 0,\ x \in \mathbb{R}^d \setminus A, \tag{2.15}$$

subject to the initial condition

$$u(0,x) = 0 \qquad \text{for } x \in \mathbb{R}^d \setminus A$$

and boundary condition

$$\lim_{x \to y} u(t,x) = 1 \qquad \text{for } t > 0,\ y \in B \text{ regular.}$$

Bessel functions

Bessel functions appear naturally in partial differential equations theory. For a broad overview for the theory of Bessel functions we refer to the treatise by G. N. Watson [34].

The Bessel function of the first kind of order $\nu \ge 0$ is defined as

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}. \tag{2.16}$$

The Bessel function of the second kind of order $\nu > 0$ is defined as

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}(J_\nu(x)\cos \nu\pi - J_{-\nu}(x)), \tag{2.17}$$

for $n \in \mathbb{N}_0$:

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x). \tag{2.18}$$

Furthermore, the modified Bessel functions (of imaginary argument) are defined in the following way.

The modified Bessel function of the first kind of order ν is defined as

$$I_\nu(x) = i^{-\nu} J_\nu(ix). \tag{2.19}$$

The modified Bessel function of the second kind of order ν is defined as

$$K_\nu(x) = \frac{1}{2\pi} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \tag{2.20}$$

and again for $n \in \mathbb{N}_0$

$$K_n(x) = \lim_{\nu \to n} K_\nu(x). \tag{2.21}$$

The following asymptotic expansions of the Bessel functions when x tends to 0 and ∞ will be useful:
when $x < 1$ we have

$$\begin{aligned} J_\nu(x) &\simeq \tfrac{x^\nu}{2^\nu \nu!}, && x \to 0 \quad \text{for any } \nu \geq 0, \\ Y_0(x) &\simeq \tfrac{2}{\pi} \log \tfrac{x}{2}, && x \to 0, \\ Y_\nu(x) &\simeq \tfrac{(\nu-1)!}{\pi} \left(\tfrac{2}{x}\right)^\nu, && x \to 0 \quad \text{for any } \nu > 0 \end{aligned} \tag{2.22}$$

and for $x \to \infty$ we have

$$\begin{aligned} J_\nu(x) &\simeq \left(\tfrac{2}{\pi x}\right)^{1/2}, && x \to \infty \quad \text{for any } \nu \geq 0, \\ Y_\nu(x) &\simeq \left(\tfrac{2}{\pi x}\right)^{1/2}, && x \to \infty \quad \text{for any } \nu \geq 0. \end{aligned} \tag{2.23}$$

Lipschitz Manifolds and rectifiable sets

A function $f : A \subseteq \mathbb{R}^d \to \mathbb{R}^n$ is called *Lipschitzian* if there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \qquad \forall\, x, y \in A.$$

Any Lipschitzian function f is continuous.

A set $A \subseteq \mathbb{R}^d$ is a d–dimensional *Lipschitz manifold* if A is locally representable as the subgraph of a Lipschitzian function, i.e. for any $a \in A$ there exists a neighborhood $U \subset \mathbb{R}^d$, $a \in U$, a unit vector $u \in \mathbb{R}^d$ and a Lipschitzian function $\phi :\, u^\perp \to \mathbb{R}$ such that

$$A \cap U = \{x + tu :\, x \in u^\perp,\, t \leq \phi(x)\} \cap U,$$

where $u^\perp$ denotes the $(d-1)$–dimensional subspace of $\mathbb{R}^d$ perpendicular to u.

The topological boundary ∂A of a d–dimensional Lipschitz manifold $A \subseteq \mathbb{R}^d$ is called $(d-1)$*–dimensional Lipschitz manifold*. Consequently, it is locally representable as a graph of a Lipschitzian function, i.e. for any $a \in \partial A$ there exists a neighborhood $U \subset \mathbb{R}^d$, $a \in U$, a unit vector $u \in \mathbb{R}^d$ and a Lipschitzian function $\phi:\ u^\perp \to \mathbb{R}$ such that

$$\partial A \cap U = \{x + \phi(x)u :\ x \in u^\perp\} \cap U.$$

A set $W \subseteq \mathbb{R}^d$ is called k*–rectifiable*, $k \in \{0, 1, \ldots, d\}$, if it is a Lipschitzian image of a bounded subset of $\mathbb{R}^k$.

Following notation of Federer [7], we say that $W \subseteq \mathbb{R}^d$ is $(\mathcal{H}^k, k)$–*rectifiable* if all following assumptions are fulfilled

- W is $\mathcal{H}^k$–measurable,
- $\mathcal{H}^k(W) < \infty$,
- $W = \bigcup_{i=0}^{\infty} W_i$, $\mathcal{H}^k(W_0) = 0$, W_i is k–rectifiable, $i \geq 1$.

Any bounded $(d-1)$–dimensional Lipschitz manifold A is locally $(d-1)$–rectifiable (i.e., to any $a \in A$ there exists a neighborhood U such that $A \cap U$ is $(d-1)$–rectifiable).

Chapter 3

Wiener sausage

Let $\{B(t) : t \geq 0\}$ be the standard d-dimensional Brownian motion in $\mathbb{R}^d$ starting at the origin, i.e. $B(0) = o$. Given a radius $r \geq 0$ and a time $t > 0$, consider the set

$$S_{r,t} = \{B(s) : 0 \leq s \leq t\} \oplus B(o, r),$$

which is the set of all points with distance at most r to the trajectory of the Brownian motion up to time t and is called *Wiener sausage*.

It is used in physics; the space visited by a moving spherical particle along Brownian trajectory up to time t corresponds to a Wiener sausage. Several applications could be found in applied sciences. A particular application in technology is the modelling of sensor network (see e.g. [15], introduction to Chapter 4). Here a Boolean model of Wiener sausages corresponds to a total area scanned by sensors moving along Brownian trajectories that are randomly scattered in the space. For further applications, see e.g. references in [37].

The Wiener sausage $S_{r,t}$ is a compact subset of $\mathbb{R}^d$ almost surely. It is easy to see that it is a random closed set in the sense of Matheron [19], i.e. that it is measurable in the Matheron-Fell topology (see (2.1)). Indeed, let

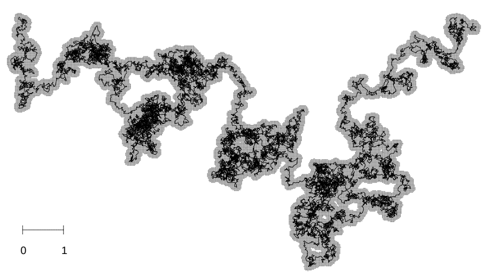

Figure 3.1: A realization of the Wiener sausage with total time $t = 20$ and dilation radius $r = 0.1$.

$\tau_B = \inf\{s \geq 0 : B(s) \in B\}$ be the first hitting time of a Borel set B, which is a (measurable) random variable. The equality of events $[S_{r,t} \cap B \neq \emptyset] = [\tau_{B \oplus B(o,r)} \leq t]$ verifies the measurability of $S_{r,t}$.

Due to elementary properties of the underlying Brownian motion we can derive several useful facts about behavior of the Wiener sausage. Indeed, the scale invariance property gives a relation between total time and dilation radius.

Lemma 3.1 *For any $r > 0$, $\alpha > 0$ and $t > 0$ it holds*

$$S_{r,\alpha t} \stackrel{\mathcal{D}}{=} \sqrt{\alpha} \cdot S_{r/\sqrt{\alpha},t}. \tag{3.1}$$

Proof: From the scale invariance property of Brownian motion we have $S_{0,\alpha t} \stackrel{\mathcal{D}}{=} \sqrt{\alpha}\, S_{0,t}$. Applying the dilation we obtain

$$S_{r,\alpha t} = S_{0,\alpha t} \oplus B(o,r) \stackrel{\mathcal{D}}{=} \sqrt{\alpha}\, S_{0,t} \oplus \sqrt{\alpha}\, B(o, r/\sqrt{\alpha}) = \sqrt{\alpha} \cdot S_{r/\sqrt{\alpha},t},$$

since the Minkowski addition is a linear operator. □

A further nice property is that any projection of the Wiener process into lower dimensional space is again a Wiener sausage

Lemma 3.2 *Let $S^d_{r,t}$ denote a Wiener sausage in the space $\mathbb{R}^d$. Let L_k denote a k–dimensional subspace of $\mathbb{R}^d$ and Π_{L_k} the orthogonal projection $\Pi_{L_k} : \mathbb{R}^d \to L_k$. Then it holds*

$$\Pi_{L_k}(S^d_{r,t}) \stackrel{\mathcal{D}}{=} S^k_{r,t}. \tag{3.2}$$

Proof: The assertion follows easily from the fact that $S_{0,t}$ is isotropic. Hence using suitable rotation the subspace L_k can be chosen parallel to coordinate system and (3.2) is derived directly from the definition (2.11). □

3.1 Geometric properties

Geometric properties of the Wiener sausage were summarized in [5]. We add here some more detailed information.

3.1.1 Mean volume

Computation of the mean volume for the Wiener sausage was first investigated by Kolmogoroff and Leontowitsch [17] for the two–dimensional case. Later, Berezhkovskii et al. [1] derived the formula for general dimension $d \geq 2$. The asymptotic behavior of the mean volume for $t \to \infty$ was studied in a former work by Spitzer [30].

Denote $V(r,t) = V_d(S_{r,t})$ the volume of the Wiener sausage. Finiteness of the expected volume $\mathrm{E}V(r,t)$ can be justified by the following argument

$$\mathrm{E}V(r,t) \leq \mathrm{E}\,\omega_d\big(r + \max\{|W_i(t')| : 0 \leq t' \leq t,\ i = 1,\dots,d\}\big)^d, \tag{3.3}$$

where $W_i(t)$ is the i–th coordinate of $B(t)$. For all i, $W_i(t)$ is a one–dimensional Wiener process (independent of W_j, $j \neq i$). It is well known that all moments of $Z_i = \max\{|W_i(t')|, 0 \leq t' \leq t\}$ are finite. Hence, also $\max_{1\leq i\leq d} Z_i \leq Z_1 + \cdots + Z_d$ has finite moments of all orders.

Other moments $\mathrm{E}V^k(r,t), k \in \mathbb{N}$ are also finite. This follows from the result

$$\mathrm{E}\exp\{\alpha\, V(r,t)\} < \infty,$$

for all $\alpha > 0$ and $r > 0$, shown by Sznitman [32].

Computation of $\mathrm{E}V(r,t)$ is described in [1], it starts with the interchange of integral and expectation which is justified by the finiteness of mean volume:

$$\mathrm{E}V(r,t) = \mathrm{E}\int_{\mathbb{R}^d} \mathrm{I}(x \in S_{r,t})\,dx = \int_{\mathbb{R}^d} \mathrm{P}(\tau^x_{B(o,r)} \leq t)\,dx.$$

The integrated probability, i.e. the distribution of the first hitting time to a ball for Brownian motion, can be regarded as the unique solution to the heat conduction problem, as already mentioned in Theorem 2.2 (see [30]):

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \tfrac{1}{2}\triangle u, && t > 0,\ x \in \mathbb{R}^d \setminus B(o,r), \\ &u(0,x) = 0, && x \in \mathbb{R}^d \setminus B(o,r), \\ &u(t,x) = 1, && t > 0,\ x \in B(o,r). \end{aligned} \tag{3.4}$$

Due to the rotational symmetry of the problem we can use polar coordinates and reduce the dimension of the equation. A classical approach how to solve (3.4) is to apply the Laplace transform. It is more convenient first to integrate over the space $\mathbb{R}^d$ and then to carry out the Laplace transform by inverse transformation.

Theorem 3.1 (Berezhkovskii et al.) *It holds*

$$\mathrm{E}V(r,t) = \omega_d r^d + \mathrm{I}\{d \geq 3\}\frac{d(d-2)}{2}\omega_d t r^{d-2} + \frac{4d\omega_d r^d}{\pi^2}\int_0^\infty \frac{1 - \exp\{-\frac{x^2 t}{2r^2}\}}{x^3(J_\nu^2(x) + Y_\nu^2(x))}\,dx, \tag{3.5}$$

where J_ν and Y_ν are Bessel functions of the first and second kinds of order $\nu = \frac{d-2}{2}$. In case of dimensions $d = 1, 3$, (3.5) can be simplified to

$$\mathrm{E}V_1(S_{r,t}) \;=\; 2r + \frac{2\sqrt{2}t}{\sqrt{\pi}}, \tag{3.6}$$

$$\mathrm{E}V_3(S_{r,t}) = \frac{4}{3}\pi r^3 + 4r^2\sqrt{2\pi t} + 2\pi r t. \tag{3.7}$$

Proof: [1], Eq. (9). □

From now on, we fix the notation $\nu = (d-2)/2$. Asymptotic behavior for the total time t tending to infinity was studied already in papers [30] and [1]. In any dimension the mean volume tends to infinity with growing t and to zero with $r \to 0$.

In the next proposition we present leading terms for these asymptotics for the case of dimensions $d \geq 3$.

Proposition 3.1 *When $d \geq 3$, the asymptotic for $\mathrm{E}V_d(S_{r,t})$ for large values of t (and small radii r) is universal*

$$\mathrm{E}V_d(S_{r,t}) \simeq \omega_d \frac{d(d-2)}{2} r^{d-2} t, \qquad \frac{t}{r^2} \to \infty.$$

Proof: This can be shown by the following argument. Set $\tau = t/2r^2$. We can use the estimate $1 - \exp\{-\tau x^2\} \leq \min\{1, \tau x^2\}$ to show that the last term in (3.5) is majorized by the second one as $\tau \to \infty$. Namely, the interchange of the limit and integral in

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\infty \frac{1 - \exp\{-\tau x^2\}}{x^3(J_\nu^2(x) + Y_\nu^2(x))} = 0$$

is justified by the Lebesgue dominated convergence theorem with the integrable majorizing function

$$\frac{\min\{1, x^2\}}{x^3(J_\nu^2(x) + Y_\nu^2(x))} \in \mathrm{L}_1(0, \infty). \tag{3.8}$$

Since $J_\nu^2(x) + Y_\nu^2(x)$ is a decreasing function of x (c.f. e.g. [34], page 487), having the limits $\lim\limits_{x\to 0} J_\nu^2(x) + Y_\nu^2(x) = \infty$ and $\lim\limits_{x\to\infty} J_\nu^2(x) + Y_\nu^2(x) = 0$, it has no zero points in $(0, \infty)$.

Hence, the function in (3.8) is continuous on $(0, \infty)$ and to show its integrability it suffices to treat only boundary points.

As $x \to 0$, we have from the asymptotic expansions for Bessel functions

$$\begin{aligned}
\frac{\min\{1, x^2\}}{x^3(J_0^2(x) + Y_0^2(x))} &\simeq \frac{c_1}{x \log^2 x/2} \\
\frac{\min\{1, x^2\}}{x^3(J_\nu^2(x) + Y_\nu^2(x))} &\simeq \frac{c_2}{x^{1+\nu}} \qquad \nu > 0,
\end{aligned}$$

which are both integrable on some interval $(0, \epsilon)$, with ϵ, c_1, c_2 positive constants.

As $x \to \infty$, the asymptotic expansions for Bessel functions is independent of the order $\nu \geq 0$ and we have

$$\frac{\min\{1, x^2\}}{x^3(J_\nu^2(x) + Y_\nu^2(x))} \simeq \frac{c_3}{x^2}.$$

The latter function is obviously integrable on (ϵ, ∞), with ϵ, c_3 positive constants. □

If the dimension d is equal to 2, the second term in (3.5) vanishes and the leading term is then more complicated. This asymptotic behavior was first studied in the pioneering work [17]. Spitzer [30] derived later the following formula:

$$\mathrm{E}V_2(S_{r,t}) \simeq \frac{2\pi t}{\log t} + \frac{2\pi t}{(\log t)^2}(2\log r + 1 + \gamma - \log 2), \quad t \to \infty \tag{3.9}$$

where γ is the Euler's number.

The expression (3.9) can be even widen by further terms of the type $t/\log^n t$. This is derived by the special type of inversion of the Laplace transform as is done in [1].

Lemma 3.3 *Set $\tau = t/2r^2$. Then the expansion of the mean volume $\mathrm{E}V_2(S_{r,t})$ for large values of τ can be expressed as*

$$\mathrm{E}V_2(S_{r,t}) \simeq \pi r^2 + \sum_{j=0}^{\infty} \frac{2\pi t\, A_j}{(\log \beta\tau)^{j+1}}, \quad \tau \to \infty, \tag{3.10}$$

where $\beta = 4\exp(-2\gamma)$, γ is the Euler number and the coefficients A_j are defined as:

$$A_j = \frac{d^j}{dx^j}\left[\frac{1}{\Gamma(1-x)}\right]_{x=-1}.$$

Proof: The hint to the proof can be found in [1]. □

3.1.2 Mean surface area

It is well known that the surface area of a convex (full-dimensional) set can be achieved as derivative of the volume of its parallel body. This approach can be generalized for certain sets. In particular, it was shown in [24] that the latter idea can be used in the case of mean surface area of the Wiener sausage.

Later, Last in [18] derived the same result as a side product of a study of mean curvatures of Brownian motion.

Theorem 3.2 (Rataj et al.) *Let $S_{r,t}$ be a d–dimensional Wiener sausage, $d \geq 2$. Then for almost all radii $r > 0$,*

$$\begin{aligned} &\mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t}) \\ &= \quad d\omega_d r^{d-1}\left(1+(d-2)^2\frac{t}{2r^2}+\frac{4d}{\pi^2}\int_0^\infty \frac{\varphi_d(x^2\frac{t}{2r^2})}{x^3(J_\nu^2(x)+Y_\nu^2(x))}\,dx\right), \end{aligned} \tag{3.11}$$

where $\varphi_d(y) = 1 - e^{-y} - 2ye^{-y}/d$ and $\nu = (d-2)/2$. Furthermore, equation (3.11) holds for all $r > 0$ when $d = 2$ or 3. Especially we have

$$\mathrm{E}\mathcal{H}^2(\partial S_{r,t}) = 4\pi r^2 + 8r\sqrt{2\pi t} + 2\pi t. \tag{3.12}$$

Proof: [24]. □

The asymptotic behavior of $\mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t})$ is interesting both for $t \to \infty$ and $r \to 0$. We summarize it in the following proposition.

Proposition 3.2 *In any dimension $d \geq 2$,*

$$\lim_{t\to\infty} \mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t}) = \infty. \tag{3.13}$$

The asymptotic for $r \to 0$ depends on the dimension in the following way:

$$\text{If } d = 2, \qquad \lim_{r\to 0} \mathrm{E}\mathcal{H}^{1}(\partial S_{r,t}) = \infty. \tag{3.14}$$

$$\text{If } d = 3, \qquad \lim_{r\to 0} \mathrm{E}\mathcal{H}^{2}(\partial S_{r,t}) = 2\pi t. \tag{3.15}$$

$$\text{If } d > 3, \qquad \lim_{r\to 0} \mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t}) = 0. \tag{3.16}$$

When the dimension d is three and higher the leading term can be presented in the following way

$$\mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t}) \simeq d\omega_d r^{d-1}(d-2)^2\tau, \quad \tau \to \infty, \tag{3.17}$$

where $\tau = t/2r^2$.

Proof: We start with the proof of (3.17). Expressions (3.15), (3.16) and (3.13) for $d \geq 3$ then follow easily.

It is sufficient to show

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\infty \frac{\varphi_d(x^2\tau)}{x^3(J_\nu^2(x)+Y_\nu^2(x))}\,dx = 0.$$

This can be shown by the Lebesgue dominated theorem; using the bounds $0 \leq \varphi_d(y) \leq \min\{y, 1\}$ we majorize

$$\frac{\varphi_d(x^2\tau)/\tau}{x^3(J_\nu^2(x)+Y_\nu^2(x))} \leq \frac{\min\{x^2, 1\}}{x^3(J_\nu^2(x)+Y_\nu^2(x))}. \tag{3.18}$$

The last function is integrable on $(0,\infty)$ (see the asymptotic expansions of Bessel functions in Chapter 2, Preliminaries). Since the function $J_\nu^2(x)+Y_\nu^2(x)$ is bounded from zero the integrated function is continuous and it suffices to treat only boundary points 0 and ∞. For $x\to 0$ the integrated function behaves like x^{d-3} which is integrable on any bounded interval $[0,A]$, $A\in\mathbb{R}$. For $x\to\infty$ the integrated function is majorized by K/x^2 (for some $K>0$) which is also integrable on any (ϵ,∞), $\epsilon>0$.

Let now the dimension $d=2$. Then the middle term in (3.11) vanishes. Set

$$F(\tau)=\int_0^\infty \frac{\varphi_2(x^2\tau)}{x^3(J_0^2(x)+Y_0^2(x))}\,dx.$$

The function φ_2 is positive and increasing on $(0,\infty)$. Hence $F(\tau)$ is also positive and increasing and using Fatou's lemma we get

$$\begin{aligned}\lim_{\tau\to\infty}F(\tau) &\geq \int_0^\infty \lim_{\tau\to\infty}\frac{\varphi_2(x^2\tau)}{x^3(J_0^2(x)+Y_0^2(x))}\,dx\\ &= \int_0^\infty \frac{1}{x^3(J_0^2(x)+Y_0^2(x))}\,dx=\infty,\end{aligned}$$

which proves (3.13) for $d=2$.

It remains to prove (3.14). It can be shown that

$$\varphi_2(\tau)\geq \tau^2\frac{1}{2e},\quad \text{for } 0\leq\tau\leq 1.$$

Applying this inequality we obtain

$$\begin{aligned}\lim_{r\to 0} r\int_0^\infty \frac{\varphi_2(x^2\tau)}{x^3(J_0^2(x)+Y_0^2(x))}\,dx &\geq \lim_{r\to 0} r\int_0^{\frac{1}{\sqrt{\tau}}} \frac{\varphi_2(x^2\tau)}{x^3(J_0^2(x)+Y_0^2(x))}\,dx\\ &\geq \lim_{r\to 0} r\frac{1}{2e}\int_0^{\frac{1}{\sqrt{\tau}}} \frac{x^4\tau^2}{x^3(J_0^2(x)+Y_0^2(x))}\,dx\\ &= \lim_{r\to 0}\frac{t^2}{8er^3}\int_0^1 \mathrm{I}_{\left[x\leq\frac{r\sqrt{2}}{\sqrt{t}}\right]}\frac{x}{J_0^2(x)+Y_0^2(x)}\,dx.\end{aligned}$$

Since according to (2.22)

$$J_0^2(x)+Y_0^2(x)\simeq\frac{4}{\pi^2}\log^2 x/2,\quad x\to 0,$$

there exists r_0 such that for $0 < r < r_0$ we have

$$\begin{aligned}\lim_{r\to 0}\frac{t^2}{8er^3}\int_0^1 \mathrm{I}_{\left[x\le\frac{r\sqrt{2}}{\sqrt{t}}\right]}\frac{x}{J_0^2(x)+Y_0^2(x)}\,dx &\ge \lim_{r\to 0}\frac{t^2}{8er^3}\int_0^1 \mathrm{I}_{\left[x\le\frac{r\sqrt{2}}{\sqrt{t}}\right]}\frac{x}{1/\sqrt{x}}\,dx\\ &= \lim_{r\to 0}\frac{t^2}{8er^3}\frac{2}{5}\left(\frac{r\sqrt{2}}{\sqrt{t}}\right)^{5/2} = \infty.\end{aligned}$$

□

3.1.3 Intrinsic volumes

Let $d_X(\cdot) = \mathrm{dist}\,(\cdot, X)$ denote the distance function to a set $X \subseteq \mathbb{R}^d$. We say that $r > 0$ is a *critical value* of d_X if there exists a point y with $d_X(y) = r$ and such that $y \in \overline{\mathrm{conv}}(X \cap B(y, r))$, where $\overline{\mathrm{conv}}(\cdot)$ denotes the closed convex hull, i.e. the smallest convex closed set containing the argument. Positive values which are not critical are called *regular*.

It can be shown (see [8]) that whenever r is a regular value of d_X then the r-parallel set $X_r = X \oplus B(o, r)$ is a d-dimensional Lipschitz manifold and the closure of its complement has positive reach. Hence, according to [27, Proposition 4] its intrinsic volumes are well defined.

Moreover, Fu in [8] showed that for any closed subset X of $\mathbb{R}^2$ or $\mathbb{R}^3$, the set of critical values of d_X has Lebesgue measure zero.

Theorem 3.3 *Let the dimension d be 2 or 3 and let $r, t > 0$ be given. Then the intrinsic volume $V_j(S_{r,t})$ is defined and finite almost surely for any $j \le d$.*

Proof: First we show that any $r > 0$ is a regular value for $d_{S_{0,t}}$ almost surely (in dimensions $d = 2$ and 3).

Assume for contradiction that for some $r > 0$ it holds

$$\mathrm{P}(r \text{ is critical for } d_{S_{0,t}}) > 0.$$

Consequently, it can be shown using the scaling invariance property of Brownian motion that there exists a whole interval of critical values containing r (see [24, Lemma 4.2]):

$$\exists c > 1,\ \mathrm{P}(s \text{ is critical of } d_{S_{0,t}}) > 0\, \forall s \in [r, cr].$$

We use the Fubini theorem to show

$$\mathrm{E}\lambda_1(s > 0 :\ s \text{ is critical of } d_{S_{0,t}}) = \int_0^\infty \mathrm{P}(s \text{ is critical of } d_{S_{0,t}})ds$$

$$\geq \int_r^{cr} \mathrm{P}(s \text{ is critical of } d_{S_{0,t}})ds > 0,$$

which is contradicting the Fu's result.

Therefore, any r is a regular value for $d_{S_{0,t}}$ almost surely and consequently, $S_{r,t}$ is a d–dimensional Lipschitz manifold. The rest of the assertion follows from [27, Proposition 4]. □

The finiteness of the expectations $\mathrm{E}V_j(S_{r,t})$ is still an open problem for $j < d-1$, up to our knowledge. The case of general dimension d is completely open up to now. Another task would be to find asymptotic formulae for $\mathrm{E}V_j(S_{r,t})$ as $r \to 0_+$ using the scaling invariance.

3.1.4 Covariogram

The covariogram of a bounded random closed subset X of $\mathbb{R}^d$ is the function

$$C_X(h) = \mathrm{E}V_d(X \cap (X-h)), \quad h \in \mathbb{R}^d.$$

If X is isotropic (i.e. its distribution is invariant with respect to rotations) then the covariogram depends only on the length of h and not on its direction, and we denote

$$\widetilde{C}_X(s) = C_X(su), \quad s \geq 0,$$

where u is any unit vector in $\mathbb{R}^d$. The covariogram of the Wiener sausage is related to a solution of a heat conduction equation in the following way.

Theorem 3.4 *We have for any $r, t > 0$,*

$$\widetilde{C}_{S_{r,t}}(s) = 2V(r,t) - \int_{\mathbb{R}^d} y_{su}(t,x)dx, \tag{3.19}$$

where the function $y_{su}(t,x)$ is the unique solution of the differential equation

$$\begin{aligned} &\tfrac{\partial y}{\partial t} = \tfrac{1}{2} \triangle y && t > 0,\ x \in \mathbb{R}^d \setminus B, \\ &y(0,x) = 0, && x \in \mathbb{R}^d \setminus B, \\ &y(t,x) = 1, && t > 0,\ x \in B, \end{aligned} \tag{3.20}$$

and B is the union of two balls of radii r and with centers at o and su. Moreover, when $d = 2,3$ for any $t > 0$, $r > 0$:

$$\widetilde{C}'_{S_{r,t}}(0) = -\frac{\omega_{d-1}}{d\omega_d} \mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t}). \tag{3.21}$$

Proof: A simple argument is used to express the covariogram $C_{S_{r,t}}(h)$ of the Wiener sausage:

$$\begin{aligned} C_{S_{r,t}}(h) &= \mathrm{E}V_d(S_{r,t} \cap (S_{r,t} - h)) = \int_{\mathbb{R}^d} \mathrm{P}(x \in S_{r,t} \cap (S_{r,t} - h))dx \\ &= \int_{\mathbb{R}^d} \mathrm{P}\left(\tau^x_{B(o,r)} \le t, \tau^x_{B(h,r)} \le t\right) dx \\ &= \int_{\mathbb{R}^d} \left(\mathrm{P}\left(\tau^x_{B(o,r)} \le t\right) + \mathrm{P}\left(\tau^x_{B(h,r)} \le t\right)\right) dx \\ &\quad - \int_{\mathbb{R}^d} \mathrm{P}\left(\tau^x_{B(o,r)\cup B(h,r)} \le t\right) dx. \end{aligned}$$

Obviously, first two integrated terms in the last expression yield the mean volume of the Wiener sausage. The last probability can be again regarded as a solution to heat conduction problem as in (3.4), this time the boundary conditions are fitted to the union of two balls. This proves (3.19).

Let u be a fixed unit vector. The *total projection* of a d–dimensional Lipschitz manifold A in the direction u is defined as

$$TP_A(u) = \frac{1}{2} \int_{\partial A} |u \cdot n(x)| \, \mathcal{H}^{d-1}(dx),$$

where $n(x)$ is the outer unit normal vector of A at x.

It can be shown that the derivative

$$\frac{d}{ds} \, |_{s=0} \, C_A(su)$$

of a "sufficiently regular" bounded deterministic set A equals minus the total projection of A in direction u; the case when A is d–dimensional Lipschitz manifold of positive reach can be found in [22, Theorem 2]. Since the mean total projection equals $\frac{\omega_{d-1}}{d\omega_d}$ times the total surface area, we get the relation

$$\mathrm{E}_u \left.\frac{d}{ds}\right|_{s=0} C_A(su) = -\frac{\omega_{d-1}}{d\omega_d} \mathcal{H}^{d-1}(\partial A), \tag{3.22}$$

where E_u denotes the isotropic mean value with respect to the direction u.

If A is a d-dimensional Lipschitz manifold the closure of complement of which has positive reach, then, applying (3.22) to $\overline{\mathbb{R}^d \setminus A}$ intersected with a sufficiently large ball, we find easily that (3.22) holds for the set A itself.

Since when $d \le 3$ for all $r > 0$ and all $t > 0$, the reach of $\overline{\mathbb{R}^d \setminus S_{r,t}}$ is positive and $S_{r,t}$ is a d–dimensional Lipschitz manifold (see the begining of Chapter 3), (3.22) is true with $A = S_{r,t}$ almost surely. Applying the mean value we get (3.21) since $S_{r,t}$ is isotropic. The interchange of derivative and expectation is justified by the integrability of $\mathcal{H}^{d-1}(\partial S_{r,t})$, see [24]. □

Remark: It seems to be impossible to derive an analytic solution to (3.20). So far, the covariogram $\widetilde{C}_{S_{r,t}}(h)$ can be obtained only via Monte–Carlo simulation methods or by numerical solution of the PDE.

3.2 Approximation of the Wiener sausage

The previous section and analytical expressions of essential properties of the Wiener sausage showed how complicated set it is. Since some of its characteristics are still impossible to derive analytically a possible way how to work with the Wiener sausage is to use approximations by easier sets. It is the underlying Brownian motion what needs to be simplified.

Throughout the whole section we will assume the total time t of the Wiener sausage be equal to 1 (see Lemma 3.1 how to enlarge an approximated Wiener sausage to general total time). Consequently, the letter t will be used for time variable as is usual in such contexts.

Let

$$B(t) = (W_1(t), \dots, W_d(t)), \qquad t \in [0,1]$$

be a d–dimensional Brownian motion, where the coordinates $\{W_i(t),\ t \in [0,1]\}$, $i = 1, \dots, d$ are independent standard Wiener processes such that $W_i(0) = 0$, $i = 1, \dots, d$. Approximating each coordinate by a piecewise linear curve $\{W_i^n(t), t \in [0,1]\}$, $n = 1, 2, \dots$ we can derive an approximation of d–dimensional Brownian motion

$$B^n(t) = (W_1^n(t), \dots, W_d^n(t)), \qquad t \in [0,1]. \tag{3.23}$$

Subsequently, the approximation S_r^n of the Wiener sausage $S_r = S_{r,1}^o$ is

$$S_r^n = \{B^n(t), t \in [0,1]\} \oplus B(o,r). \tag{3.24}$$

In the following sections two different approximations $W_i^n(t)$ will be studied.

3.2.1 Approximation by random walk

The simplest approximation of one–dimensional Wiener process is given by famous Donsker invariance principle.

Let G be an arbitrary distribution on $\mathbb{R}$ such that $\int_{\mathbb{R}} x\ G(dx) = 0$, $\int_{\mathbb{R}} x^2\ G(dx) = 1$ and let $Y_0 = 0$ and $Y_1, Y_2, \dots$ be a sequence of independent random variables each distributed according to G. Set

$$\mathbf{S}_n = \sum_{i=0}^{n} Y_i$$

(i.e. symmetric random walk) and $(W^{\mathrm{rw}}(t), t \geq 0)$ to be a continuous piecewise linear process such that

$$W^{\mathrm{rw}}(n) = \mathbf{S}_n, \quad n \in \mathbb{N}_0.$$

Furthermore, for $t \in [0,1]$ set

$$\mu_n(t) = n^{-1/2} W^{\mathrm{rw}}(nt). \tag{3.25}$$

Theorem 3.5 (Donsker) *Let* $Y_1, Y_2, \ldots$ *and* $\mu_n(t)$, $t \in [0,1]$, $n \in \mathbb{N}$ *be defined as before. Then*

$$\{\mu_n(t),\ t \in [0,1]\} \xrightarrow{\mathcal{D}} \{W(t),\ t \in [0,1]\}, \qquad n \to \infty. \tag{3.26}$$

Proof: The proof can be found for example in [2, Theorem 8.2]. □

The convergence (3.26) can be improved to convergence in probability and almost surely. Unfortunately, we have to work on different probability spaces in order to derive convergence almost surely.

Following the well-known Skorohod Embedding Theorem [13, Theorem 14.1], there exists a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ with a Wiener process W and non–negative independent random variables τ_i, $i = 1, 2, \ldots$, $\mathrm{E}\tau_1 = \mathrm{E}Y_1^2 = 1$ such that

$$\mathbf{S}_n \stackrel{\mathcal{D}}{=} \mathbf{S}_n^* = W\left(\sum_{i=1}^{n} \tau_i\right).$$

Having this representation of a random walk we can define similarly as in (3.25) a continuous piecewise linear process

$$\{W^*(t), t \geq 0\}$$

such that

$$W^*(n) = \mathbf{S}_n^*, \quad n \in \mathbb{N}$$

and for $t \in [0,1]$

$$\mu_n^*(t) = n^{-1/2} W^*(nt).$$

Theorem 3.6 (Strassen) *Let* $\mathbf{S}_n$ *be a random walk with* $\mathrm{E}\mathbf{S}_1 = 0$ *and* $\mathrm{E}\mathbf{S}_1^2 = 1$*. Then* $\forall n \in \mathbb{N}$ *there exists a probability space where a random walk* $\mathbf{S}_n^*$ *with the same distribution as* $\mathbf{S}_n$ *and Wiener processes* $\{W_{(n)}(t),\ t \in [0,1]\}$ *are defined so that*

$$\max_{0 \leq t \leq 1} |\mu_n^*(t) - W_{(n)}(t)| \to 0, \quad n \to \infty \tag{3.27}$$

in probability and

$$(\log\log n)^{-1/2} \max_{0 \leq t \leq 1} |\mu_n^*(t) - W_{(n)}(t)| \to 0, \quad n \to \infty \tag{3.28}$$

almost surely.

Proof: [13, Theorem 14.6]. □

The expression (3.28) shows that the speed of rising new increments in $\mathbf{S}_n$ is not sufficient to obtain an almost sure convergence of $\mu_n^*(t)$ to a Wiener process (in the sense given in Theorem 3.6).

The next Proposition shows how almost sure convergence can be regained.

Proposition 3.3 *Let $Y_0 = 0$ and $Y_1, Y_2, \dots$ be i.i.d. $N(0,1)$ distributed random variables. Set*

$$\mathbf{S}_n = \sum_{k=0}^{n} Y_k.$$

We define $W^n(t)$ to be a piecewise linear process on $[0,1]$ such that $W^n(0) = 0$ and

$$W^n(t) = \frac{\mathbf{S}_{k-1}}{\sqrt{2^n}} + 2^n \left(t - \frac{k-1}{2^n} \right) \frac{Y_k^i}{\sqrt{2^n}}, \qquad t \in \left[\frac{k-1}{2^n}, \frac{k}{2^n} \right], \tag{3.29}$$

$k = 1, \dots, 2^n$.

Then there exists a sequence of Wiener processes $W_{(n)}(t)$ on $[0,1]$ such that

$$\sup_{t \in [0,1]} |W^n(t) - W_{(n)}(t)| \to 0, \qquad n \to \infty \tag{3.30}$$

almost surely.

Proof: Since Y_i are independent Gaussian, the finite–dimensional distribution of W^n in $(0, 1/2^n, \dots, 1)$ coincides with the same distribution for a Wiener process. Hence there exists for any $n \in \mathbb{N}$ a Wiener process $W_{(n)}$ defined on the same probability space such that

$$W_{(n)}\left(\frac{k}{2^n}\right) = W^n\left(\frac{k}{2^n}\right) \qquad \text{for all } k = 1, \dots, 2^n. \tag{3.31}$$

We will show that given $\epsilon > 0$ it holds

$$\mathrm{P}\left(\overline{\lim_{n\to\infty}} \sup_{0 \le t \le 1} |W^n(t) - W_{(n)}(t)| > \epsilon \right) = 0$$

almost surely. We have

$$\begin{aligned} &\mathrm{P}\left(\overline{\lim_{n\to\infty}} \sup_{0 \le t \le 1} |W^n(t) - W_{(n)}(t)| > \epsilon \right) \\ &\le \ \mathrm{P}\left(\sup_{0 \le t \le 1} |W^n(t) - W_{(n)}(t)| > \epsilon \text{ for infinitely many } n \right). \end{aligned} \tag{3.32}$$

We can estimate

$$
\begin{aligned}
&\mathrm{P}\left(\sup_{0\le t\le 1}|W^n(t)-W_{(n)}(t)|>\epsilon\right)\\
&\le \sum_{k=1}^{2^n}\mathrm{P}\left(\sup_{\frac{k-1}{2^n}\le t\le \frac{k}{2^n}}|W^n(t)-W_{(n)}(t)|>\epsilon\right)\\
&= 2^n\,\mathrm{P}\left(\sup_{0\le t\le \frac{1}{2^n}}|W^n(t)-W_{(n)}(t)|>\epsilon\right)\\
&\le 2^n\,\mathrm{P}\left(\sup_{0\le t\le \frac{1}{2^n}}|X^{0,2^{-n},0}(t)|+\frac{Y_1}{\sqrt{2^n}}>\epsilon\right)\\
&\le 2^n\left[\mathrm{P}\left(\sup_{0\le t\le \frac{1}{2^n}}|X^{0,2^{-n},0}(t)|>\epsilon\right)+\mathrm{P}\left(Y_1>\epsilon\sqrt{2^n}\right)\right],
\end{aligned}
$$

where $X^{0,2^{-n},0}(t)$ is the Brownian bridge on $[0,1/2^n]$. Further, we use the representation $X^{0,2^{-n},0}(t)\overset{\mathcal{D}}{=}(1/2^n-t)W(t)$, maximal inequality for Wiener process and Feller inequality (see Lemma 2.1) to obtain

$$
\begin{aligned}
&\mathrm{P}\left(\sup_{0\le t\le 1}|W^n(t)-W_{(n)}(t)|>\epsilon\right)\\
&\le 2^n\left[\mathrm{P}\left(\sup_{0\le t\le \frac{1}{2^n}}\left|\left(\frac{1}{2^n}-t\right)W(t)\right|>\epsilon\right)+\mathrm{P}\left(Y_1>\epsilon\sqrt{2^n}\right)\right]\\
&\le 2^n\left[\mathrm{P}\left(\sup_{0\le t\le \frac{1}{2^n}}\left|\frac{1}{2^n}W(t)\right|>\epsilon\right)+\mathrm{P}\left(Y_1>\epsilon\sqrt{2^n}\right)\right]\\
&\le 2^n\left[4\,\mathrm{P}\left(W\left(\frac{1}{2^n}\right)>\epsilon 2^n\right)+\mathrm{P}\left(Y_1>\epsilon\sqrt{2^n}\right)\right]\\
&\le 2^n\left[4\exp\left\{-\frac{2^n\epsilon^2}{2}\right\}+\exp\left\{-\frac{2^n\epsilon^2}{2}\right\}\right]=5\,2^n\exp\left\{-\frac{2^n\epsilon^2}{2}\right\}:=a_n.
\end{aligned}
$$

The latter inequality give

$$
\sum_{n=1}^{\infty}\mathrm{P}\left(\sup_{0\le t\le 1}|W^n(t)-W_{(n)}(t)|>\epsilon\right)\le\sum_{n=1}^{\infty}a_n. \tag{3.33}
$$

Next we apply the ratio criterion to ensure that the row $\sum_{n=1}^{\infty}a_n$ is convergent:

$$
\frac{a_{n+1}}{a_n}=2\exp\{-\epsilon^2 2^{n-1}\}.
$$

Since ϵ is fixed we can find n_0 such that $\frac{a_{n+1}}{a_n}<1$ for any $n>n_0$. The proof is finished by using Borel – Cantelli Lemma which ensures that the probability in (3.32) is 0 for any $\epsilon>0$. □

Remark: In fact, any speed of adding new terms Y_i into approximation $W^n(t)$ which is higher then $n \log\log n$ is sufficient to derive an almost surely convergence in the given sense. The presented speed of 2^n coincides with the approximation given in the next Section.

3.2.2 Haar-Schauder approximation

In this section, following [14, pp. 56–59], an almost surely convergent approximation of the Wiener process is given. The method can be used as a self–contained construction of the Wiener process. Its main idea lies in the following fact.

Let $W(t)$ be a standard Wiener process, fix $0 \leq s < t < \infty$. Conditioned on $W(s) = x, W(t) = y$ the distribution of $W\left(\frac{t+s}{2}\right)$ is normal with mean $\mu = \frac{x+y}{2}$ and variance $\sigma^2 = \frac{t-s}{4}$. For the proof and more information on this we refer to [14].

Define Haar functions by $H_1(t) = 1,\ 0 \leq t \leq 1$ and

$$H_{2^m+k}(t) = \begin{cases} 2^{m/2} & t \in \left[\frac{k-1}{2^m}, \frac{2k-1}{2^{m+1}}\right), \\ -2^{m/2} & t \in \left[\frac{2k-1}{2^{m+1}}, \frac{k}{2^m}\right), \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2, \ldots, 2^m$ and $m = 0, 1, \ldots$. The Schauder function is given by

$$R_k(t) = \int_0^t H_k(s)ds,\ k \in \mathbb{N}.$$

Given a sequence $\{Y_n\}_{n\in\mathbb{N}}$ of i.i.d. N(0,1)–distributed random variables it is known that the distribution of the Wiener process $\{W(t), t \in [0,1]\}$ corresponds to a distribution of Haar–Schauder series

$$W(t) = \sum_{k=1}^{\infty} Y_k R_k(t), \qquad t \in [0,1]. \tag{3.34}$$

The series converges almost surely uniformly. Hence the approximation $W^n(t)$ is simply given by partial sums

$$W^n(t) = \sum_{k=1}^{n} Y_k R_k(t), \qquad t \in [0,1]. \tag{3.35}$$

3.2.3 Speed of convergence

In this section we are going to survey how far the approximation S_r^n of a Wiener sausage given by (3.24) is from its theoretical counterpart, a realization of Wiener sausage with same parameters, i.e. our goal is to obtain the distribution and the asymptotical behavior of Hausdorff distance

$$d_H(S_r^n, S_r),$$

as n tends to infinity.

The results obtained below can be used either for the approach given in Proposition 3.3 or for that given by Haar–Schauder approximation.

As we mentioned above the first method has the advantage in easy application to practical computing while the second one has the nice property that the approximated Brownian motion runs through the same edge points when increasing the approximation level n.

Advantageously, the distribution of the error remains unchanged in both situations.

Let

$$E_n = \sup_{t\in[0,1]} |B^n(t) - B_{(n)}(t)|, \tag{3.36}$$

where $B^n(t) = (W_1^n(t), \dots, W_d^n(t))$ is the approximation of d–dimensional Brownian motion given either by (3.29) or (3.35) and $B_{(n)}(t)$ are Brownian motions in $\mathbb{R}^d$ which are identical in the case of Haar-Schauder approximation.

We state an auxiliary Lemma first.

Lemma 3.4 *Let $(X, \widehat{X})$ and $(Y, \widehat{Y})$ be independent non–negative random vectors with marginal distribution functions F_X, $F_{\widehat{X}}$, F_Y, $F_{\widehat{Y}}$ such that*

$$F_X(x) \le F_{\widehat{X}}(x), \qquad x \ge 0,$$

$$F_Y(x) \le F_{\widehat{Y}}(x), \qquad x \ge 0.$$

Then it holds

$$\mathrm{P}(X + Y \le x) \le \mathrm{P}(\widehat{X} + \widehat{Y} \le x), \qquad \textit{for any } x \ge 0,$$

which can be rewritten as

$$\int_0^\infty F_X(x-y)dF_Y(y) \le \int_0^\infty F_{\widehat{X}}(x-y)dF_{\widehat{Y}}(y), \qquad \textit{for any } x \ge 0.$$

Proof: We have

$$\int_0^\infty F_X(x-y)dF_Y(y) \le \int_0^\infty F_{\widehat{X}}(x-y)dF_Y(y).$$

Since the convolution is a symmetric operation we can further proceed to obtain

$$\begin{aligned}\int_0^\infty F_{\widehat{X}}(x-y)dF_Y(y) &= \int_0^\infty F_Y(y-x)dF_{\widehat{X}}(x)\\ &\leq \int_0^\infty F_{\widehat{Y}}(y-x)dF_{\widehat{X}}(x)\\ &= \int_0^\infty F_{\widehat{X}}(x-y)dF_{\widehat{Y}}(y).\end{aligned}$$

□

Proposition 3.4 *Let S_r^n, $n = 1, 2, \ldots$ be an approximation of a Wiener sausage S_r defined in* (3.24) *and let E_n be given by equation* (3.36)*. Then*

$$\mathrm{P}(d_H(S_r^n, S_r) \leq y) \geq \mathrm{P}(E_n \leq y). \tag{3.37}$$

Moreover,

$$\mathrm{P}(E_n \leq y) \geq \left(\int_{\frac{\ln 2}{2^{n+1}}}^{y^2 - \frac{(d-1)\ln 2}{2^{n+1}}} \cdots \int_{\frac{\ln 2}{2^{n+1}}}^{y^2 - \sum_{i=1}^{d-2} x_i - \frac{\ln 2}{2^{n+1}}} G_n\left(y^2 - \sum_{i=1}^{d-1} x_i\right) \ldots \right.$$
$$\left. \ldots\, dG_n(x_1) \ldots dG_n(x_{d-1}) \right)^{2^n}, \tag{3.38}$$

where

$$G_n(z) = \left(1 - 2e^{-2^{n+1}z}\right) \cdot \mathrm{I}\left[z \geq 2^{-(n+1)} \ln 2\right]$$

and

$$\mathrm{P}(E_n \leq y) \leq \left(\int_0^{dy^2} \cdots \int_0^{dy^2 - \sum_{i=1}^{d-2} x_i} J_n\left(dy^2 - \sum_{i=1}^{d-1} x_i\right) \ldots \right.$$
$$\left. \ldots\, dJ_n(x_1) \ldots dJ_n(x_{d-1}) \right)^{2^n}, \tag{3.39}$$

where

$$J_n(z) = 1 - e^{-2^{n+1}z}.$$

Proof: It can be easily verified that

$$\begin{aligned} d_H(S_r^n, S_r) &\leq d_H\big(\{B^n(t),\ t\in[0,1]\}, \{B_{(n)}(t),\ t\in[0,1]\}\big) \\ &\leq \sup_{t\in[0,1]} |B^n(t) - B_{(n)}(t)|, \end{aligned}$$

which proves (3.37).

We can express E_n as

$$\begin{aligned} E_n &= \max_{1\leq i\leq 2^n} \sup_{\frac{i-1}{2^n}\leq t\leq \frac{i}{2^n}} |B^n(t) - B_{(n)}(t)| \\ &\stackrel{\mathcal{D}}{=} \max_{1\leq i\leq 2^n} \sup_{0\leq t\leq 2^{-n}} \left| 2^n t \frac{Y_i}{\sqrt{2^n}} - \left(X_i^{0,2^{-n},0}(t) + 2^n t \frac{Y_i}{\sqrt{2^n}} \right) \right| \\ &= \max_{1\leq i\leq 2^n} \sup_{0\leq t\leq 2-n} \left| X_i^{0,2^{-n},0}(t) \right|, \end{aligned}$$

where Y_i, $i = 1, \ldots, 2^n$, $n \in \mathbb{N}$ are independent standard Gaussian distributed variables and $X_i^{0,2^{-n},0}(t)$, $i = 1, \ldots, 2^n$, $n \in \mathbb{N}$ are independent d–dimensional Brownian bridges.

We can estimate the distribution of E_n by the following arguments

$$\begin{aligned} \mathrm{P}(E_n \leq y) &= \left(\mathrm{P}\left(\sup_{0\leq t\leq 2^{-n}} \left| X^{0,2^{-n},0}(t) \right| \leq y \right) \right)^{2^n} \\ &\geq \left(\mathrm{P}\left(\sum_{k=1}^{d} \left(\sup_{0\leq t\leq 2^{-n}} \left| Y_k^{0,2^{-n},0}(t) \right| \right)^2 \leq y^2 \right) \right)^{2^n}, \end{aligned}$$

where $Y_k^{0,2^{-n},0}(t)$, $k = 1, \ldots, d$ are independent one–dimensional Brownian bridges. Since their distributions are similar we shall omit the subscript k in what follows.

Set

$$F_n(z) := \mathrm{P}\left(\left(\sup_{0\leq t\leq 2^{-n}} \left| Y^{0,2^{-n},0}(t) \right| \right)^2 \leq z \right),$$

which is the same for all coordinates $k = 1, \ldots, d$. We have for $z \geq 0$

$$\begin{aligned} F_n(z) &= \mathrm{P}\left(\sup_{0\leq t\leq 2^{-n}} \left| Y^{0,2^{-n},0}(t) \right|^2 \leq z \right) \\ &= 1 - \mathrm{P}\left(\sup_{0\leq t\leq 2^{-n}} \left| Y^{0,2^{-n},0}(t) \right| \geq \sqrt{z} \right) \end{aligned}$$

$$\begin{aligned}
&= 1 - \mathrm{P}\left(\sup_{0\le t\le 2^{-n}} Y^{0,2^{-n},0}(t) \ge \sqrt{z} \quad \vee \inf_{0\le t\le 2^{-n}} Y^{0,2^{-n},0}(t) \le -\sqrt{z} \right) \\
&= 1 - \mathrm{P}\left(\sup_{0\le t\le 2^{-n}} Y^{0,2^{-n},0}(t) \ge \sqrt{z} \quad \vee \sup_{0\le t\le 2^{-n}} -Y^{0,2^{-n},0}(t) \ge \sqrt{z} \right) \\
&\ge 1 - \mathrm{P}\left(\sup_{0\le t\le 2^{-n}} Y^{0,2^{-n},0}(t) \ge \sqrt{z} \right) \\
&\quad - \mathrm{P}\left(\sup_{0\le t\le 2^{-n}} -Y^{0,2^{-n},0}(t) \ge \sqrt{z} \right) \\
&= 1 - 2\,\mathrm{P}\left(\sup_{0\le t\le 2^{-n}} Y^{0,2^{-n},0}(t) \ge \sqrt{z} \right),
\end{aligned}$$

since $-Y^{0,2^{-n},0}(t)$ is again a Brownian bridge. We conclude using (2.10)

$$F_n(z) \ge G_n(z),$$

where we set

$$G_n(z) := \left(1 - 2\exp\left\{-2^{n+1}z\right\}\right) \cdot \mathrm{I}\left[z \ge 2^{-(n+1)} \cdot \ln 2\right].$$

Then $G_n(z)$ is a distribution function which is absolutely continuous with respect to Lebesgue measure and its density function is given by

$$g_n(z) = 2^{n+2} \exp\left\{-2^{n+1}z\right\} \cdot \mathrm{I}\left[z \ge 2^{-(n+1)} \cdot \ln 2\right].$$

To complete the proof of (3.38) we use convolution theorem and the assertion of Lemma 3.4 iteratively $(d-1)$–times.

The second inequality (3.39) can be shown similarly. Since we have for any $0 \le i \le d$

$$\sup_{0\le t\le 2^{-n}} \left|X^{0,2^{-n},0}(t)\right|^2 \ge \sup_{0\le t\le 2^{-n}} \left|Y_i^{0,2^{-n},0}(t)\right|^2$$

we conclude by summing up over all i that

$$\sup_{0\le t\le 2^{-n}} \left|X^{0,2^{-n},0}(t)\right|^2 \ge \frac{1}{d}\sum_{i=1}^{d} \sup_{0\le t\le 2^{-n}} \left|Y_i^{0,2^{-n},0}(t)\right|^2.$$

Then we have

$$\mathrm{P}(E_n \le y) \quad = \quad \left(\mathrm{P}\left(\sup_{0\le t\le 2^{-n}} \left|X^{0,2^{-n},0}(t)\right| \le y\right)\right)^{2^n}$$

$$\leq \quad \left(\mathrm{P} \left(\sum_{k=1}^{d} \left(\sup_{0 \leq t \leq 2^{-n}} \left| Y_k^{0,2^{-n},0}(t) \right| \right)^2 \leq dy^2 \right) \right)^{2^n}.$$

Again, let

$$F_n(z) := \mathrm{P} \left(\left(\sup_{0 \leq t \leq 2^{-n}} \left| Y^{0,2^{-n},0}(t) \right| \right)^2 \leq z \right).$$

$$\begin{aligned} F_n(z) &= \mathrm{P} \left(\sup_{0 \leq t \leq 2^{-n}} \left| Y^{0,2^{-n},0}(t) \right|^2 \leq z \right) \\ &= 1 - \mathrm{P} \left(\sup_{0 \leq t \leq 2^{-n}} \left| Y^{0,2^{-n},0}(t) \right| \geq \sqrt{z} \right) \\ &= 1 - \mathrm{P} \left(\sup_{0 \leq t \leq 2^{-n}} Y^{0,2^{-n},0}(t) \geq \sqrt{z} \quad \vee \right. \\ &\qquad \left. \vee \inf_{0 \leq t \leq 2^{-n}} Y^{0,2^{-n},0}(t) \leq -\sqrt{z} \right) \\ &\leq 1 - \mathrm{P} \left(\sup_{0 \leq t \leq 2^{-n}} Y^{0,2^{-n},0}(t) \geq \sqrt{z} \right) \\ &= 1 - \exp\left(-2^{n+1} z\right) := J_n(z). \end{aligned}$$

The rest of the proof is same as in the case of inequality " $\geq$ ".

□

Theorem 3.7 *Let the dimension $d = 2$. Then*

$$\mathrm{P}(d_H(S_r^n, S_r) \leq y) \geq b_n. \tag{3.40}$$

where $b_n \to 1$ as $n \to \infty$ and it holds

$$1 - b_n \simeq 2^{2n+3} y^2 e^{-2^{n+1} y^2}, \qquad n \to \infty. \tag{3.41}$$

Furthermore, for distribution of E_n we have the following estimates

$$a_n \geq \mathrm{P}(E_n \leq y) \geq b_n, \tag{3.42}$$

where

$$1 - a_n \simeq 2^{2n+2} y^2 e^{-2^{n+2} y^2}, \qquad n \to \infty. \tag{3.43}$$

Proof: Let G_n and J_n denote the same distribution functions as in Proposition 3.4; g_n and j_n their density functions respectively. We have

$$\begin{aligned}
&(\mathrm{P}(E_n \le y))^{2^{-n}} \\
\ge\ & \int_{\frac{\ln 2}{2^{n+1}}}^{y^2-\frac{\ln 2}{2^{n+1}}} G_n(y^2-x) g_n(x)\, dx \\
=\ & 2^{n+2} \int_{\frac{\ln 2}{2^{n+1}}}^{y^2-\frac{\ln 2}{2^{n+1}}} \exp\left\{-2^{n+1}x\right\} - 2\exp\left\{-2^{n+1}y^2\right\} dx \\
=\ & 2^{n+2}\left[-2^{-(n+1)}\exp\left\{-2^{n+1}x\right\}\right]_{\frac{\ln 2}{2^{n+1}}}^{y^2-\frac{\ln 2}{2^{n+1}}} \\
& -2^{n+3}\exp\left\{-2^{n+1}y^2\right\}\left(y^2-2^{-n}\ln 2\right) \\
=\ & 1-\exp\left\{-2^{n+1}y^2\right\}\left(4+2^{n+3}y^2-8\ln 2\right) \\
\ge\ & 1-\exp\left\{-2^{n+1}y^2\right\}\left(2^{n+3}y^2\right).
\end{aligned}$$

The opposite inequality is the following

$$\begin{aligned}
&(\mathrm{P}(E_n \le y))^{2^{-n}} \\
\le\ & \int_0^{2y^2} J_n(2y^2-x) j_n(x)\, dx \\
=\ & 2^{n+1}\int_0^{2y^2} \exp\left\{-2^{n+1}x\right\} - \exp\left\{-2^{n+2}y^2\right\} dx \\
=\ & \left[\exp\left\{-2^{n+1}x\right\}\right]_0^{2y^2} - 2^{n+2}y^2\exp\left\{-2^{n+2}y^2\right\} \\
=\ & 1-\exp\left\{-2^{n+2}y^2\right\}\left(1+2^{n+2}y^2\right).
\end{aligned}$$

To show (3.41) we first introduce a substitution $k = 2^n$. It will be sufficient if

$$\lim_{k\to\infty}\frac{1-\left(1-8ky^2e^{-2ky^2}\right)^k}{8k^2y^2e^{-2ky^2}} = 1.$$

We have

$$\begin{aligned}
\lim_{k\to\infty}\frac{1-\left(1-8ky^2e^{-2ky^2}\right)^k}{8k^2y^2e^{-2ky^2}} &= \lim_{k\to\infty}\frac{1-\exp\left\{k\log\left(1-8ky^2e^{-2ky^2}\right)\right\}}{8k^2y^2e^{-2ky^2}} \\
&= \lim_{k\to\infty}\frac{-k\log\left(1-8ky^2e^{-2ky^2}\right)}{8k^2y^2e^{-2ky^2}}
\end{aligned}$$

$$= \lim_{k\to\infty} \frac{k \sum_{N=1}^{\infty} \left(8ky^2 e^{-2ky^2}\right)^N}{8k^2y^2e^{-2ky^2}} = 1.$$

Again, let $k = 2^n$.

$$\lim_{k\to\infty} \frac{1 - \left(1 - (1+4ky^2)e^{-4ky^2}\right)^k}{4k^2y^2e^{-4ky^2}}$$
$$= \lim_{k\to\infty} \frac{1 - \exp\left\{k \log\left(1 - (1+4ky^2)e^{-4ky^2}\right)\right\}}{4k^2y^2e^{-4ky^2}}$$
$$= \lim_{k\to\infty} \frac{-k \log\left(1 - (1+4ky^2)e^{-4ky^2}\right)}{4k^2y^2e^{-4ky^2}}$$
$$= \lim_{k\to\infty} \frac{k \sum_{N=1}^{\infty} \left((1+4ky^2)e^{-4ky^2}\right)^N}{4k^2y^2e^{-4ky^2}} = 1,$$

which proves the rest of (3.42). □

Theorem 3.8 *Let the dimension $d = 3$ Then*

$$\mathrm{P}(d_H(S_r^n, S_r) \le y) \ge b_n, \tag{3.44}$$

where $b_n \to 1$ as $n \to \infty$ and it holds

$$1 - b_n \simeq 2^{3n+3} y^4 e^{-2^{n+1}y^2}, \qquad n \to \infty. \tag{3.45}$$

Furthermore, for distribution of E_n we have the following estimates

$$a_n \ge \mathrm{P}(E_n \le y) \ge b_n, \tag{3.46}$$

where

$$1 - a_n \simeq 9 \cdot 2^{3n+2} y^4 e^{-3\cdot 2^{n+1}y^2}, \qquad n \to \infty. \tag{3.47}$$

Proof Let G_n and J_n denote the same distribution functions as in Proposition 3.4, g_n and j_n be their density functions respectively. We have

$$(\mathrm{P}(E_n \le y))^{2^{-n}}$$
$$\ge \int_{\frac{\ln 2}{2^{n+1}}}^{y^2 - \frac{\ln 2}{2^n}} \left(1 - e^{-2^{n+1}(y^2-x)}\left(4 + 2^{n+3}(y^2-x) - 8\ln 2\right)\right) 2^{n+2} e^{-2^{n+1}x}\, dx$$
$$= 2^{n+2}\left(-2^{-(n+1)}\right)\left[e^{-2^{n+1}x}\right]_{\frac{\ln 2}{2^{n+1}}}^{y^2 - \frac{\ln 2}{2^n}}$$

$$
\begin{aligned}
& -\left(4+2^{n+3}y^2-8\ln 2\right)2^{n+2}e^{-2^{n+1}y^2}\left(y^2-3\cdot 2^{-(n+1)}\cdot\ln 2\right) \\
& +2^{n+2}e^{-2^{n+1}y^2}2^{n+3}\int\limits_{2^{-(n+1)}\cdot\ln 2}^{y^2-2^{-n}\cdot\ln 2} x\,dx \\
= \; & -2\left(e^{-2^{n+1}y^2+2\ln 2}-e^{-\ln 2}\right) \\
& -e^{-2^{n+1}y^2}\left(2^{n+4}\,y^2+2^{2n+5}\,y^2-2^{n+5}(\ln 2)\,y^2\right) \\
& -e^{-2^{n+1}y^2}\left(-24\ln 2-3\cdot 2^{n+4}(\ln 2)\,y+3\cdot 2^4\ln^2 2\right) \\
& +2^{2n+5}e^{-2^{n+1}y^2}\left(\frac{(y^2-2^{-n}\ln 2)^2}{2}-\frac{(2^{-(n+1)}\ln 2)^2}{2}\right) \\
= \; & 1-e^{-2^{n+1}y^2}\left[8-24\ln 2+36\ln^2 2+(1-3\ln 2)2^{n+4}y^2+2^{2n+4}\,y^4\right].
\end{aligned}
$$

The opposite inequality is the following

$$
\begin{aligned}
& (\mathrm{P}(E_n\le y))^{2^{-n}} \\
\ge \; & \int\limits_0^{3y^2}\left(1-e^{-2^{n+1}(3y^2-x)}\left(1+2^{n+1}(3y^2-x)\right)\right)2^{n+1}e^{-2^{n+1}x}\,dx \\
= \; & \int\limits_0^{3y^2}2^{n+1}e^{-2^{n+1}x}\,dx-2^{n+1}e^{-3\cdot 2^{n+1}y^2}\int\limits_0^{3y^2}\left(1+2^{n+1}(3y^2-x)\right)\,dx \\
= \; & \left[-e^{-2^{n+1}x}\right]_0^{3y^2} \\
& \qquad -2^{n+1}e^{-3\cdot 2^{n+1}y^2}\left(3y^2\left(1+3\cdot 2^{n+1}y^2\right)-2^{n+1}\left[\frac{x^2}{2}\right]_0^{3y^2}\right) \\
= \; & 1-e^{-3\cdot 2^{n+1}y^2}\left(1+3\cdot 2^{n+1}y^2+\left(3\cdot 2^{n+1}y^2\right)^2-9\cdot 2^n y^4\right).
\end{aligned}
$$

Again, to show (3.45) we apply the substitution $k=2^n$. For the sake of brevity we set $C=8-24\ln 2+36\ln^2 2$.

We have as $k\to\infty$

$$
\begin{aligned}
& 1-\left(1-e^{-2ky^2}\left(C+(1-3\ln 2)8ky^2+8k^2y^4\right)\right)^k = \\
= \; & 1-\exp\left\{k\log\left(1-e^{-2ky^2}\left(C+(1-3\ln 2)8ky^2+8k^2y^4\right)\right)\right\} \\
\simeq \; & -k\log\left(1-e^{-2ky^2}\left(C+(1-3\ln 2)8ky^2+8k^2y^4\right)\right) \\
\simeq \; & ke^{-2ky^2}\left(C+(1-3\ln 2)8ky^2+8k^2y^4\right) \\
\simeq \; & 8k^3y^4e^{-2ky^2}.
\end{aligned}
$$

It remains to show (3.47). As $k \to \infty$,

$$\begin{aligned} 1 - \left(1 - e^{-6ky^2}\left(1 + 6ky^2 + \left(6ky^2\right)^2 - 9ky^4\right)\right)^k &= \\ = \quad 1 - \exp\left\{k \log\left(1 - e^{-6ky^2}\left(1 + 6ky^2 + \left(6ky^2\right)^2 - 9ky^4\right)\right)\right\} \\ \simeq \quad -k \log\left(1 - e^{-6ky^2}\left(1 + 6ky^2 + \left(6ky^2\right)^2 - 9ky^4\right)\right) \\ \simeq \quad ke^{-6ky^2}\left(1 + 6ky^2 + \left(6ky^2\right)^2 - 9ky^4\right) \\ \simeq \quad 36k^3y^4e^{-6ky^2}. \end{aligned}$$

□

3.2.4 Application to the convergence of approximated covariogram

The approximations investigated above, namely the easiest one from Proposition 3.3, will be in this section applied to the estimation of the covariogram of the Wiener sausage (for numerical results see Section 4.3.2).

Throughout this section we will denote by $C_{S_r}(h)$, $h \geq 0$ the covariogram of the Wiener sausage $S_{r,1}$ (compare with Section 3.1.4). It is defined as

$$C_{S_r}(h) = \mathrm{E}V_d(S_r \cap (S_r + u \cdot h)),$$

where u is an arbitrary unit vector.

We shall assume for simplicity $\sigma = 1$. For $i = 1, \ldots, d$ we let

$$W_i^n(t),\ t \in [0,1] \tag{3.48}$$

be d independent approximations of Wiener process constructed in the similar way as in Proposition 3.3.

Following the invariance principle of Donsker (Theorem 3.5) it holds

$$\left\{W_i^n(t),\ t \in [0,1]\right\} \xrightarrow{\mathcal{D}} \left\{W_i(t),\ t \in [0,1]\right\}, \qquad n \to \infty, \tag{3.49}$$

$i = 1, \ldots, d.$.

In other words, the approximations $W_i^n(t)$, $i = 1, \ldots, d$ converge in distribution to independent standard Wiener processes W_i on $[0,1]$. It follows

$$\{B^n(t),\ t \in [0,1]\} = \{(W_1^n(t), \ldots, W_d^n(t)),\ t \in [0,1]\} \xrightarrow{\mathcal{D}} \{B(t),\ t \in [0,1]\}.$$

Proposition 3.5 *It holds*

$$C_{S_r^n}(h) \to C_{S_r}(h), \qquad n \to \infty, \qquad h \geq 0. \tag{3.50}$$

First we need an auxiliary result. The assertion of the next Lemma is well known for Wiener process and discrete symmetric random walk. We are giving a proof for the case of symmetric continuous random walk to have the text complete although the technique used inside is very standard.

Lemma 3.5 *Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $Y_i \sim N(0,1)$. Then the inequality*

$$\mathrm{P}\left[\max_{1\le k\le n}|\mathbf{S}_k| \ge x\right] \le 2\,\mathrm{P}[|\mathbf{S}_n|\ge x], \quad x\ge 0 \tag{3.51}$$

holds for $\mathbf{S}_k = Y_1+\cdots+Y_k,\ k=1,\ldots,n$.

Proof: According to strong invariance principle there exists a Wiener process $W(t)$ in $\mathbb{R}$ such that

$$\max_{1\le k\le n}|\mathbf{S}_k - W(k)| \to 0, \quad n\to\infty \tag{3.52}$$

in probability. Given $\epsilon,\delta>0$ there exists n_o such that for $n>n_o$ we have using (3.52) and maximal inequality for Wiener process (see Lemma 2.1):

$$\begin{aligned}
&\mathrm{P}(\max_{1\le k\le n}|\mathbf{S}_k|\ge x) = \\
&= \mathrm{P}(\max_{1\le k\le n}|\mathbf{S}_k|\ge x,\ \max_{1\le k\le n}|\mathbf{S}_k - W(k)|>\delta) \quad + \\
&\quad + \quad \mathrm{P}(\max_{1\le k\le n}|\mathbf{S}_k|\ge x,\ \max_{1\le k\le n}|\mathbf{S}_k - W(k)|\le\delta) \\
&\le \epsilon + \mathrm{P}(\max_{1\le k\le n}|W(k)|\ge x-\delta) \\
&\le \epsilon + \mathrm{P}(\max_{1\le t\le n}|W(t)|\ge x-\delta) \\
&\le \epsilon + 2\,\mathrm{P}(|W(n)|\ge x-\delta) \\
&\le \epsilon + 2\epsilon + 2\,\mathrm{P}(|W(n)|\ge x-\delta,\ \max_{1\le k\le n}|\mathbf{S}_k - W(k)|<\delta) \\
&\le 3\epsilon + 2\,\mathrm{P}(|\mathbf{S}_n|\ge x-2\delta).
\end{aligned}$$

Since ϵ, δ were arbitrary the Lemma is proved. □

Proof of Proposition 3.5: First we show that the mapping $A\mapsto V_d(A\oplus B(o,r))$ is continuous in Hausdorff metric.

Set

$$V_A(r) = V_d(A\oplus B(o,r))$$

and let A, B be two arbitrary compact subsets of $\mathbb{R}^d$ such that $d_H(A,B)<\delta$, $\delta>0$. Then $A\subseteq B\oplus B(o,\delta)$ and $B\subseteq A\oplus B(o,\delta)$ and it holds

$$|V_A(r) - V_B(r)| \le \max_{C\in\{A,B\}} V_C(r+\delta) - V_C(r). \tag{3.53}$$

Applying the co–area formula [7, Section 3.2.34] to the distance function we obtain

$$V_A(r) = V_A(0) + \int_0^r \mathcal{H}^{d-1}\left(\triangle_A^{-1}\{s\}\right) ds, \qquad r > 0. \tag{3.54}$$

Hence, the function V_A is absolutely continuous and applying this result to inequality (3.53) we arrive at continuity of the mapping $A \mapsto V_d(A \oplus B(o,r))$.

Consequently, we have by the mapping theorem for convergence in distribution [2, Theorem 2.7]

$$V_d(S_r^n) \xrightarrow{\mathcal{D}} V_d(S_r), \qquad n \to \infty.$$

Our aim is to show that

$$\mathrm{E}V_d(S_r^n) \to \mathrm{E}V_d(S_r), \qquad n \to \infty. \tag{3.55}$$

This convergence holds if $V_d(S_r^n)$, $n = 1, 2, \ldots$ are uniformly integrable random variables ([2, Theorem 3.5]). A sufficient condition for the uniform integrability is that

$$\sup_{n \in \mathbb{N}} \mathrm{E}(V_d(S_r^n))^2 < \infty. \tag{3.56}$$

We can write

$$V_d(S_r^n) \le \omega_d \left(r + \max_{t \in [0,1]} |B^n(t)|\right)^d \le \omega_d \left(r + \sum_{i=1}^d \max_{t \in [0,1]} |W_i^n(t)|\right)^d.$$

Since the distributions of $|W_i^n(t)|$, $i = 1, \ldots, d$ are identical we get

$$\begin{aligned}\mathrm{E}(V_d(S_r^n))^2 &\le \omega_d^2 \mathrm{E}\left[r + \sum_{i=1}^d \max_{1 \le k \le n} \frac{|\mathbf{S}_k^i|}{\sqrt{n}}\right]^{2d} \\ &= \omega_d^2 \sum_{\substack{k_0,\ldots,k_d \ge 0: \\ k_0 + \ldots + k_d = 2d}} \frac{(d+1)!}{k_0! \ldots k_d!} r^{k_0} \prod_{i=1}^d \mathrm{E} \max_{1 \le k \le n} \left(\frac{|\mathbf{S}_k|}{\sqrt{n}}\right)^{k_i},\end{aligned}$$

where $\mathbf{S}_k \stackrel{\mathcal{D}}{=} \mathbf{S}_k^i \sim N(0,k)$. Use the assertion of Lemma 3.5 to get the upper bound

$$\mathrm{E} \max_{1 \le k \le n} \left(\frac{|\mathbf{S}_k|}{\sqrt{n}}\right)^m = \frac{1}{n^{m/2}} \int_0^\infty \mathrm{P}\left[\max_{1 \le k \le n} |\mathbf{S}_k| \ge x^{1/m}\right] dx$$

$$\leq \frac{2}{n^{m/2}} \int_0^\infty \mathrm{P}[|\mathbf{S}_n| \geq x^{1/m}]\, dx = \frac{2}{n^{m/2}} \int_0^\infty m y^{m-1}\, \mathrm{P}[|\mathbf{S}_n| \geq y]\, dy$$
$$= \frac{2}{n^{m/2}} \mathrm{E}|\mathbf{S}_n|^m \leq m!, \qquad m \in \mathbb{N},$$

where the latter inequality follows from the fact that $\mathbf{S}_n \sim N(0, n)$. Hence, condition (3.56) is verified, and the convergence (3.55) of mean volumes holds.

If we started with the approximation $S_r^n \cup (S_r^n + h \cdot u)$ of the union of Wiener sausage and its shift we would derive using the same approach,

$$\mathrm{E}V_d(S_r^n \cup (S_r^n + h \cdot u)) \to \mathrm{E}V_d(S_r \cup (S_r + h \cdot u)) \quad n \to \infty.$$

Since the covariogram can be expressed as

$$C_{S_r}(h) = 2\mathrm{E}V_d(S_r) - \mathrm{E}V_d(S_r \cup (S_r + h \cdot u))$$

the assertion is proved. □

Remark: According to invariance principle of Donsker (Theorem 3.5), Proposition 3.5 holds (with slight changes in the proof) for any choice of symmetric random walk $\mathbf{S}_n^i$ in (3.48). We choose in Section 4.3.2 the random variables Y_n^i standard normally distributed to reduce the error of approximation and to increase the speed of the algorithm.

Chapter 4

Boolean model of Wiener sausages

Probably the most frequently used model among stationary random sets is the Boolean model, cf. e.g. [33]. It arises naturally as a system of randomly scattered particles in Euclidean space of arbitrary dimension provided that the distributions of location are independent of each other. Boolean model is a generalization of Poisson point process and it can be derived as a so–called germ–grain model. This means that the process of germs forms a stationary Poisson point process and grains (random closed sets) are assigned to germs. In this context the term distribution of a typical grain is used.

In the case when the typical grain is distributed as the Wiener sausage we speak about the Boolean model of Wiener sausages. Its properties are studied in the present work [4] and most of the results mentioned in this chapter can be found therein.

The Boolean model of Wiener sausages is a random structure that appears as a model to a network of sensors. These sensors are moving according

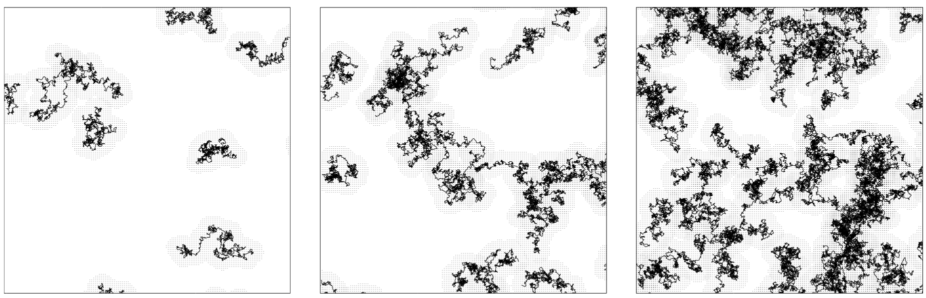

Figure 4.1: Three realizations of Boolean models of Wiener sausages for $t = 10$ and $r = 1$. The intensity λ is chosen to fit volume fractions 0.25, 0.5 and 0.75, respectively.

to Brownian path. Their initial locations in the area are chosen at random. Each sensor can detect the target within the range $r > 0$. The total detection area up to time t forms a Boolean model of Wiener sausages, see e.g. [15].

We start with a definition of the Boolean model of Wiener sausages. Assume $\varphi = \{x_n\}_{n=1}^{\infty}$ to be a stationary Poisson point process in $\mathbb{R}^d$ with intensity $\lambda > 0$ (see e.g. [33] for more details). Consider an independent identically distributed collection of Wiener sausages $\{(S_{r,t})_n\}_{n=1}^{\infty}$ (each starting at the origin) which are independent of the process φ. Introduce the *Boolean model* Ξ *of Wiener sausages* by putting

$$\Xi = \bigcup_{n=1}^{\infty} \big(x_n + (S_{r,t})_n\big). \tag{4.1}$$

If the local finiteness property

$$\mathrm{E}\lambda^d(B \oplus \check{S}_{r,t}) < \infty \tag{4.2}$$

(where $\check{C} = \{-c : c \in C\}$) is fulfilled for any compact set B then Ξ is a random closed set (see [33]).

Since $\mathrm{E}V_d(S_{r,t} \oplus B(o,r')) = \mathrm{E}V_d(S_{r+r',t}) < \infty$ for all $r' > 0$, (4.2) is fulfilled.

The isotropy of $S_{r,t}$ and stationarity of φ imply that Ξ is stationary and isotropic. It means that the probability distribution of Ξ is invariant with respect to rigid motions.

Alternatively, Ξ can be introduced directly as a union of particles lying in a Poisson process Φ of Wiener sausages (Poisson process on the space of compact sets).

$$\Xi = \cup\{x + S_{r,t} : x + S_{r,t} \in \Phi\}.$$

Assume Λ is the intensity measure of Φ. The assumption (4.2) then takes the well known form

$$\Lambda(\mathcal{K}_B) < \infty,$$

from which it can be more easily seen that only finitely many grains intersect a compact set B.

Moreover, it is well known that the intensity Λ can be disintegrated so that

$$\Lambda(dF) = \lambda\,\lambda^d(dx)\Lambda_0(dF_0), \tag{4.3}$$

where $F = x + F_0$ and we came back to the intensity λ of the process φ and Λ_0 the distribution of typical grain. In this particular case, Λ_0 corresponds to the distribution of the Wiener sausage.

4.1 Capacity functional, volume fraction

The *capacity functional* $T_\Xi(C) = P(\Xi \cap C \neq \emptyset)$, $C \subset \mathbb{R}^d$ compact, plays the same role in the theory of random sets as the distribution function of random variables in the classical probability theory. Namely, it defines the distribution law of Ξ uniquely. It is known that the capacity functional of the Boolean model is given by

$$T_\Xi(C) = 1 - e^{-\lambda \mathrm{E}\, V_d(S_{r,t} \oplus \check{C})} \tag{4.4}$$

for all compact C.

Following [30], we can compute the expected volume of $S_{r,t} \oplus \check{C}$ using Fubini's theorem as

$$\mathrm{E}\, V_d(S_{r,t} \oplus \check{C}) = \int\limits_{\mathbb{R}^d} \mathrm{P}(x \in S_{r,t} \oplus \check{C}) dx = \int\limits_{\mathbb{R}^d} \mathrm{P}\left(\tau^x_{C \oplus B(o,r)} \leq t\right) dx\,, \tag{4.5}$$

where τ^x_A is introduced in Chapter 2. Set $u(t,x) = \mathrm{P}\left(\tau^x_{C \oplus B(o,r)} < t\right)$, $x \in \mathbb{R}^d$, $t \geq 0$. Following Theorem 2.2, $u(t,x)$ is the unique solution to the following heat conduction problem:

$$\begin{array}{ll} \frac{\partial u}{\partial t} = \frac{\sigma^2}{2} \triangle u, & t > 0,\ x \in \mathbb{R}^d \setminus (C \oplus B(o,r)), \\ u(0,x) = 0, & x \in \mathbb{R}^d \setminus (C \oplus B(o,r)), \\ u(t,x) = 1, & t \geq 0,\ x \in \partial(C \oplus B(o,r)). \end{array} \tag{4.6}$$

Here any boundary point $x \in \partial(C \oplus B(o,r))$ is regular, since there always exists a cone Q with vertex at x such that $Q \cap (C \oplus B(o,r))$ has positive measure. The regularity of x follows from the fact that Q is recurrent for Brownian motion starting at x (for more details see the proof of [24, Lemma 4.1]).

For an arbitrary compact set C the problem (4.6) has to be solved by numerical methods. In particular case, if $C = \{o\}$ an analytical solution is given by the expected volume of the Wiener sausage, see (3.5).

The *volume fraction* p_Ξ of the Boolean model Ξ is defined by

$$p_\Xi = \mathrm{P}(o \in \Xi) = \mathrm{E}\, V_d(\Xi \cap [0,1]^d).$$

This is the one–point coverage probability of Ξ. It follows from relations $p_\Xi = T_\Xi(\{o\})$, (4.4) and (3.5) that

$$p_\Xi = 1 - \exp\left\{ - \lambda \left(\omega_d r^d + \frac{d(d-2)}{2} \omega_d\, \sigma^2 r^{d-2} t + \frac{4d\, \omega_d\, r^d}{\pi^2} \int\limits_0^\infty \frac{1 - e^{-\frac{\sigma^2 y^2 t}{2r^2}}}{y^3 \left(J^2_\nu(y) + Y^2_\nu(y)\right)}\, dy \right) \right\}. \tag{4.7}$$

4.2 Contact distributions

For an introduction to problems concerning contact distributions for random sets we refer to [11].

Let $C \subset \mathbb{R}^d$ be a compact set, $o \in C$. In this context C is usually called *structuring element*. Taking C fixed we can define the C–distance (the distance relative to C) of a point x to a set $A \subset \mathbb{R}^d$ by

$$d_C(x, A) := \inf\{\rho \geq 0 : (x + \rho C) \cap A \neq \emptyset\}.$$

For the case when the set on the right side is empty (o is the boundary point of C) it is taken $d_C(x, A) = \infty$. Clearly, $d_C(x, A) \leq \rho$ if and only if x is contained in the *generalized outer parallel set* $A \oplus \rho\check{C}$.

If the C–distance $d_C(x, A)$ of a point $x \notin A$ is attained at a unique point y (i.e. y lies in the boundary ∂A) then we define the *contact direction vector* $u_C(x, A)$ by

$$u_C(x, A) := \frac{y - x}{d_C(x, A)}.$$

The points $x \in \mathbb{R}^d \setminus A$ for which $d_C(x, A)$ is attained in more than one point form the *exoskeleton* $\mathrm{exo}_C(A)$ of A.

Let Ξ be a stationary random closed set such that its volume fraction obeys $p_\Xi < 1$ (we exclude the trivial case $\Xi = \mathbb{R}^d$ since the contact distribution is not well defined then). The *contact distribution function* of Ξ is introduced conditionally on the event $\{o \notin \Xi\}$ as

$$H_C(\rho) = P(d_C(o, \Xi) \leq \rho | o \notin \Xi), \qquad \rho > 0,$$

provided that the probability of the complement event $\{o \in \Xi\}$ is positive when $p_\Xi > 0$. It can be shown that

$$H_C(\rho) = P(\Xi \cap \rho C \neq \emptyset | o \notin \Xi) = \frac{T_\Xi(\rho C) - T_\Xi(\{o\})}{1 - T_\Xi(\{o\})},$$

cf. [33]. If Ξ is the Boolean model of Wiener sausages then $T_\Xi(\{o\})$ is given by relation (4.7). The value of $T_\Xi(\rho C)$ can be assessed numerically if C is a general compact set. However, if C is a unit ball then the spherical contact distribution function $H_s(\rho) := H_{B(o,1)}(\rho)$ can be given explicitly.

Theorem 4.1 *Let Ξ be a Boolean model of Wiener sausages with intensity $\lambda > 0$ and with spherical contact distribution function $H_s(\rho)$. Then it holds*

$$H_s(\rho) = 1 - e^{-\lambda M(d,\sigma^2,r,\rho,t)}, \tag{4.8}$$

where

$$M(d, \sigma^2, r, \rho, t) =$$

$$
\begin{aligned}
&= \mathrm{E}V_d(S_{r+\rho,t}) - \mathrm{E}V_d(S_{r,t}) \\
&= \omega_d\big((r+\rho)^d - r^d\big) + \frac{d(d-2)}{2}\omega_d\,\sigma^2\big((r+\rho)^{d-2} - r^{d-2}\big)t \\
&\quad + \frac{4d\,\omega_d}{\pi^2}(r+\rho)^d \int_0^\infty \frac{1 - e^{-\frac{\sigma^2 y^2 t}{2(r+\rho)^2}}}{y^3\,(J_\nu^2(y) + Y_\nu^2(y))}\,dy \\
&\quad - \frac{4d\,\omega_d}{\pi^2} r^d \int_0^\infty \frac{1 - e^{-\frac{\sigma^2 y^2 t}{2r^2}}}{y^3\,(J_\nu^2(y) + Y_\nu^2(y))}\,dy\,.
\end{aligned}
$$

Proof: According to (4.4) we have

$$T_\Xi(\rho B(o,1)) = T_\Xi(B(o,\rho)) = 1 - e^{-\lambda \mathrm{E}\,V_d(S_{r,t}\oplus B(o,\rho))} = 1 - e^{-\lambda \mathrm{E}\,V_d(S_{r+\rho,t})}$$

which together with relation (3.5) yields the formula (4.8). □

Following [11], it can be shown that the contact distribution function H_C of Ξ is absolutely continuous if o is an interior point of C. Moreover, the density $H_s'(\rho)$ can be given explicitly in the special case of spherical contact distribution function as

$$H_s'(\rho) = \frac{\mathrm{E}\mathcal{H}^{d-1}(\partial(\Xi \oplus \rho B(o,1)) \cap [0,1]^d)}{1 - p_\Xi}\,. \tag{4.9}$$

Since $\Xi \oplus \rho B(o,1)$ is again a Boolean model of Wiener sausages with dilation radius $(r+\rho)$ and the same total time t combining (4.9) with the result (4.20) given later in section 4.4 we obtain

$$H_s'(\rho) = e^{-\lambda(\mathrm{E}V_d(S_{r+\rho,t}) - \mathrm{E}V_d(S_{r,t}))}\,\lambda\,\mathrm{E}\mathcal{H}^{d-1}(\partial S_{r+\rho,t}), \tag{4.10}$$

which can be also assessed by differentiation of (4.8).

4.3 Covariance function

The *covariance function* of the isotropic Boolean model Ξ can be introduced by

$$C_\Xi(h) = \mathrm{P}(o, h\cdot u \in \Xi), \tag{4.11}$$

where u is an arbitrary unit vector in $\mathbb{R}^d$ and $h \geq 0$; see [33] and [20]. It follows from (4.4) and the formula of total probability that

$$C_\Xi(h) = 2p_\Xi - T_\Xi(\{o, h\cdot u\}) = 2p_\Xi - 1 + e^{-\lambda \mathrm{E}\,V_d(S_{r,t}\cup(S_{r,t}+u\cdot h))}\,. \tag{4.12}$$

Moreover, from relations (4.4) and (4.5) we get that the mean volume

$$\mathrm{E}\,V_d(S_{r,t} \cup (S_{r,t} + u\cdot h)) = \int_{\mathbb{R}^d} \mathrm{P}\left(\tau^x_{B(o,r)\cup B(h,r)} \leq t\right)\,dx \tag{4.13}$$

in formula (4.12) can be computed by integrating the solution of the heat conduction problem (4.6) where

$$C \oplus B(r,o) = \{o, h \cdot u\} \oplus B(o,r) = B(o,r) \cup B(h,r)\,.$$

The mean volume in (4.13) is related to the *covariogram*

$$C_{S_{r,t}}(h) = \mathrm{E}V_d(S_{r,t} \cap (S_{r,t} + u \cdot h))$$

of the Wiener sausage $S_{r,t}$ by

$$\mathrm{E}V_d(S_{r,t} \cup (S_{r,t} + h \cdot u)) = 2\mathrm{E}V_d(S_{r,t}) - \mathrm{E}\,V_d(S_{r,t} \cap (S_{r,t} + u \cdot h)) \qquad (4.14)$$

where $\mathrm{E}V_d(S_{r,t})$ is given in (3.5).

As far as it is known up to now, it is difficult to find an explicit analytical solution to (4.6) on the complement of the union of two spheres. Hence, numerical methods can be used to solve (4.6) and get a graph of the covariance function C_Ξ. In Section 4.3.1, we perform this numerical analysis by means of finite elements method. Alternatively, a large number of Monte Carlo simulations of Wiener sausages can lead to precise estimates of C_Ξ as it is done in Section 4.3.2. Both attitudes to this problem are studied in the work [4].

4.3.1 Numerical solution of the heat conduction problem and approximation formulae

For the cases of $d = 2, 3$ the finite element method can be used to compute an approximate solution u to the problem (4.6). Using rotational and axial symmetry in 3D resp. axial symmetries in 2D to reduce the complexity of the problem the solution is obtained in a more efficient way (for the better insight to this problem we refer to [4]). The results that are presented were calculated using the software package FEMLAB/COMSOL [9].

In the following an approximation $\widetilde{C_\Xi}$ for the covariance function C_Ξ is given. For a fixed volume fraction p_Ξ the covariance function is approximated

$$C_\Xi(h) \approx \widetilde{C_\Xi} := 2p_\Xi - 1 + (1 - p_\Xi)^{\kappa(h,t)}, \qquad (4.15)$$

where $\kappa(h,t)$ is given by (4.18) and (4.19) below. Let $d = 2, 3$ and set

$$\begin{aligned} A_r(h,t) &:= \mathrm{E}V_d(S_{r,t} \cup (S_{r,t} + h \cdot u)) \\ &= V_d(B(o,r) \cup B(h \cdot u, r)) + \int\limits_{\mathbb{R}^d \setminus (B(o,r) \cup B(h,r))} u(t,x)\,dx, \end{aligned}$$

where $u(t,x)$ denotes the solution of (4.6) for some given $h \geq 0$, $r > 0$ and $C = \{o, h \cdot u\}$. Substituting the intensity λ from (4.7) to (4.12) one gets

$$C_\Xi(h) = 2p_\Xi - 1 + (1 - p_\Xi)^{A_r(h,t)/A_r(0,t)}.$$

For $t = 0$, analytic calculations of $V_d(B(o,r) \cup B(h \cdot u, r))$ give respectively

$$\frac{A_r(h,0)}{A_r(0,0)} = \kappa(h,0)$$
$$:= \begin{cases} \frac{2}{\pi}\left(\pi - \arccos\left(\frac{h}{2r}\right) + \frac{h}{2r}\sqrt{1 - \frac{h}{2r}}\right), & \text{if } h \leq 2r, \\ 2, & \text{otherwise,} \end{cases} \tag{4.16}$$

for $d = 2$ and

$$\frac{A_r(h,0)}{A_r(0,0)} = \kappa(h,0) := \begin{cases} \frac{1}{2}\left(\frac{h}{2r}\right)^3 - 2\left(\frac{h}{2r}\right)^2 + \frac{5}{2}\frac{h}{2r} + 1, & \text{if } h \leq 2r, \\ 2, & \text{otherwise,} \end{cases} \tag{4.17}$$

for $d = 3$.

For $t > 0$ and $d = 2, 3$ a closed formula for $A_r(h,t)/A_r(0,t)$ is not known yet. The following approximation $\kappa(h, \nu(t))$ was introduced in [4, Section 4.2].

$$\frac{A_r(h,t)}{A_r(0,t)} \approx \kappa(h,0)$$
$$= \begin{cases} \frac{1}{2}\left(\frac{h}{\nu(t)}\right)^3 - 3\left(\frac{h}{\nu(t)}\right)^2 + 3\frac{h}{\nu(t)} + 1, & \text{if } h \leq \nu(t), \\ 2, & \text{otherwise,} \end{cases} \tag{4.18}$$

with

$$\nu(t) = \begin{cases} 3.124\, t^{0.3925} + 2.794, & d = 2, \\ 3.744\, t^{0.2182} + 1.454, & d = 3. \end{cases} \tag{4.19}$$

4.3.2 Estimation by Monte–Carlo simulations

There are two ways leading to estimates of the covariance C_Ξ from simulations. The first way is to use the definition (4.11) and simulate many realizations of the Boolean model Ξ in a finite observation window estimating the two-point coverage probability from each of them and then averaging over all realizations. The second way is to simulate one Wiener sausage many times and estimate its covariogram. By expressions (4.12) and (4.14), this would lead to an estimate of the covariance C_Ξ. We would prefer the second approach since it leads to more precise results.

In order to do so, simulate N independent copies $\{(S^n_{r,t})_k\}_{k=1}^N$ of the approximated Wiener sausage $S^n_{r,t}$ for sufficiently large approximation parameter n and compute the volume of intersection $S^n_{r,t} \cap (S^n_{r,t} + h \cdot u)$ by averaging over N realizations.

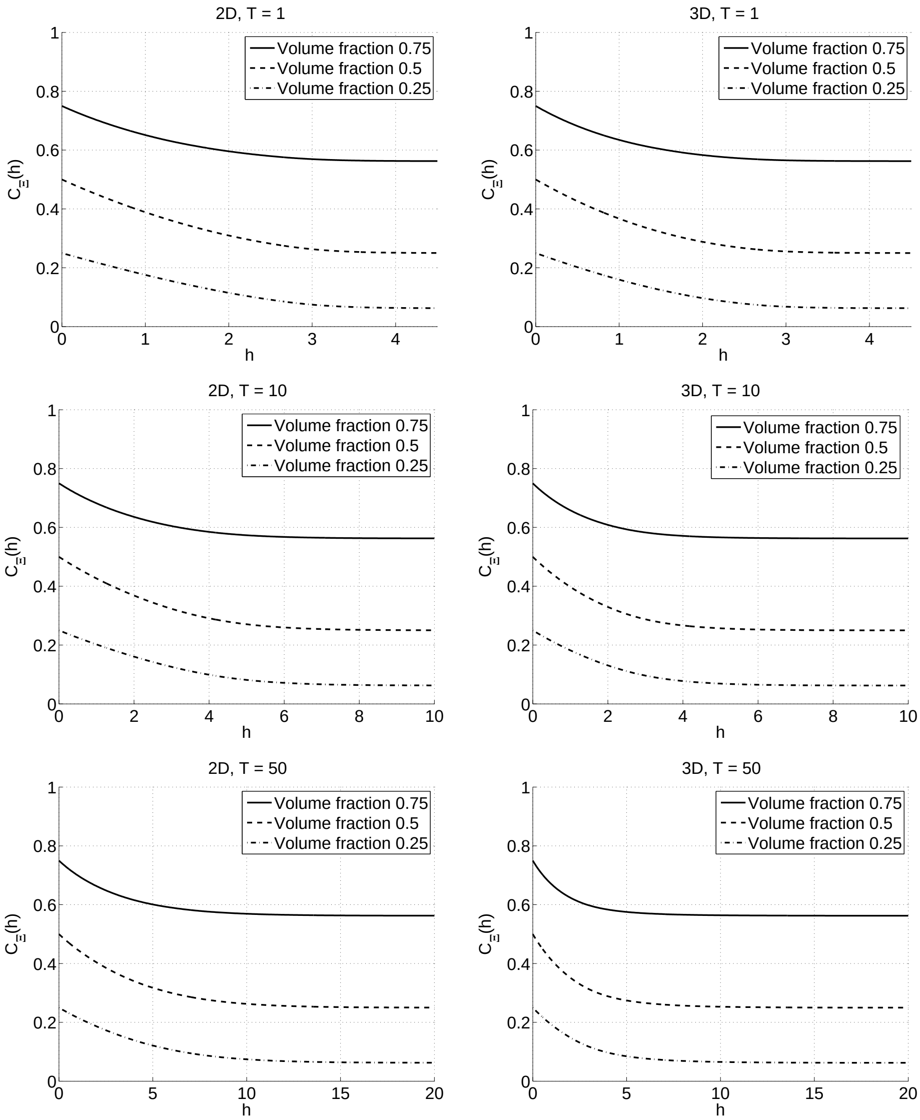

Figure 4.2: Estimated covariance functions (by Monte–Carlo simulations) of the planar and spatial Boolean model of Wiener sausages with $r = 1$ and total times $t = 1$, $t = 10$ and $t = 50$. In each case, the intensity λ is chosen to fit volume fractions 0.25, 0.5 and 0.75, respectively.

The results of Section 3.2.4 (namely the relation (3.50)) together with the law of large numbers allow us to estimate the covariance function of the Boolean model Ξ from sufficiently many approximations $S^n_{r,t}$.

In Figure 4.2, such estimates are given in two and three dimensions. In each case, 10000 approximations of Wiener sausages with $r = 1$ and $t = 1$, $t = 10$ and $t = 50$, respectively were simulated. To evaluate the volume of $S^n_{r,t}$ and its covariogram numerically, realizations of Wiener sausages have to be discretized on a quadratic (cubic) grid in $\mathbb{R}^2$ ($\mathbb{R}^3$), and the number of pixels (voxels) belonging to $S^n_{r,t}$ has to be counted. Hence, besides the error of the approximation of the Wiener sausage, the discretization error occurs.

Given the total runtime t, the maximal shift distance h_{max} for which the covariogram $C_{S_{r,t}}$ was computed is given by

$$h_{max} = 2 \cdot F^{-1}_{0,t}(0.99),$$

where $F^{-1}_{0,t}$ is the quantile function of the normal distribution $N(0,t)$. This value h_{max} yields a good empirical upper bound for the range of dependence of the covariance function C_Ξ. It means that $C_\Xi(h) \approx p^2_\Xi$ is approximately constant for $h > h_{max}$.

4.4 Specific surface area

The specific surface area S_Ξ is defined as the mean surface area of Ξ per unit volume, i.e.

$$S_\Xi = \frac{\mathrm{E}\mathcal{H}^{d-1}(\partial\Xi \cap B)}{\lambda^d(B)},$$

where B is an arbitrary bounded borel set such that $\lambda^d(B) > 0$. It is well defined if the boundary of Ξ is smooth enough. Since the boundary $\partial S_{r,t}$ is almost surely $(d-1)$–dimensional Lipschitz manifold when $d = 2, 3$ (see [8]) and $(\mathcal{H}^{d-1}, d-1)$–rectifiable for almost all r in higher dimensions we can state due to a result of Wieacker the following Theorem.

Theorem 4.2 *For the dimensions $d = 2, 3$ the specific surface area S_Ξ is well defined and it is equal to*

$$S_\Xi = \lambda\, \mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t})\, e^{-\mathrm{E}V_d(S_{r,t})}, \tag{4.20}$$

where λ is the intensity of Ξ, $\mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t})$ resp. $\mathrm{E}V_d(S_{r,t})$ is the mean surface area resp. the mean volume of the Wiener sausage $S_{r,t}$.

When $d \geq 4$, (4.20) holds for almost all radii r.

Proof: For the notation used in the proof see the introduction to Chapter 4. Introduce a Boolean model $\widetilde{\Xi}$ of Wiener sausage boundaries

$$\widetilde{\Xi} = \bigcup_{n=1}^{\infty} x_n + \partial(S_{r,t})_n.$$

Those are according to [24, Corollary 4.1] $(d-1)$–dimensional Lipschitz manifolds when $d = 2, 3$.

For higher dimensions, we use the co–area theorem [7, 3.2.22]. Taking $f(x) = \operatorname{dist}(x, S_{0,t})$ the assumption given there assures that $\partial S_{r,t}$ is $(\mathcal{H}^{d-1}, d-1)$–rectifiable set for almost all r.

Therefore, together with (4.2) all assumptions of [36, Theorem 8] are fulfilled and hence

$$\mathrm{E}\mathcal{H}^{d-1}(\widetilde{\Xi} \cap B) = \int_{\mathcal{F}_K} \mathcal{H}^{d-1}(F \cap B)\widetilde{\Lambda}(dF),$$

where K is a compact subset such that $B \subset K$ and $\widetilde{\Lambda}$ is the intensity measure of the underlying Poisson process of $\widetilde{\Xi}$. Following (4.3) we can rewrite

$$\begin{aligned}
\int_{\mathcal{F}_K} \mathcal{H}^{d-1}(F \cap B)\widetilde{\Lambda}(dF) &= \lambda\, \mathrm{E}_{F_0} \int_{\mathbb{R}^d} \mathcal{H}^{d-1}(x + F_0 \cap B)\lambda^d(dx) \\
&= \lambda\, \mathrm{E}_{F_0} \int_{\mathbb{R}^d} \mathcal{H}^{d-1}(F_0 \cap B - x)\lambda^d(dx) \\
&= \lambda\, \mathrm{E}_{F_0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{I}_{F_0}(y)\, \mathrm{I}_{(B-x)}(y)\mathcal{H}^{d-1}(dy)\lambda^d(dx) \\
&= \lambda\, \mathrm{E}_{F_0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{I}_{F_0}(y)\, \mathrm{I}_{(B-y)}(x)\mathcal{H}^{d-1}(dy)\lambda^d(dx),
\end{aligned}$$

where E_{F_0} denotes the expectation with a respect to distribution of typical grain, i.e. boundary of Wiener sausage.

Set $\mu_{F_0} = \mathcal{H}^{d-1}|_{F_0}$. Interchanging integrals in the last equality yields

$$\begin{aligned}
\int_{\mathcal{F}_K} \mathcal{H}^{d-1}(F \cap B)\widetilde{\Lambda}(dF) &= \lambda\, \mathrm{E}_{F_0} \int_{\mathbb{R}^d} \mathrm{I}_{F_0}(y)\lambda^d(B-y)\mu_{F_0}(dy) \\
&= \lambda\, \lambda^d(B)\, \mathrm{E}_{F_0}\mathcal{H}^{d-1}(F_0).
\end{aligned}$$

Therefore rewriting the last equation we obtain

$$\mathrm{E}\mathcal{H}^{d-1}(\widetilde{\Xi} \cap B) = \lambda\, \mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,t})\lambda^d(B). \tag{4.21}$$

The only difference from (4.20) is in the multiplier $e^{-\mathrm{E}V_d(S_{r,t})}$ which corresponds to volume fraction of Ξ ($e^{-\mathrm{E}V_d(S_{r,t})} = 1 - p_\Xi$). The specific area of Ξ differs from that for $\widetilde{\Xi}$ only by those parts of boundaries that are covered by other grain in Ξ. Heuristically, the probability that a point is covered by Ξ is p_Ξ and the independence property of Poisson process yields the formula.

More formally, the theory of Palm distributions is used to passage from (4.21) to (4.20).

Denote by Q_F the Palm distribution of Φ at F. We have

$$\begin{aligned}\mathrm{E}\mathcal{H}^{d-1}(\widetilde{\Xi} \cap B) &= \mathrm{E}\int_{\mathcal{K}}\int_{B} \mathrm{I}_{\partial F}(x)\mathcal{H}^{d-1}(dx)\Phi(dF) \\ &= \int_{\mathcal{K}}\int_{\mathcal{N}(\mathcal{K})}\int_{B} \mathrm{I}_{\partial F}(x)\mathcal{H}^{d-1}(dx)Q_F(d\Phi)\Lambda(dF) \\ &= \int_{\mathcal{K}}\int_{B} \mathrm{I}_{\partial F}(x)\mathcal{H}^{d-1}(dx)\Lambda(dF), \end{aligned} \tag{4.22}$$

where $\mathcal{N}(\mathcal{K})$ is the space of locally finite integer valued measures on $\mathcal{K}$, see e.g. [13, Chapter 12].

On the contrary, for the Boolean model of whole Wiener sausages we have

$$\begin{aligned}&\mathrm{E}\mathcal{H}^{d-1}(\partial\Xi \cap B) \\ &= \mathrm{E}\int_{\mathcal{K}}\int_{B} \mathrm{I}_{\partial F}(x)\,\mathrm{I}\{x \notin G,\ \forall G \in \Phi - \delta_F\}\mathcal{H}^{d-1}(dx)\Phi(dF) \\ &= \int_{\mathcal{K}}\int_{\mathcal{N}(\mathcal{K})}\int_{B} \mathrm{I}_{\partial F}(x)\,\mathrm{I}\{x \notin G,\ \forall G \in \Phi - \delta_F\}\mathcal{H}^{d-1}(dx)Q_F(d\Phi)\Lambda(dF) \\ &= \int_{\mathcal{K}}\int_{\mathcal{N}(\mathcal{K})}\int_{B} \mathrm{I}_{\partial F}(x)\,\mathrm{I}\{x \notin G,\ \forall G \in \Phi\}\mathcal{H}^{d-1}(dx)Q(d\Phi)\Lambda(dF) \\ &= \int_{\mathcal{K}}\int_{B} \mathrm{I}_{\partial F}(x)\,\mathrm{P}(x \notin \Xi)\mathcal{H}^{d-1}(dx)\Lambda(dF),\end{aligned}$$

where we used Slivnyak's theorem (cf. e.g. [33]). Since Ξ is stationary, $\mathrm{P}(x \notin \Xi) = 1 - p_\Xi$ does not depend on x and hence can be written in front of the integral. We finally obtain

$$\mathrm{E}\mathcal{H}^{d-1}(\partial\Xi \cap B) = (1 - p_\Xi)\int_{\mathcal{K}}\int_{B} \mathrm{I}_{\partial F}(x)\mathcal{H}^{d-1}(dx)\Lambda(dF). \tag{4.23}$$

The proof is completed by comparison of (4.22) with (4.23). □

Chapter 5

Wiener sausage with random time

Up to now the Wiener sausage was defined for given deterministic parameters $r > 0$ (the radius of dilation) and $t > 0$ (the total time for which the Brownian particle moves).

A straightforward generalization of the Wiener sausage is to take one of the parameters r, t random. From the introduction to Chapter 3, especially using Lemma 3.1, we know that the Wiener sausage of any dilation radius r can be reconstructed from $S_{1,t}$ and vice versa, $S_{r,t}$ of any total time $t > 0$ can be reconstructed from $S_{r,1}$. It is therefore enough to consider for example only total time t to be random and dilation radius r to be deterministic since we can apply the argument mentioned in the previous paragraph.

Two important cases appear depending on the fact whether the distribution of the total time is dependent or independent on the distribution of underlying Brownian motion.

5.1 Independent time

Let $f(t)$ be the probability density function of a random variable $T \geq 0$ which is independent of $(B(t),\ t \geq 0)$ a d–dimensional Brownian motion with $B(0) = o$. We assume $\int_0^\infty t f(t) dt = T_0 < \infty$, $\int_0^\infty t^2 f(t) dt < \infty$, i.e. finite expectation and variance of the random variable T.

We define the Wiener sausage $S_{r,T}$ with random total time T as

$$S_{r,T} = \{B(t) :\ 0 \leq t \leq T\} \oplus B(o, r).$$

Again, it is a random closed set in the Matheron sense (c.f. [19]) since for any Borel set B the event

$$[S_{r,T} \cap B \neq \emptyset] \quad = \quad [\tau_{B \oplus B(o,r)} \leq T] = \bigcap_{q \in \mathbb{Q}} [\tau_{B \oplus B(o,r)} < T + q]$$

$$= \bigcap_{p\in\mathbb{Q}} \bigcap_{q\in\mathbb{Q}} [\tau_{B\oplus B(o,r)} < p < T + q]$$
$$= \bigcap_{p\in\mathbb{Q}} \bigcap_{q\in\mathbb{Q}} \left([\tau_{B\oplus B(o,r)} < p] \cap [T > p - q]\right)$$

is measurable.

Compactness (a.s.) of $S_{r,T}$ is verified by the argument given below that the mean volume $\mathrm{E}V_d(S_{r,T})$ is finite. Since for any elementary event $\omega \in \Omega$ $S_{r,T(\omega)}(\omega)$ is an r–neighborhood of a Wiener trajectory, it cannot happen that its volume is finite while the set itself is unbounded.

Proposition 5.1 *The mean volume* $\mathrm{E}V_d(S_{r,T})$ *is finite and equals*

$$\mathrm{E}V_d(S_{r,T}) = \omega_d r^d + \mathrm{I}\{d \geq 3\}\frac{d(d-2)}{2}\omega_d T_0 r^{d-2} + \frac{4d\omega_d r^d}{\pi^2}\int_0^\infty \frac{1 - L_T\left(\frac{x^2}{2r^2}\right)}{x^3(J_\nu^2(x) + Y_\nu^2(x))}\,dx, \quad (5.1)$$

where J_ν *and* Y_ν *are Bessel functions of the first and second kind of order* $\nu = \frac{d-2}{2}$ *and* $L_T(\cdot)$ *is the Laplace transform of the random time* T.

In case of dimensions $d = 1, 3$, (5.1) *can be simplified to*

$$\mathrm{E}V_1(S_{r,T}) = 2r + \frac{2\sqrt{2}T_0}{\sqrt{\pi}}, \quad (5.2)$$

$$\mathrm{E}V_3(S_{r,T}) = \frac{4}{3}\pi r^3 + 4r^2\sqrt{2\pi}\int_0^\infty \sqrt{t}f(t)\,dt + 2\pi r T_0. \quad (5.3)$$

Proof: Conditioning on the independent random time T we get

$$\begin{aligned}\mathrm{E}V_d(S_{r,T}) &= \mathrm{E}\,\mathrm{E}\,[V_d(S_{r,T})|\,T = t] \\ &= \int_0^\infty \mathrm{E}V_d(S_{r,t})f(t)dt \\ &= \omega_d r^d + \mathrm{I}\{d \geq 3\}\frac{d(d-2)}{2}\omega_d T_0 r^{d-2} \\ &\quad + \frac{4d\omega_d r^d}{\pi^2}\int_0^\infty \frac{1 - \int_0^\infty e^{-\frac{x^2 t}{2r^2}} f(t)dt}{x^3(J_\nu^2(x) + Y_\nu^2(x))}\,dx,\end{aligned}$$

where we applied equation (3.5) and Fubini theorem. The rest of the assertion is an easy consequence of Theorem 3.1. □

The same approach can be used to derive the expected surface area $\mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,T})$.

Proposition 5.2 *For almost all radii $r > 0$,*

$$\begin{aligned} &\mathrm{E}\mathcal{H}^{d-1}(\partial S_{r,T}) \qquad (5.4)\\ &= d\omega_d r^{d-1}\left(1 + (d-2)^2\frac{T_0}{2r^2} + \frac{4d}{\pi^2}\int_0^\infty \frac{\int_0^\infty \varphi_d(x^2\frac{t}{2r^2})f(t)\,dt}{x^3(J_\nu^2(x)+Y_\nu^2(x))}\,dx\right), \end{aligned}$$

where $\varphi_d(y) = 1 - e^{-y} - 2ye^{-y}/d$ and $\nu = (d-2)/2$. Furthermore, equation (5.4) holds for all $r > 0$ when $d = 2$ or 3. Especially we have

$$\mathrm{E}\mathcal{H}^2(\partial S_{r,T}) = 4\pi r^2 + 8r\sqrt{2\pi}\int_0^\infty \sqrt{t}f(t)\,dt + 2\pi T_0. \qquad (5.5)$$

5.2 Path–dependent random time

In this section we define a special case of Wiener sausage where the underlying Brownian motion is terminated at the time it reaches the boundary of ball $B(o,R)$ with given fixed radius $R > 0$. In other words, it is terminated at the first exit time off the boundary of the ball $B(o,R)$.

We use the same notation as before

$$\tau_{B(o,R)^C} = \inf\left\{t:\ B(t) \in B(o,R)^C\right\}$$

for the first exit time off a ball $B(o,R)$, where $(B(t), t \geq 0)$ is a d–dimensional Brownian motion starting at the origin.

For any $r \geq 0$, $R > 0$ we set

$$S_r^R = \left\{B(t):\ 0 \leq t \leq \tau_{B(o,R)^C}\right\} \oplus B(o,r).$$

S_r^R is a random compact set. Again its measurability can be seen from $[S_r^R \cap b] = [\tau_{B(o,r)} \leq \tau_{B(o,R)^C}]$. Both random times in the expression are random variables. Hence, using the same approach as in the last section we obtain the measurability of S_r^R. Its boundedness is obvious, since $S_r^R \subseteq B(o, r+R)$.

The argument given above also guarantees the finiteness of the expected volume of S_r^R. It can be assessed by the equations

$$\begin{aligned} \mathrm{E}V_d(S_r^R) &= \mathrm{E}\int_{\mathbb{R}^d} \mathrm{I}(x \in S_r^R)\,dx \\ &= \int_{\mathbb{R}^d} \mathrm{P}(x \in S_r^R)\,dx \end{aligned}$$

$$= \int_{\mathbb{R}^d} \mathrm{P}\left(\tau^x_{B(o,r)} \le \tau^x_{B(x,R)^C}\right) dx,$$

where $\tau^x_{(\cdot)}$ is the first hitting time for the Brownian motion started at x. We used the symmetry of Brownian motion and started B from x instead of the origin.

Since S_r^R is obviously bounded the finiteness of its (mean) surface area follows.

Proposition 5.3

$$\mathcal{H}^{d-1}(\partial S_r^R) \le \frac{d}{r}\,\omega_d\,(R+r)^d \tag{5.6}$$

Proof: Set $X = \{B(t) : 0 \le t \le \tau_{B(o,R)^C}\}$. The assertion is a combination of two arguments. Since X is obviously bounded, according to [25] (Corollary 2 and inequalities given above) it holds

$$\mathcal{H}^{d-1}(\partial S_r^R) \le (V_X)'_-(r)$$

for any $r > 0$, where $(V_X)'_-(r)$ is the left hand side derivative of $V_X(r) = V_d(X \oplus B(o,r))$.

We will slightly modify the proof of [24, Lemma 4.4], where the inequality

$$V'_X(r) \le \left(\frac{r}{s}\right)^{d-1} V'_X(s) \tag{5.7}$$

is given. We will obtain the same result for the left hand side derivative $(V_X)'_-(r)$ which exists for any $r > 0$ (see [31]). Following [16], it holds

$$V_X(\lambda\, a) - V_X(\lambda\, b) \le \lambda^d (V_X(a) - V_X(b))$$

for any $0 \le b \le a$ and $\lambda \ge 1$. Thus

$$(V_X)'_-(\lambda\, a) = \lim_{b \to a_-} \frac{V_X(\lambda\, a) - V_X(\lambda\, b)}{\lambda(a-b)} \le$$
$$\le \lambda^{d-1} \lim_{b \to a_-} \frac{V_X(a) - V_X(b)}{a-b} = \lambda^{d-1}(V_X)'_-(a).$$

The derivative $V'_X(r)$ exists for almost all $r > 0$ (see (3.54)). Therefore, for almost all $r > 0$ it holds $(V_X)'_-(r) = V'_X(r)$ which finally gives

$$(R+r)^d \omega_d \ge V_X(r) - V_X(0) = \int_0^r V'_X(s)ds = \int_0^r (V_X)'_-(s)ds$$
$$\ge (V_X)'_-(r) \int_0^r \left(\frac{s}{r}\right)^{d-1} ds = \frac{r}{d}(V_X)'_-(r).$$

□

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