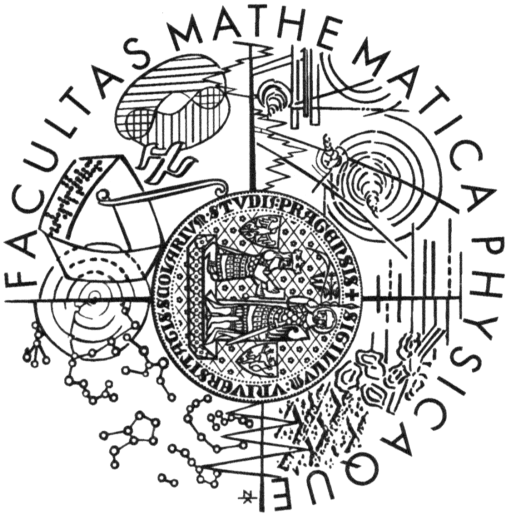
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DISSERTATION

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Model for stochastic claims reserving based on individual loss information and incorporating non-proportional reinsurance

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I hereby declare that this submission is my own work and that, to the best of my knowledge, it contains no material previously published or written by another person, except where due acknowledgment has been made in the text.

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List of Abbreviations

CAS	Casualty Actuarial Society
CEA	Comité Européen des Assurances
CEIOPS	Committee of European Insurance and Occupational Pensions Supervisors
CoC	Cost of capital
CRO Forum	Chief Risk Officer Forum
FSA	Financial Services Authority
GDV	German Solvency model
GLM	General linear model
IAA	International Actuarial Association
IAIS	International Association of Insurance Supervisors
IASB	International Accounting Standards Board
IBNR	Incurred But Not Reported
MSR	Minimum Solvency Requirement
ODP	Over dispersed Poisson
P&C	Property & Casualty, non-life
P&L	Profit and Loss
QIS	Quantitative Impact Study
S&P	Standard & Poor's
SCR	Solvency Capital Requirement
SL	Stop Loss (reinsurance)
SST	Swiss Solvency Test
XL	Excess of Loss (reinsurance)

List of Symbols used in the Model

h	index used for sums and products (general)
i	index denoting accident years
j	index denoting development years
k	index denoting individual losses
l	index denoting development years with first payments
a_s	priority of an XL treaty for accident year s
$\bar{a}_s$	priority of a SL treaty for accident year s
$\bar{a}_{s,j}^{O}$	priority of a SL treaty for accident year s less closed losses at development year j
$A_{s,j,r-1}$	help expression defined in (3.41)
b_j	development factors for large paid claim numbers
$\hat{b}_j$	estimate of parameter b_j
c_j	multiplicative development factors in the model for annual (small) losses
$\hat{c}_j$	estimate of parameter c_j
d_j^l, d_j	development factors for individual claim amounts
$\hat{d}_j$	estimate of parameter d_j
$d_j^{(2)}$	coefficient defined in (3.33)
$D_{i,j}$	cumulative development factors for individual claim amounts, defined in (3.43)
$D_{j,i-1}^{(2)}$	cumulative development factors of second order defined in (B.2)
$E_{s,j+1}$	error terms in the multiplicative model for annual (small) losses
$\tilde{E}_{s,j,k}^l$	error terms of development of individual claim amounts
Φ	distribution function of a standardized normal distribution $N(0, 1)$
$H_{s,j,r}^{(i)}$	help expressions defined on page 89
$\tilde{H}_{s,j,r}^{(i)}$	help expression defined in (B.8)
$H_{s,j,r}^{O(i)}$	help expressions defined on page 87
$\tilde{H}_{s,j,r}^{O(i)}$	help expression defined in (B.5)
$I_{s,j}$	all available information on losses from accident year s available at the end of development year j
$J_{s,j,k}^l$	indicator whether an individual loss $Y_{s,j,k}^l$ is open at the end of development year j
$K_{j,r-1}^{a}(Y)$	help expression defined in (4.11)
$K_{j,r-1}^{a(2)}(Y)$	help expression defined in (D.2)

$\tilde{\kappa}_{0,j}$	linear coefficient in the expression for variance of loss numbers, defined in (3.9)
$\kappa_{j,r-1}$	linear coefficient in the expression for conditional variance of ultimate loss numbers, defined in (3.13)
L_s	limit of an XL treaty for accident year s
λ_s	mean number of losses in accident year s having first payment in the development year 0
$\hat{\lambda}$	estimate of parameter λ_s (assumed independent of s)
m_s	expected value of the first payment of a loss from accident year s
$\hat{m}_s$	estimate of parameter m_s
$m_s^{(2)}$	second moment of the first payment of a loss from accident year s
$N_{s,j}$	number of paid losses from accident year s having first payment in development year $\leq j$
$\tilde{N}_{s,j}$	number of paid losses from accident year s having first payment in development year j
q_{j+1}	probability that a claim will be closed in development year $j+1$
$Q_{l,j}$	probability that a claim with the first payment in dev. year l is open at the end of dev. year j
$\tilde{Q}^A_{s,j,r}$	help expression defined in (3.47)
$\tilde{Q}^{\beta}_{s,j,r}$	help expression defined in (3.44)
$\tilde{Q}^{\beta(2)}_{s,j,r}$	help expression defined in (B.1)
$\tilde{Q}^D_{s,j,r}$	help expression defined in (3.46)
r	number of the last development year in the model
r_X	reinsurance effect, defined in (4.3)
$r_X(a)$	reinsurance effect for an XL reinsurance with priority a, defined in (4.4)
$r_X^{(n)}$	reinsurance effect of n-th power, defined in (4.5)
$r_X^{(n)}(a)$	reinsurance effect of n-th power for an XL reinsurance with priority a, defined in (4.6)
R_t^1, R_t^2	help expressions in computation of net loss reserve after an XL reinsurance
Res_t	reserve at the end of the year t
Res_t^{SL}	net reserve at the end of the year t after a SL reinsurance
Res_t^{XL}	net reserve at the end of the year t after an XL reinsurance
s_s	standard deviation of the first payment of a loss from accident year s
$\hat{s}_s$	estimate of parameter s_s
$v^a_{t-j,j,l-1,r}$	help expression defined in (4.17)
$v^{a(2)}_{t-j,j,l-1,r}$	help expression defined in (D.3)
$W^l_{t-j,j,r,k}$	help variable (multiplicative dev. of individual losses) defined in (4.12)
$W^l_{t-j,l,r,k}$	help variable (multiplicative dev. of individual losses) defined in (4.14)
$X_{s,j}$	cumulative paid annual losses from accident year s in development year j
$X^C_{s,j}$	cumulative paid annual closed losses from accident year s in dev. year j
$X^O_{s,j}$	cumulative paid annual open losses from accident year s in dev. year j
$X^{sm}_{s,j}$	cumulative paid annual small losses from accident year s in dev. year j
$X^{(2)}_{s,j}$	sum of squared paid ind. claims from accident year s in dev. year j
$X^{C(2)}_{s,j}$	sum of squared closed paid ind. claims from accident year s in dev. year j

$X_{s,j}^{O(2)}$	sum of squared open paid ind. claims from accident year s in dev. year j
$\tilde{X}_{s,j}^{SL}$	net cumulative paid annual losses after a SL reinsurance
$\tilde{X}_{s,j}^{XL}$	net cumulative paid annual losses after an XL reinsurance
$Y_{s,j,k}^{l}$	individual paid claim amount from accident year s, development year j, having first payment in dev. year l
$\tilde{Y}_{s,j,k}^{l}$	net paid individual losses from accident year s, development year j, having first payment in dev. year l

Chapter 1

Introduction

1.1 Problem setting and the aim of the work

The new solvency framework (Solvency II) that is currently being developed defines principles how capital adequacy and solvency in insurance companies should be determined. In particular, it states that reinsurance and risk transfer should be adequately considered (see [20], principle 11). As reinsurance is one of the most effective instruments to manage risk and hence required risk capital and solvency of a company it is very important that a company has a process in place that manages reinsurance and risk transfer effectively. For this an appropriate consideration of reinsurance in actuarial computation of IBNR reserves and in determination of underwriting risk capital (premium and reserve risk) is indispensable.

However, most of the models used in practice to compute actuarial IBNR reserves consider reinsurance in a very simple way that can even lead, especially for non-proportional reinsurance, to distorted results. Similarly, most of the currently used solvency models don't consider reinsurance adequately as well. In practice, large companies use often some sort of internal model that determines non-life premium risk capital by means of Monte Carlo simulation. However, there is no standard how to create such a model and often these models don't cope with reinsurance in reserve risk in a proper way.

The aim of this work is to close this gap and to develop a stochastic model for computing reserves in a non-life insurance company that can cope especially with non-proportional reinsurance in a more proper way. In particular, the work aims to

1. develop a stochastic non-life reserving model that is
   - based on individual losses, and
   - incorporates reinsurance recoveries, especially of non-proportional reinsurance treaties, in an adequate manner, and hence
2. enables to develop a stochastic model that can be used to
   - compute or simulate required risk capital for underwriting risk (premium and reserve risk),
   - model the influence of especially non-proportional reinsurance on the underwriting risk capital, and hence
   - optimize reinsurance program with respect to net underwriting result and required net risk capital.

1.2 Structure of the work

This work is divided in 7 chapters and is structured in the following way.

Chapter 2 gives a brief introduction on the solvency frameworks and the most important stochastic reserving models. It is shown that the currently used models can't be generalized to cope with reinsurance in a straightforward way.

In Chapter 3 we develop a stochastic model based on individual loss information that can be later expanded to cope with non-proportional reinsurance. First, an intuitive consideration is given that is based on the model used in [29], then models for loss numbers and severities are developed and the moments computed. These models are then put together. Finally, the formulas for loss reserve with and without risk margin are derived. At the end of the chapter we discuss how the parameters of the model can be estimated.

Chapter 4 deals with reinsurance. First, main functions and types of reinsurance are shortly introduced. Then the model from Section 3 is adjusted to cope with the most usual non-proportional reinsurance types (excess of loss and stop loss). For this the concept of reinsurance relief effect is introduced that enables to write closed-form expressions and at the end to get an approximation of the results that would have otherwise to be simulated. Finally a framework is developed that shows how to use the results in order to optimize reinsurance with respect to net underwriting result and required net risk capital.

In Chapter 5 we give an example of how the model can be used in practice. We use anonymized data on individual losses from general liability insurance from the years 1997 – 2006. As general liability is a long-tail line of business, we compute tail factors as well. The model can be enlarged to incorporate tail factors in a straightforward way. We are aware that the development factors for loss severities can in reality depend on the loss severity itself. That's why we split the losses into small and large losses. For small losses we use the simpler model from [29] based on annual losses and for large losses we use the model developed in sections 3 and 4. First, the model is parametrized, then loss reserve distribution is simulated and net loss reserves for excess of loss treaties with various priorities are computed.

Chapter 6 summarizes the most important results of the work.

Some of the rather complex expressions and derivations, especially for higher moments, are given in Appendix. Throughout the work (if not specified else) the notion gross and net are understood with respect reinsurance.

Chapter 2

Stochastic reserving and solvency models in a non-life insurance company

In this chapter we give a brief introduction of the solvency framework and the most important stochastic reserving models.

2.1 Solvency framework and reinsurance

2.1.1 Development of solvency frameworks

The new Solvency II framework for insurers' capital requirements that is now being developed should remove some lacks that have been identified in the rules of the current Solvency I framework. One of the main drawbacks of the current solvency system is the lack of risk sensitivity and inappropriate consideration of risk mitigation techniques. The new Solvency II system will be principle based, leaving room for the development of internal models for determination of risk capital. Within the scope of the first phase of the new Solvency II process several studies and working papers have been published. A good overview of the recent status is given in [3] and [4].

The main principles that the new solvency regime should fulfill have been formulated in [20]. The methods that the supervisor might use to assess the current financial position of an insurer have been discussed by International Actuarial Association (IAA) in [18]. In particular, the three pillar approach, the appropriate time horizon, confidence level, types of risks, risk measures and dependencies have been dealt with. The IAA working party believes that the solvency should be determined as a 'total balance sheet requirement'. Further, it's recommended to use a time horizon of one year, a high level of confidence, such as 99%, as well as the present value of the remaining future obligations (liabilities) determined e.g. as the best estimate plus margin with a moderate level of confidence such as 75%.

Under the solvency II requirement not only Solvency Capital Requirement (SCR) should be determined, but also the Minimal Solvency Requirement (MSR). In the Second and Third

Quantitative Impact Study (QIS2 and QIS3) the insurers have been invited to use also their internal models within the scope of SCR. For more information we refer to [5] and [6].

Beside the European solvency framework several other national solvency models have been developed. A good overview and comparison of various national solvency frameworks is given in [32], [12] or Chapter 10 in [22]. In particular, the German solvency model is described in [15] and the description of the Swiss solvency framework can be found in [34] and [14].

Solvency models of various national legislative frameworks can be divided into static and dynamic ones.

Static solvency models. Static models are based on static data from the P&L account and the balance sheet. Simple factor models apply a small number of factors and are not very risk-sensitive. An example of such a model is Solvency I. Models that take better account of the risk being measured use a great number of factors that should describe the risk of the insurer, typical factors are e.g. standard deviations or coefficients of variation of loss amounts of various lines of business. Among factor models that are risk-based e.g. S&P model, German GDV model, British model for non-life insurance (FSA), American risk based capital model, Canadian or Australian model can be mentioned.

Factor models usually take account of reinsurance through an application of risk factors directly to the values of premium or reserves net of reinsurance. Many of the models also incorporate in some way the credit risk from reinsurance (e.g. by considering the rating of the reinsurance companies or maximum exposure to an individual reinsurer).

Dynamic solvency models. Dynamic solvency models quantify risk in a prospective way. They can be divided into principle based models and scenario based models. The latter ones quantify the impact of various scenarios on the net asset value taking into account the discounted financial cash flows. An example of such a model is the Swiss solvency test (SST), which is based on principles and incorporates scenarios as well. Dynamic models can cope with reinsurance in a much accurate way than factor models.

Internal models. Some of the solvency jurisdictions (in particular Solvency II in Europe, Swiss Solvency Test, Individual Capital Assessment in United Kingdom, C3 Phase II in USA, Canada (Segregated Funds) and Australia) are considering or using internal models. The following definition of an internal model is provided by Jewel (see [17]):

> 'A model is a set of verifiable mathematical relationships or logical procedures which is used to represent observed measurable real-world phenomena, to communicate alternate hypotheses about the cause of the phenomena and to predict future behavior of the phenomena for the purpose of decision-making.'

An excellent guidance to risk professionals and supervisors regarding internal models for insurer risk assessment and capital requirements has been developed by IAA in [17]. A good summary showing some pitfalls connected with internal models is given in [24].

Measurement of technical liabilities. In Solvency II framework it's stated that loss reserves[1] should be computed as best estimate plus risk margin. There are two main approaches to compute risk margin — cost of capital approach and percentile approach. For a general discussion on the measurement of technical liabilities and risk margin we refer to [19] or Chapter 4 of [22].

The cost of capital margin is defined as the cost of future required (marginal regulatory) capital to run off the existing liabilities. An explanation of the underlying ideas and rational of the cost of capital approach for risk margins is given in [7] and [13]. Examples of how to calculate the Swiss Solvency Test cost of capital margin can be found in [21]. This approach is recommended by CEA and CRO forum and is used by Swiss Solvency Test.

Percentile approach defines risk margin as e.g. the 75% percentile of the discounted ultimate future payments less the best estimate. This approach is recommended by European Commission in [19].

2.1.2 Solvency and reinsurance

Within the Solvency I framework reinsurance is accounted for in a very simple way. In the formula for minimum solvency requirement computed from premium as well as from losses a coefficient of net proportion is applied. This coefficient is limited to the minimum value of 50%. This can be inappropriate for reinsurance programs with high proportion of the reinsurer. Moreover, for non-proportional treaties designed to cover high losses or extreme events the mitigating effect of reinsurance on risk capital can be understated.

In the Solvency II framework the International Actuarial Association (IAA) mentions in connection with reinsurance the following principle [18]:

> 'The solvency assessment method should recognize appropriately the impact of various risk transfer or risk sharing mechanisms used by the insurer.'

The International Association of Insurance Supervisors (IAIS) has published in connection with reinsurance the following principles [20]:

> 'Any allowance for reinsurance in a capital adequacy and solvency regime should consider the effectiveness of the risk transfer and make allowance for the likely security of the reinsurance counter-party.'

A deeper discussion of the consideration of reinsurance is also given in Chapter 6 of [22].

In general, we can summarize these principles in the following points:

- Reinsurance and risk transfer should be appropriately accounted for when determining solvency capital requirement.

[1] In this work we use the notion 'loss reserve' for the non-life technical provisions for outstanding claims. These are defined in Article 28 of EU Insurance Accounts Directive as 'the total estimated ultimated cost to an insurance undertaking of settling all claims arising from events which have occurred up to the end of the financial year, whether reported or not, less amount already paid in respect of such claims'.

- When considering risk transfer, its efficiency and the counterparty credit risk have to be accounted for.

The consultancy company EMB states in its newsletter [8] to Solvency II, to the topic of computing the SCR in the QIS3:

> 'Rückversicherung — ein Hauptsteuerungshebel für das Risiko wird nicht adäquat berücksichtigt: Jegliche nicht-proportionale Rückversicherung wird nicht richtig abgebildet. Somit kann man auch nicht analysieren, ob die Rückversicherung als Kapitalersatz effizient für das Unternehmen ist.'

In other words, the reinsurance is in QIS3 still not adequately accounted for. Consequently, it cannot be analyzed, whether the reinsurance is efficient as a substitution of capital for the company.

In this work we develop a model that can overcome this lack and considers (non-proportional) reinsurance in a more proper way. In the following subsection we give a brief overview of stochastic reserving models to show that most of these models cannot be adapted in a straightforward way to cope explicitly with non-proportional reinsurance.

2.2 Stochastic reserving models

A good introduction to the topic of reserving is given e.g. in [27]. [2] summarizes the current state of knowledge regarding the estimation of the distribution of future payments and gives an overview of the terminology. The paper gives an overview of the currently used models with the emphasis given on the criteria for selecting an appropriate model and model reasonability checks and a practical illustration. Moreover, five areas for future research are mentioned, one of them being 'making use of individual claim detail'. An overview of the typical models used to measure the expected unpaid loss liability is also given in [1], with emphasis given on the papers from the CAS Forum 1994, and in the introduction part of [9] or [10].

The log-linear models and especially the generalized linear models (GLM) are introduced in detail in the papers by England and Verrall - in [9] a brief overview of the existing stochastic claims reserving models and GLM framework with a comparison of various predictive structures is given, [11] discusses bootstrapping and Bayesian techniques in the context of GLM, [10] introduces a wide range of stochastic reserving models with the extension to consider parametric curves and smoothing models for the shape of the development run-off, compares the models and considers the framework of bootstrapping and dynamic financial analysis.

The generalization of chain ladder by Mack to give the standard errors of the reserve distribution is introduced e.g. in [28]. Mack develops distribution-free formula for standard error of chain ladder reserve estimates and compares the results to the results of other parametric methods. Murphy gives in [31] an analysis of link ratio estimators and frames the models in a least square regression models. As a by-product he derives the formulas for variances of point estimates of loss reserves. A general description of bootstrap technique in claim reserving is given in [33], where bootstrap is applied on data for different claim reserving methods. Hayne

discusses in [16] the application of collective risk theory in stochastic reserving. In a later paper Hayne extends his model to include parameter risk. Venter discusses in [37] a model that incorporates methods to reduce modeled error variance.

In the following text we give a brief overview of the most frequently used stochastic reserving models. In the most of them the generalization of the model to explicitly cope with reinsurance (through other method than one of the mentioned in previous section) is not easily possible.

2.2.1 Log-linear models

Log-linear models focus on the logarithm of the incremental claim amounts C_{ij}, $Y_{ij} = \log(C_{ij})$, and define the lognormal class of models

$$Y_{ij} = m_{ij} + \epsilon_{ij}, \quad \epsilon_{ij} \sim i.i.d.\ N(0, \sigma^2).$$

From this model it follows that $Y_{ij} \sim N(m_{i,j}, \sigma^2)$. For the mean term m_{ij} the following linear model is defined,

$$m_{ij} = c + \alpha_i + \beta_j.$$

Alternatively, a more general model can be defined (Hoerl curve),

$$m_{ij} = c + \alpha_i + \beta_i \log(j) + \gamma_i j,$$

which enables to model the development of losses behind the last column of the development triangle. Sometimes, a mixture of the two models is used. To avoid over-parametrization, an additional constraint, e.g. $\alpha_1 = \beta_1 = 0$, has to be set. The predictive values produced by this model are close, bu not identical, to those from the simple chain ladder model. Parameters of the predictor structure of the mean term are estimated by maximum likelihood method. In this case, as the data are normally distributed, it is equivalent to minimizing the residual sum of squares. The main limitation of these models results from the logarithmic transform, as the incremental claim amounts have to be positive. In case of incurred losses this assumption won't be fulfilled.

2.2.2 Wright's model

A very interesting alternative is the model of Wright. This model doesn't pose the assumption of a lognormal distribution any more and the systematic and random components of the underlying model for the data are based on a risk theoretic model of the claims generating process. The model uses a compound distribution assumption for incremental payments. From the standard risk theory we have

$$\mathrm{E}C_{ij} = \mathrm{E}N_{ij}\mathrm{E}X_{ij},$$

and

$$\mathrm{Var}(C_{ij}) = \mathrm{E}N_{ij}\mathrm{Var}(X_{ij}) + (\mathrm{E}X_{ij})^2\mathrm{Var}(N_{ij}).$$

Additionally, the model for $\mathrm{E}N_{ij}$ and $\mathrm{E}X_{ij}$ and the respective relationship between the mean and the variance is specified. Wright assumes a Poisson distribution for claim numbers and defines

$$\mathrm{E}N_{ij} = e_i a_j \kappa_i j^{A_i} e^{-b_i j},$$

and

$$\mathrm{Var}[N_{ij}] = \mathrm{E}N_{ij},$$

where e_i is a measure of exposure, a_j is a known adjustment term, b_i are factors describing run-off and κ_i and A_i are parameters of the model.

For the incremental claims amounts X_{ij}, Wright considers a distribution of Gamma type with

$$\mathrm{E}X_{ij} = e^{\delta(i+j)}kj^{\lambda},$$

and

$$\mathrm{Var}X_{ij} = (\mathrm{E}X_{ij})^2 v,$$

where the term $e^{\delta(i+j)}$ models possible inflation and k, v and λ are parameters. Combining the equations we get for the expected value of incremental loss amounts

$$\mathrm{E}C_{ij} = m_{ij} = e_i a_j \kappa_i j^{A_i} e^{-b_i j} e^{\delta(i+j)} kj^{\lambda},$$

and for the variance

$$\mathrm{Var}C_{ij} = (1+v)kj^{\lambda}e^{\delta(i+j)}\mathrm{E}C_{ij}.$$

Wright showed that the model can be re-parametrized to represent a generalized linear model. An adaptation of the model to model non-proportional reinsurance is not easily possible, because the model is based on incremental claim amounts and non-proportional reinsurance typically operates on cumulative individual or annual claims from a particular accident year.

2.2.3 Generalized linear models (GLM) and Over-dispersed Poisson models (ODP)

The distributional assumptions of GLM are usually expressed in terms of the first two moments for each 'unit' u of a random variable X,

$$\mathrm{E}X_u = m_u \quad \text{and} \quad \mathrm{Var}X_u = \frac{\phi V(m_u)}{w_u}, \tag{2.1}$$

where ϕ denotes a scale parameter, $V(m_u)$ is the so called variance function (a function of the mean) and w_u are weights (often set to 1). The choice of distribution dictates the values of ϕ and $V(m_u)$.

The ODP model assumes, in particular, that the incremental claims, C_{ij}, are distributed as independent over-dispersed Poisson random variables, with mean and variance

$$\mathrm{E}C_{ij} = m_{ij} \quad \text{and} \quad \mathrm{Var}(C_{ij}) = \phi m_{ij}.$$

The predictive structure for the mean m_{ij} is defined as

$$\log(m_{ij}) = c + \alpha_i + \beta_j,$$

with an additional constraint, e.g. $\alpha_1 = \beta_1 = 0$. Parameters are estimated with the maximum likelihood method. This definition results in forecast values that are consistent with the chain ladder method. In contrast to log-linear models the single incremental claims are allowed to be

negative, however, the expected value has still to be positive. In practice the ODP models are applied using the so called bootstrapping algorithm. First, the so called Pearson residuals are computed as

$$r_{ij} = \frac{C_{ij} - \hat{m}_{ij}}{\sqrt{\hat{\phi}\hat{m}_{ij}}},$$

where $\hat{m}_{ij}$ and $\hat{\phi}$ are the estimates of m_{ij} and ϕ. Then a large amount of pseudo-values of these residuals is simulated. Their distribution results from the assumption on the distribution of C_{ij}. The pseudo-data are computed from the simulated Pearson residuals using the inverse formula

$$C_{ij}^B = r_{ij}^B \sqrt{\hat{\phi}\hat{m}_{ij}} + \hat{m}_{ij},$$

where r_{ij}^B are the Pearson residuals simulated by the bootstrap. The model used to obtain the residuals can be fitted to each triangle of pseudo-data to obtain new fitted values $\hat{m}_{ij}^B$. As the forecast values are the same as in the chain ladder model, some short-cuts can be made and the implementation can be made in a spreadsheet. Bootstrap forecasts $\tilde{C}_{ij} = \hat{m}_{ij}^B$ don't contain process error. To add the process error, forecast values, C_{ij}^*, can be simulated from an ODP distribution with mean $\hat{m}_{ij}^B$ and variance $\hat{\phi}\hat{m}_{ij}^B$.

2.2.4 Mack's model

If we need to allow for negative incremental claims, which is typical for incurred losses, we can use the model of Thomas Mack. Mack's model is defined as a recursive model based on cumulative claims D_{ij} with the response mean a variance defined as

$$\mathrm{E}[D_{ij}|D_{ij-1}] = \lambda_j D_{ij-1}, \quad \mathrm{Var}[D_{ij}|D_{ij-1}] = \sigma_j^2 D_{ij-1}, \quad j \geq 2.$$

Equivalently, the model can be defined by means of the development factors, $f_{ij} = D_{ij}/D_{ij-1}$, as

$$\mathrm{E}[f_{ij}|D_{ij-1}] = \lambda_j, \quad \mathrm{Var}[f_{ij}|D_{ij-1}] = \frac{\sigma_j^2}{D_{ij-1}}, \quad j \geq 2.$$

The model assumes, that the cumulative claims at time j are conditionally independent of the cumulative claims at time $j-1$. The model is defined as distribution-free, when using bootstrapping, however, additional assumption on the distribution of D_{ij} has to be made. The model for the development factors can be interpreted as a GLM model defined by (2.1) with $X_u = f_{ij}, m_u = \lambda_u, w_u = D_{i,j-1}, V(m_u) = 1$ and non-constant scale parameters $\phi_j = \sigma_j^2$. Pearson residuals for the bootstrapping procedure are defined as

$$r_{ij} = \frac{\sqrt{w_{ij}}(f_{ij} - \hat{\lambda}_j)}{\hat{\sigma}_j},$$

giving pseudo-data

$$f_{ij}^B = r_{ij}^B \frac{\hat{\sigma}_j}{\sqrt{w_{ij}}} + \hat{\lambda}_j,$$

where r_{ij}^B are the Pearson residuals generated by the bootstrap and $\hat{\lambda}_j$ and $\hat{\sigma}_j$ are the estimates of λ_j and σ_j. The model used to obtain the residuals can be fitted to each triangle of pseudo-data to obtain new fitted development factors $\tilde{\lambda_j}$. Again, due to the equivalency of the predictions to the ones of the chain ladder method, some short-cuts in the implementation can be maid. The forecasts are then obtained recursively from

$$D_{i,j}^* | D_{i,j-1}^* \sim N(\tilde{\lambda}_l D_{i,j-1}^*, \hat{\sigma}_j^2 D_{i,j-1}^*).$$

2.2.5 Multiplicative model based on cumulative annual claims

In the following paragraph we briefly describe model for estimating prudence in claims provisions and capital adequacy based on annual losses which has been introduced in [29]. This model is based on the triangle of cumulative paid losses (Δ),

$$\begin{array}{llll} X_{t-r,0} & X_{t-r,1} & \ldots & X_{t-r,r} \\ \ldots & \ldots & & \\ X_{t-1,0} & X_{t-1,1} & & \\ X_{t,0}. & & & \end{array}$$

We assume that all losses are fully developed after the development year r. According to the assumption of chain ladder method,

$$\mathrm{E}\left[X_{s,j+1} \middle| X_{s,0}, \ldots, X_{s,j}\right] = X_{s,j} c_j,$$

a multiplicative model is defined,

$$X_{s,j+1} = X_{s,j}\left(1 + (c_j - 1)E_{s,j+1}\right), \tag{2.2}$$

where $E_{s,j}$ are independent random variables with identical second and third moment for fixed j. Moreover, they are independent of $\{X_{s,j-1}\}$, $j = 1, 2, \ldots, r$ and satisfy $\mathrm{E}E_{s,j} = 1$, for $s = t - r, \cdots, t$, and $j = 1, 2, \ldots, r$. The best estimate of the undiscounted loss reserve at time t can be expressed as

$$\sum_{j=0}^{r-1} X_{t-j,j}(c_j \cdots c_{r-1} - 1). \tag{2.3}$$

IASB requires that the best estimate should be enhanced by risk margin. It is assumed that risk margin is computed with a confidence level of 75%. Then the undiscounted loss reserve including risk margin has the form

$$R_t = \sum_{j=0}^{r-1} \mathrm{E}\left[X_{t-j,\infty} - X_{t-j,j} \middle| \Delta\right] + z_{0.75} \sqrt{\mathrm{Var}\left(\sum_j (X_{t-j,\infty} - X_{t-j,j} \middle| \Delta\right)}, \tag{2.4}$$

where $z_{0.75}$ is e.g. quantile of normal distribution. If we instead define risk margin as the margin that market participants would require for bearing risk (third part of the current exit value according to IASB) we interpret the parameter $z_{0.75}$ simply as a parameter expressing the level of prudence, which can be calibrated according to the market.

In [29] the factors c_j are estimated by chain ladder method

$$\hat{c}_j = \frac{1}{r-j} \sum_{s=t-r}^{t-j} \frac{X_{s,j+1}}{X_{s,j}}.$$

In the section on parameter estimates we develop for annual small losses another estimate of c_j through expressing the model in the form of linear regression. A similar method has been used in [31].

If we denote

B_t	earned risk premium in the year t at the end of the period (determ.),
u_0	initial allocated capital,
u_t	capital allocated in the year t for the next period,
$F = 1 + i$,	where i is the interest rate of the financial means,
$G = 1 + (1 - \gamma)i$,	where γ is mean time of loss payments during the year, and
$X_{t-j,j} = {}^{j}X_t$	the diagonal elements,

we get for risk reserve at the end of the year $t + 1$ the relation

$$U_{t+1} = u_t F + B_{t+1} + R_t F - R_{t+1} - \mathrm{Pmt}_{t+1} G, \tag{2.5}$$

where Pmt_{t+1} is the payment in the calendar year $t + 1$,

$$\mathrm{Pmt}_{t+1} = {}^{0}X_{t+1} + \sum_{j=1}^{r-1} ({}^{j}X_{t+1} - {}^{j-1}X_t) + {}^{r-1}X_t E_{t+1,r}(c_{r-1} - 1). \tag{2.6}$$

For the one-period model the initial capital u_0 is determined from the requirement $P(U_1 < 0) = \varepsilon$. Applying NP2-approximation gives

$$0 = \mathrm{E}U_1 - z_{1-\varepsilon}\sqrt{\mathrm{E}\overline{U}_1^2} + \frac{\mathrm{E}\overline{U}_1^3}{6\mathrm{E}\overline{U}_1^2}(z_{1-\varepsilon}^2 - 1).$$

In [29] the value $\varepsilon = 0.005$ is assumed, as this value is often used by regulators.

Most of the models estimate both process and estimation variance, some of them concentrated originally on process variance and were extended to cope with estimation variance later. The variance of the prediction can be expressed as

$$\mathrm{E}\left[X_{ij} - \hat{X}_{ij}\right]^2 = \mathrm{E}\left[\left(X_{ij} - \mathrm{E}X_{ij}\right) - \left(\hat{X}_{ij} - \mathrm{E}X_{ij}\right)\right]^2.$$

Assuming independence of future and past observations this can be approximately expressed as

$$\mathrm{E}\left[X_{ij} - \hat{X}_{ij}\right]^2 \cong \mathrm{E}\left[X_{ij} - \mathrm{E}X_{ij}\right]^2 + \mathrm{E}\left[\hat{X}_{ij} - \mathrm{E}\hat{X}_{ij}\right]^2,$$

in other words, prediction variance $\cong$ process variance (process risk) + estimation variance (parameter risk).

2.2.6 Reserving models and reinsurance in practice

In practice two methods are usually used to cope with reinsurance in reserving. If the company has data available on net basis the reserves are computed directly from net data. This can be inappropriate if there were significant changes in the reinsurance structure as the assumptions of the actuarial loss reserving methods may not be fulfilled. The second method is to apply a gross-to-net coefficient to gross loss reserves. This method can be inappropriate for non-proportional reinsurance treaties. In this case, large losses are sometimes coped with individually.

In the following chapters we construct a similar model to the one in [29] which uses all available information on individual losses. With such a model, we are then able to model ceded and net losses even if the reinsurance programs or parameters vary over the years. Thus, the model can be used to manage the required risk capital through optimizing the reinsurance structure and parameters.

Chapter 3

Model for estimating prudence in claims provisions based on individual losses

In this chapter we develop a stochastic reserving model that is based on individual loss information. We start with a short intuitive consideration and proceed with the development of the model for loss numbers and loss severities. Then we put the two models together and derive formulas for loss reserve. At the end we discuss the estimation of the parameters of the model.

3.1 Intuitive consideration

Intuitive consideration is based on modeling cumulative amounts of paid losses, $X_{s,j}$, by means of a compound distribution,

$$X_{s,j} = \sum_{k=1}^{\tilde{N}_{s,j}} Y_{s,j,k}, \quad j = 0, \cdots, r,$$

and using a multiplicative model for individual losses,

$$Y_{s,j+1,k} = Y_{s,j,k} \left(1 + (d_j - 1)\tilde{E}_{s,j+1,k}\right).$$

However, we have to model also losses occurring in the triangle for the first time at a later development year (i.e. losses having their first payment at development year $j > 0$ as we are working with paid losses[1]). Let's have an example of a loss development triangle. In this example the eighth loss from the accident year 2004 has the first payment in the development year $j = 2$.

[1] In a model for incurred losses, these losses would be the so called pure IBNR losses, whereas the multiplicative formula would define the IBNeR (incurred but not enough reported) losses.

Number of losses:

	0	1	2
2004	5	2	1
2005	4	2	
2006	5		

Paid losses from 2004:

	0	1	2
1	10	14	15
2	8	11	12
3	11	14	14
4	6	8	9
5	11	13	14
6	-	10	12
7	-	9	10
8	-	-	9

That's why we have to begin with the model for claim numbers.

3.2 Model for number of claims

Let's have a triangle of cumulative claim numbers,

$$\begin{array}{llll} N_{t-r,0} & N_{t-r,1} & \ldots & N_{t-r,r} \\ \ldots & \ldots & & \\ N_{t-1,0} & N_{t-1,1} & & \\ N_{t,0}, & & & \end{array}$$

and assume that after the development year r no new claims are reported or have payments. This assumption can be loosen simply by extending the model by tail factors. We do this in Chapter 5 where practical results are shown. We base the model on the incremental claim numbers,

$$\begin{array}{llll} \tilde{N}_{t-r,0} & \tilde{N}_{t-r,1} & \ldots & \tilde{N}_{t-r,r} \\ \ldots & \ldots & & \\ \tilde{N}_{t-1,0} & \tilde{N}_{t-1,1} & & \\ \tilde{N}_{t,0}, & & & \end{array}$$

where $\tilde{N}_{s,j}$ is the number of claims occurred in the year s that have the first payment in the development year j, i.e.

$$\begin{aligned} \tilde{N}_{s,j} &= N_{s,j} - N_{s,j-1}, \quad j = 1, \ldots r, \\ \tilde{N}_{s,0} &= N_{s,0}. \end{aligned}$$

For $\tilde{N}_{s,j}$, $j > 0$, we assume conditional Poisson distribution with parameter equal to a multiple of the number of claims from the accident year s occurred in the triangle up to the previous development year, $N_{s,j-1}$,

$$\tilde{N}_{s,j} | N_{s,j-1} \sim \text{Po}(b_{j-1} N_{s,j-1}), \quad j = 1, \ldots, r,$$

where $0 \leq b_j$. Coefficients b_j should be decreasing (at least starting from some development year) and converge to 0 with increasing j. In particular, it should hold $b_j < 1$ for some j', $j \geq j'$.

For the number of losses in the first development year, $\tilde{N}_{s,0}$, we assume Poisson distribution with parameter λ_s,

$$\tilde{N}_{s,0} \sim \mathrm{Po}(\lambda_s).$$

This assumption can be loosen easily, in most of the results we need just the moments of $\tilde{N}_{s,0}$. The dependence of λ_s on the year of loss occurrence s can be in practice modeled by a deterministic trend. It can also be modeled with a deterministic one-time jump if there was a change in process of loss settlement or some other known structural change.

The expression for the mean of the random variable $\tilde{N}_{s,j}$ conditional on the previous development year follows directly from the assumption of Poisson distribution,

$$\mathrm{E}\left[\tilde{N}_{s,j}|N_{s,j-1}\right] = b_j N_{s,j-1}, \quad s > 0, \tag{3.1}$$

$$\mathrm{E}\tilde{N}_{s,0} = \lambda_s.$$

Similarly, for the variance it holds

$$\mathrm{Var}\left[\tilde{N}_{s,j}|N_{s,j-1}\right] = b_j N_{s,j-1}, \quad s > 0, \tag{3.2}$$

$$\mathrm{Var}[\tilde{N}_{s,0}] = \lambda_s.$$

In the following subsection we determine the expressions for the moments of $N_{s,j}$ conditionally on $N_{s,j-1}$.

3.2.1 Moments of claim numbers conditional on claim numbers in previous development year

For the conditional mean $\mathrm{E}[N_{s,j}|N_{s,j-1}]$ we can use (3.1) and write

$$\mathrm{E}\left[N_{s,j}|N_{s,j-1}\right] = \mathrm{E}\left[N_{s,j-1} + \tilde{N}_{s,j}|N_{s,j-1}\right] = N_{s,j-1} + N_{s,j-1}b_{j-1} = N_{s,j-1}(1 + b_{j-1}). \tag{3.3}$$

Similarly, for the conditional variance $\mathrm{Var}[N_{s,j}|N_{s,j-1}]$ we have

$$\mathrm{Var}\left[N_{s,j}|N_{s,j-1}\right] = \mathrm{Var}\left[N_{s,j-1} + \tilde{N}_{s,j}|N_{s,j-1}\right] = \mathrm{Var}\left[\tilde{N}_{s,j}|N_{s,j-1}\right],$$

and using (3.2) we get

$$\mathrm{Var}[N_{s,j}|N_{s,j-1}] = N_{s,j-1}b_{j-1}. \tag{3.4}$$

For the second conditional moment we use the relation

$$\mathrm{E}\left[N_{s,j}^2|N_{s,j-1}\right] = \mathrm{Var}[N_{s,j}|N_{s,j-1}] + \left(\mathrm{E}[N_{s,j}|N_{s,j-1}]\right)^2,$$

from which it follows that

$$\mathrm{E}\left[N_{s,j}^2|N_{s,j-1}\right] = N_{s,j-1}b_{j-1} + N_{s,j-1}^2(1 + b_{j-1})^2. \tag{3.5}$$

For the determination of the third moment we need the formula for the third moment of a random variable N with Poisson distribution. To get this, we compute

$$\mathrm{E}N(N-1)(N-2) = \sum_{k=3}^{\infty} k(k-1)(k-2)e^{-\lambda}\frac{\lambda^k}{k!}.$$

This can be simplified as

$$\mathrm{E}N(N-1)(N-2) = \sum_{k=3}^{\infty} e^{-\lambda}\frac{\lambda^k}{(k-3)!} = \lambda^3 \sum_{k'=0}^{\infty} e^{-\lambda}\frac{\lambda^{k'}}{k'!} = \lambda^3.$$

From this we get for the third moment of a random variable with Poisson distribution

$$\mathrm{E}N^3 = \mathrm{E}N(N-1)(N-2) + 3\mathrm{E}N^2 - 2\mathrm{E}N = \lambda^3 + 3\lambda^2 + \lambda.$$

Having this, we can compute the third conditional moment $\mathrm{E}[N_{s,j}^3|N_{s,j-1}]$. By expressing $N_{s,j}$ as $N_{s,j-1} + \tilde{N}_{s,j}$ we get

$$\mathrm{E}\left[N_{s,j}^3|N_{s,j-1}\right] = \mathrm{E}\left[(N_{s,j-1} + \tilde{N}_{s,j})^3|N_{s,j-1}\right],$$

which is after expanding the brackets in the expected value equal to

$$\mathrm{E}\left[N_{s,j-1}^3 + 3N_{s,j-1}^2\tilde{N}_{s,j} + 3N_{s,j-1}\tilde{N}_{s,j}^2 + \tilde{N}_{s,j}^3|N_{s,j-1}\right].$$

It follows easily that this is equal to

$$N_{s,j-1}^3 + 3N_{s,j-1}^2\mathrm{E}\left[\tilde{N}_{s,j}|N_{s,j-1}\right] + 3N_{s,j-1}\mathrm{E}\left[\tilde{N}_{s,j}^2|N_{s,j-1}\right] + \mathrm{E}\left[\tilde{N}_{s,j}^3|N_{s,j-1}\right].$$

Variable $\tilde{N}_{s,j}$ has a Poisson distribution with parameter $b_{j-1}N_{s,j-1}$. Using for $\mathrm{E}[\tilde{N}_{s,j}^3|N_{s,j-1}]$ the formula for the third moment of the Poisson distribution developed above and putting in for $\mathrm{E}[\tilde{N}_{s,j}|N_{s,j-1}]$ and $\mathrm{E}[\tilde{N}_{s,j}^2|N_{s,j-1}]$ we get

$$\mathrm{E}\left[N_{s,j}^3|N_{s,j-1}\right] = N_{s,j-1}^3(1+b_{j-1})^3 + 3N_{s,j-1}^2 b_{j-1}(1+b_{j-1}) + N_{s,j-1}b_{j-1}. \tag{3.6}$$

With these results we can recursively compute the unconditional moments of $N_{s,j}$.

3.2.2 Unconditional moments of claim numbers

The unconditional expected value of $N_{s,j}$ can be computed recursively using the previous result for $\mathrm{E}[N_{s,j}|N_{s,j-1}]$. Using the theorem on conditional expected values we have

$$\mathrm{E}N_{s,j} = \mathrm{E}\left[\mathrm{E}[N_{s,j}|N_{s,j-1}]\right] = \mathrm{E}\left[N_{s,j-1}(1+b_{j-1})\right] = (1+b_{j-1})\mathrm{E}N_{s,j-1}.$$

Similarly, we continue by expressing $\mathrm{E}[N_{s,j-1}]$ in the same way and get

$$\mathrm{E}N_{s,j} = (1+b_{j-1})(1+b_{j-2})\mathrm{E}N_{s,j-2}.$$

It can be seen easily that after further recursion we get for $j \geq 1$ the expression

$$\mathrm{E}N_{s,j} = \lambda_s \prod_{k=0}^{j-1}(1 + b_k). \tag{3.7}$$

The expression for the unconditional variance of $N_{s,j}$ can be determined in a similar way. We can show that the variance $\mathrm{Var}[N_{s,j}]$ is a linear function of parameter λ_s, which is a consequence of the Poisson distribution assumption. The variance can be expressed as

$$\mathrm{Var}[N_{s,j}] = \mathrm{E}N_{s,j}^2 - \left(\mathrm{E}N_{s,j}\right)^2, \tag{3.8}$$

where we have already developed the expression for $\mathrm{E}N_{s,j}$. The term $\mathrm{E}N_{s,j}^2$ can be computed similarly using the theorem on conditional expected values that gives

$$\mathrm{E}N_{s,j}^2 = \mathrm{E}\left[\mathrm{E}\left[N_{s,j}^2 | N_{s,j-1}\right]\right].$$

Substituting for $\mathrm{E}[N_{s,j}^2 | N_{s,j-1}]$ from (3.5) we obtain

$$\mathrm{E}N_{s,j}^2 = \mathrm{E}\left[N_{s,j-1}b_{j-1} + N_{s,j-1}^2(1 + b_{j-1})^2\right],$$

from which it follows

$$\mathrm{E}N_{s,j}^2 = b_{j-1}\mathrm{E}N_{s,j-1} + (1 + b_{j-1})^2\mathrm{E}N_{s,j-1}^2.$$

Using recursion for $\mathrm{E}N_{s,j-1}^2$ we expand the expression to

$$\begin{aligned}\mathrm{E}N_{s,j}^2 = b_{j-1}\mathrm{E}N_{s,j-1} \; + \; (1 + b_{j-1})^2\Big(b_{j-2}\mathrm{E}N_{s,j-2} \; + \\ + \; (1 + b_{j-2})^2\big(b_{j-3}\mathrm{E}N_{s,j-3} \; + \; \cdots \; + \\ + \; (1 + b_1)^2(b_0\mathrm{E}N_{s,0} \; + \; (1 + b_0)^2\mathrm{E}N_{s,0}^2)\ldots\big)\Big),\end{aligned}$$

where from the assumption of Poisson distribution for $N_{s,0} = \tilde{N}_{s,0}$ we can replace

$$\mathrm{E}N_{s,0} = \lambda_s,$$

$$\mathrm{E}N_{s,0}^2 = \mathrm{Var}[N_{s,0}] + (\mathrm{E}N_{s,0})^2 = \lambda_s + \lambda_s^2.$$

After putting in for $\mathrm{E}N_{s,0}$ and $\mathrm{E}N_{s,0}^2$ and expanding the brackets we get

$$\mathrm{E}N_{s,j}^2 = b_{j-1}\lambda_s \prod_{i=0}^{j-2}(1 + b_i) +$$

$$+ \; (1 + b_{j-1})^2 b_{j-2}\lambda_s \prod_{i=0}^{j-3}(1 + b_i) +$$

$$+ \prod_{k=j-2}^{j-1}(1+b_k)^2 b_{j-3}\lambda_s \prod_{i=0}^{j-4}(1+b_i) + \ldots$$

$$\ldots + \prod_{k=2}^{j-1}(1+b_k)^2 b_1 \lambda_s (1+b_0) +$$

$$+ \prod_{k=1}^{j-1}(1+b_k)^2 \lambda_s b_0 + \prod_{k=0}^{j-1}(1+b_k)^2(\lambda_s + \lambda_s^2),$$

which can be rewritten as

$$\mathrm{E}N_{s,j}^2 = \lambda_s \left[\sum_{h=0}^{j-1} \left(b_h \prod_{k=h+1}^{j-1}(1+b_k)^2 \prod_{i=0}^{h-1}(1+b_i) \right) + \prod_{k=0}^{j-1}(1+b_k)^2 \right] + \lambda_s^2 \prod_{k=0}^{j-1}(1+b_k)^2.$$

Putting this expression into (3.8) gives for the variance the relation

$$\mathrm{Var}[N_{s,j}] = \lambda_s \left[\sum_{h=0}^{j-1} \left(b_h \prod_{k=h+1}^{j-1}(1+b_k)^2 \prod_{i=0}^{h-1}(1+b_i) \right) + \prod_{k=0}^{j-1}(1+b_k)^2 \right].$$

We define $\prod_1^0 := 1$, so that the relation holds for $j \geq 1$. Defining

$$\tilde{\kappa}_{0,j-1} = \sum_{h=0}^{j-1} \left(b_h \prod_{k=h+1}^{j-1}(1+b_k)^2 \prod_{i=0}^{h-1}(1+b_i) \right) + \prod_{k=0}^{j-1}(1+b_k)^2, \tag{3.9}$$

the variance can be expressed as

$$\mathrm{Var}[N_{s,j}] = \tilde{\kappa}_{0,j-1}\lambda_s. \tag{3.10}$$

Thus, we see that the variance $\mathrm{Var}[N_{s,j}]$ is a linear function of λ_s. With these results it is easy to see that for the stochastic variable $N_{s,j}$ holds

$$\mathrm{E}N_{s,0} = \mathrm{Var}[N_{s,0}],$$

$$\mathrm{E}N_{s,j} \leq \mathrm{Var}[N_{s,j}].$$

The first expression follows directly from the assumption of Poisson distribution of $N_{s,0}$. To show the inequality we can see that due to

$$b_h \geq 0$$

the first term in (3.9) is greater than or equal to zero. Thus we have

$$\tilde{\kappa}_{0,j-1} \geq \prod_{k=0}^{j-1}(1+b_k)^2 \geq \prod_{k=0}^{j-1}(1+b_k).$$

Hence, in spite of the assumption of the conditional Poisson distribution for incremental loss numbers, we get for cumulative loss numbers a distribution with variance greater or equal to the expected value.

3.2.3 Moments of ultimate claim numbers conditional on claim numbers known at calender year t

In a similar way as in the previous section we can derive the formulas for the mean and the variance of the ultimate claim numbers $N_{t-j,r}$ conditional on the values $N_{t-j,j}$, known at calender year t. For the expected value we get

$$\mathrm{E}\left[N_{t-j,r}|N_{t-j,j}\right] = N_{t-j,j}\prod_{k=j}^{r-1}(1+b_k), \tag{3.11}$$

for $j = 0, \cdots, r-1$. If we define $\prod_i^{i-1} := 1$, the expression holds even for $j = r$. For the variance holds

$$\mathrm{Var}\left[N_{t-j,r}|N_{t-j,j}\right] = \kappa_{j,r-1}N_{t-j,j}, \tag{3.12}$$

where

$$\kappa_{j,r-1} = \sum_{h=j}^{r-1}\left(b_h\prod_{k=h+1}^{r-1}(1+b_k)^2\prod_{i=j}^{h-1}(1+b_i)\right), \tag{3.13}$$

where we, again, define $\prod_i^{i-1} := 1$ and, additionally, $\sum_i^{i-1} := 0$, so that the expression holds even for $j = r-1$ and $j = r$. We can see that the variance is a linear function of $N_{t-j,j}$. For the second moment we get

$$\mathrm{E}\left[N_{t-j,r}^2|N_{t-j,j}\right] = \kappa_{j,r-1}N_{t-j,j} + N_{t-j,j}^2\prod_{k=j}^{r-1}(1+b_k)^2. \tag{3.14}$$

3.3 Severity model

In order to model loss severities we have to differentiate paid losses according to the year, in which they are emerging in the triangle, i.e. the year in which the losses have their first payment. Let's denote

$$Y_{s,j,k}^{l}, \quad l = 0, ..., j, \ k = 1, ..., \tilde{N}_{s,l},$$

individual losses from the accident year $s = t-r, \cdots, t$, that are in the development year $j = 0, \cdots, r$, and have had their first payment in the development year l. We assume, according to the collective risk model, that loss severities are independent of each other and independent of the number of losses $\tilde{N}_{s,l}$.

For losses occurring in the triangle for the first time (i.e. having their first payment), $Y_{s,j,k}^{j}$, we assume that they have mean m_s and variance s_s^2. We will denote the second moment $m_s^{(2)}$, so that we have $s_s^2 = m_s^{(2)} - m_s^2$. For simplicity we don't assume the dependence of the moments on the development year, but the model can be generalized. If we want to compute whole distribution of the loss reserve, we have to make additional assumption of the distribution of $Y_{s,j,k}^{j}$. In the example in Chapter 5 we use the log-normal distribution.

For losses that have already occurred in the triangle in the previous years it would be natural to assume, similarly to the model without information on individual losses in [29], a multiplicative model

$$Y^l_{s,j+1,k} = Y^l_{s,j,k}\left(1 + (d^l_j - 1)\tilde{E}^l_{s,j+1,k}\right), \quad l = 0, ..., j,$$

where the error terms $\tilde{E}^l_{s,j,k}$ are mutually independent, independent on $Y^l_{s,j-1,k}$, $l = 0, ..., j-1$, and $N_{s,j-1}$, having $\mathrm{E}\tilde{E}^l_{s,j,k} = 1$ and having for fixed j equal second and third moments.

The problem with this approach is that for closed losses we will typically have $\tilde{E}^l_{s,j+1,k} = 0$, $\tilde{E}^l_{s,j+2,k} = 0$, ..., so that we can't assume independence of the terms $\tilde{E}^l_{s,j,k}$. In other words, we have for closed losses

$$Y^l_{s,j+1,k} = Y^l_{s,j,k},$$

and for open losses we can assume the multiplicative model

$$Y^l_{s,j+1,k} = Y^l_{s,j,k}\left(1 + (d^l_j - 1)\tilde{E}^l_{s,j+1,k}\right), \quad l = 0, ..., j, \tag{3.15}$$

where the error terms $\tilde{E}^l_{s,j,k}$ are mutually independent, independent on $Y^l_{s,j-1,k}$, $l = 0, ..., j-1$, and $N_{s,j-1}$, having $\mathrm{E}\tilde{E}^l_{s,j,k} = 1$ and having for fixed j equal second and third moments. We denote the second moment of the error terms $s^2_{E_j} = \mathrm{E}[\tilde{E}^l_{s,j,k}]^2$. We neglect the possibility that a closed loss can be reopened.

Additionally, in the following text we assume that the development factors d^l_j don't depend on the development year l in which a loss occurred in the triangle, i.e.

$$d^l_j = d^{l'}_j, \quad l, l' = 0, \ldots, j. \tag{3.16}$$

This assumption is made due to simplicity of the derived formulas, the model can be generalized. In reality the parameter d_j can depend on loss severity as well, in the model we can cope with this problem e.g. by splitting the losses according to the severity in small, large and very large losses and to develop triangles for each of these groups.

Let's denote with $J^l_{s,j,k}$ the stochastic variable that indicates whether the loss $Y^l_{s,j,k}$ is open at the and of the development year j, i.e.

$$J^l_{s,j,k} = \begin{cases} 1, & \text{if } Y^l_{s,j,k} \text{ is open at the end of development year } j, \\ 0, & \text{if } Y^l_{s,j,k} \text{ is closed at the end of development year } j. \end{cases}$$

For closed losses we then have, due to the assumption that they can't be reopened,

$$J^l_{s,j+1,k} | \{J^l_{s,j,k} = 0\} \quad = 0.$$

For open losses we assume that a loss that is open at the and of the development year j will be closed in the development year $j+1$ with probability q_{j+1}, i.e.

$$J^l_{s,j+1,k} | \{J^l_{s,j,k} = 1\} \quad = \begin{cases} 1, & \text{with probability} \quad 1 - q_{j+1}, \\ 0, & \text{with probability} \quad q_{j+1}, \end{cases}$$

for $j = l, ..., r-1$ and for new losses

$$J^l_{s,l,k} = 1 \quad = \begin{cases} 1, & \text{with probability} \quad 1 - q_l, \\ 0, & \text{with probability} \quad q_l. \end{cases}$$

We assume that $J^l_{s,j+1,k}$ depends only on $J^l_{s,j,k}$. The expected value of $J^l_{s,j+1,k}$ can be computed as

$$\mathrm{E}\left[J^l_{s,j+1,k}\right] = \mathrm{E}\left[J^l_{s,j+1,k}|J^l_{s,j,k}=1\right]\mathrm{P}\left(J^l_{s,j,k}=1\right)+\mathrm{E}\left[J^l_{s,j+1,k}|J^l_{s,j,k}=0\right]P\left(J^l_{s,j,k}=0\right),$$

where for the expected values we have

$$\mathrm{E}\left[J^l_{s,j+1,k}|J^l_{s,j,k}=0\right]=0, \tag{3.17}$$

and

$$\mathrm{E}\left[J^l_{s,j+1,k}|J^l_{s,j,k}=1\right]=0\cdot\ q_{j+1}+1\cdot(1-q_{j+1})=1-q_{j+1}, \tag{3.18}$$

and for the probabilities we have

$$\mathrm{P}\left(J^l_{s,j,k}=1\right)=(1-q_l)\cdots(1-q_j):=Q_{l,j}, \tag{3.19}$$

and

$$\mathrm{P}\left(J^l_{s,j,k}=0\right)=q_l+(1-q_l)q_{l+1}+\cdots+(1-q_l)\cdots(1-q_{j-1})q_j=1-Q_{l,j}. \tag{3.20}$$

For the expected value of $J^l_{s,j+1,k}$ then follows

$$\mathrm{E}\left[J^l_{s,j+1,k}\right]=(1-q_{j+1})Q_{l,j}=Q_{l,j+1}. \tag{3.21}$$

For the second moment we get in a similar way

$$\mathrm{E}\left[J^l_{s,j+1,k}\right]^2=Q_{l,j+1}.$$

Using $J^l_{s,j,k}$ we can put the models for open and closed losses together in one formula,

$$Y^l_{s,j+1,k}=Y^l_{s,j,k}\left(1+(d_j-1)J^l_{s,j,k}\tilde{E}^l_{s,j+1,k}\right),\quad l=0,\ldots,j, \tag{3.22}$$

where the error terms $\tilde{E}^l_{s,j,k}$ are mutually independent and independent on $Y^l_{s,j-1,k}$, $N_{s,j-1}$, $J^l_{s,j-1,k}$ and $J^l_{s,j,k}$, having $\mathrm{E}\tilde{E}^l_{s,j,k}=1$ and having for fixed j equal second and third moments.

3.4 Model of cumulative annual losses

Having developed the models for loss numbers and loss severities we can express cumulative annual paid losses as

$$X_{s,j}=\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}Y^l_{s,j,k},\quad j=0,\cdots,r. \tag{3.23}$$

In the following text we often use the decomposition of the variable $X_{s,j+1}$ into losses that have already occurred in the triangle up to the development year j and losses that occur in the triangle for the first time in the last development year $j+1$,

$$X_{s,j+1}=\sum_{l=0}^{j+1}\sum_{k=1}^{\tilde{N}_{s,l+1}}Y^l_{s,j+1,k}=\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}Y^l_{s,j+1,k}+\sum_{k=1}^{\tilde{N}_{s,j+1}}Y^{j+1}_{s,j+1,k}. \tag{3.24}$$

It follows from the severity model that

$$X_{s,j+1} = \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j,k}(1 + (d_j - 1)J^l_{s,j,k}\tilde{E}^l_{s,j+1,k}) + \sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k}. \tag{3.25}$$

Alternatively, we can decompose $X_{s,j+1}$ in open and closed losses,

$$X_{s,j+1} = X^O_{s,j+1} + X^C_{s,j+1}, \tag{3.26}$$

where for annual cumulative open losses from the accident year s in the development year j we have

$$X^O_{s,j} = \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j,k} J^l_{s,j,k}, \quad j = 0, \cdots, r, \tag{3.27}$$

and, similarly, for annual cumulative closed losses from the accident year s in the development year j we have

$$X^C_{s,j} = \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j,k}(1 - J^l_{s,j,k}), \quad j = 0, \cdots, r. \tag{3.28}$$

In contrast to the models that don't use the information on individual losses, we will base the conditional expected values in the later development years not on $X_{s,j}$, but on the whole information on losses from the accident year s known in the development year j, i.e.

$$N_{s,j},\ Y^l_{s,j,k},\ J^l_{s,j,k}, \quad l = 0, ..., j,\ k = 1, ..., \tilde{N}_{s,l}.$$

We denote this information $I_{s,j}$. It is easy to see that $I_{s,j} \subset I_{s,j+1}$.

3.4.1 Expected values conditional on previous year - overall losses

For the mean value $\mathrm{E}[X_{s,j+1}|I_{s,j}]$ we use the decomposition introduced at the beginning of this section and write

$$\mathrm{E}\left[X_{s,j+1}|I_{s,j}\right] = \mathrm{E}\left[\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j,k}(1 + (d_j - 1)J^l_{s,j,k}\tilde{E}^l_{s,j+1,k}) + \sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k} \middle| I_{s,j}\right],$$

which is equal to

$$\mathrm{E}\left[\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j,k}(1 + (d_j - 1)J^l_{s,j,k}\tilde{E}^l_{s,j+1,k}) \middle| I_{s,j}\right] + \mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k} \middle| I_{s,j}\right]. \tag{3.29}$$

Using the independence of the error terms $\tilde{E}^l_{s,j+1,k}$ on $I_{s,j}$ for $l = 1, ..., j, k = 1, ..., \tilde{N}_{s,j}$ we can rewrite the first term in (3.29) as

$$\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j,k}\left(1 + (d_j - 1)J^l_{s,j,k}\mathrm{E}\left[\tilde{E}^l_{s,j+1,k}\right]\right),$$

which is, due to the assumption $\mathrm{E}[\tilde{E}^l_{s,j+1,k}] = 1$, equal to

$$\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j,k} + (d_j - 1)\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j,k}J^l_{s,j,k}.$$

It can be seen easily that this can be expressed as

$$X_{s,j} + (d_j - 1)X^O_{s,j}.$$

Using the decomposition $X_{s,j} = X^O_{s,j} + X^C_{s,j}$, this can be alternatively expressed as

$$X^C_{s,j} + d_j X^O_{s,j}.$$

The second term of (3.29) is an expected value of a variable with a compound distribution, which is a product of the expected value of the number of losses and the expected value of the loss severity distribution (which is in this case independent of $I_{s,j}$),

$$\mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k}\middle| I_{s,j}\right] = \mathrm{E}\left[\tilde{N}_{s,j+1}\middle| I_{s,j}\right]\mathrm{E}Y^{j+1}_{s,j+1,k} = b_j N_{s,j} m_s.$$

Summarized, we have for the expected value of $X_{s,j+1}$ conditional on the information in the previous development year the expression

$$\mathrm{E}[X_{s,j+1}|I_{s,j}] = X_{s,j} + (d_j - 1)X^O_{s,j} + b_j N_{s,j} m_s, \tag{3.30}$$

or equally

$$\mathrm{E}[X_{s,j+1}|I_{s,j}] = X^C_{s,j} + d_j X^O_{s,j} + b_j N_{s,j} m_s. \tag{3.31}$$

In other words, the expected value of $X_{s,j+1}$ conditional on $I_{s,j}$ is equal to the losses in development year j plus the incremental development of the losses that are at development year j open plus the mean value of losses that occur in the triangle at development year $j + 1$.

If we denote

$$X^{(2)}_{s,j+1} = \sum_{l=0}^{j+1}\sum_{k=1}^{\tilde{N}_{s,l}} \left(Y^l_{s,j+1,k}\right)^2,$$

then we can compute the expected value $\mathrm{E}[X^{(2)}_{s,j+1}|I_{s,j}]$ in a similar way and get

$$\mathrm{E}\left[X^{(2)}_{s,j+1}\middle| I_{s,j}\right] = X^{(2)}_{s,j} + (d^{(2)}_j - 1)X^{O(2)}_{s,j} + b_j N_{s,j} m^{(2)}_s, \tag{3.32}$$

where

$$d^{(2)}_j = \mathrm{E}\left[1 + (d_j - 1)\tilde{E}^l_{s,j+1,k}\right]^2 \tag{3.33}$$

are the second moments of the one-step multiplicative development of large losses and

$$X^{O(2)}_{s,j} = \sum_{l=0}^{j+1}\sum_{k=1}^{\tilde{N}_{s,l}} \left(Y^l_{s,j+1,k}J^l_{s,j+1,k}\right)^2.$$

3.4.2 Expected values conditional on previous year - open losses

In a similar way to the case of overall losses we decompose open losses at development year $j+1$ into the losses that have occurred in the triangle up to the development year j and the losses that come into the triangle in the development year $j+1$ and get

$$\mathrm{E}\left[X^{O}_{s,j+1}|I_{s,j}\right] = \mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y^{l}_{s,j+1,k}J^{l}_{s,j+1,k}\middle| I_{s,j}\right] + \mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k}J^{j+1}_{s,j+1,k}\middle| I_{s,j}\right]. \tag{3.34}$$

The first term in (3.34) can be expressed with the multiplicative severity model as

$$\mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y^{l}_{s,j,k}\left(1+(d_j-1)J^{l}_{s,j,k}\tilde{E}^{l}_{s,j+1,k}\right)J^{l}_{s,j+1,k}\middle| I_{s,j}\right],$$

which is, due to the independence of the error terms $\tilde{E}^{l}_{s,j+1,k}$ on $I_{s,j}$ and on $J^{l}_{s,j+1,k}$, for $l = 1, ..., j$, and $k = 1, ..., \tilde{N}_{s,j}$, equal to

$$\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y^{l}_{s,j,k}\left(1+(d_j-1)J^{l}_{s,j,k}\mathrm{E}\left[\tilde{E}^{l}_{s,j+1,k}\right]\right)\mathrm{E}\left[J^{l}_{s,j+1,k}|I_{s,j}\right].$$

Due to the assumption that $\mathrm{E}[\tilde{E}^{l}_{s,j+1,k}] = 1$ and decomposing the sum into the terms with $J^{l}_{s,j,k} = 0$ and the terms with $J^{l}_{s,j,k} = 1$ we get

$$\sum_{l=0}^{j}\sum_{\substack{k=1 \\ \& J^{l}_{s,j,k}=0}}^{\tilde{N}_{s,l}} Y^{l}_{s,j,k}\left(1+(d_j-1)J^{l}_{s,j,k}\right)\mathrm{E}\left[J^{l}_{s,j+1,k}|J^{l}_{s,j,k}=0\right] +$$

$$+\sum_{l=0}^{j}\sum_{\substack{k=1 \\ \& J^{l}_{s,j,k}=1}}^{\tilde{N}_{s,l}} Y^{l}_{s,j,k}\left(1+(d_j-1)J^{l}_{s,j,k}\right)\mathrm{E}\left[J^{l}_{s,j+1,k}|J^{l}_{s,j,k}=1\right].$$

Here we used the assumption that $J^{l}_{s,j+1,k}$ depend only on $J^{l}_{s,j,k}$. Putting in for the expected values the relations (3.17) and (3.18) it can be seen easily that the first part is equal to 0 and the second part becomes

$$d_j X^{O}_{s,j}(1-q_{j+1}).$$

For the second term of the expression (3.34), we use again the assumption of a compound distribution, the independence of $Y^{l}_{s,j+1,k}$ and $J^{l}_{s,j+1,k}$, and the fact that

$$\mathrm{E}\left[J^{j+1}_{s,j+1,k}|I_{s,j}\right] = \mathrm{E}\left[J^{j+1}_{s,j+1,k}\right] = (1-q_{j+1}),$$

and get

$$\mathrm{E}\left[\tilde{N}_{s,j+1}|I_{s,j}\right]\mathrm{E}\left[Y^{j+1}_{s,j+1,k}\right]\mathrm{E}\left[J^{j+1}_{s,j+1,k}\right],$$

which is equal to

$$b_j N_{s,j} m_s (1 - q_{j+1}).$$

Summarizing, we have for the expected value of cumulative open losses, $X^O_{s,j+1}$, conditional on the information from the previous development year j the expression

$$\mathrm{E}\left[X^O_{s,j+1} | I_{s,j}\right] = d_j X^O_{s,j}(1 - q_{j+1}) + b_j N_{s,j} m_s (1 - q_{j+1}). \tag{3.35}$$

In other words, the expected value of $X^O_{s,j+1}$ conditional on $I_{s,j}$ is equal to the open losses in development year j inclusive their development to the development year $j + 1$ times the probability that these losses stay open plus the mean value of losses that occur in the triangle in development year $j + 1$ times the probability that they stay open till the end of the year.

Similarly to the case of the overall losses, the expected value of

$$X^{O(2)}_{s,j} = \sum_{l=0}^{j+1} \sum_{k=1}^{\tilde{N}_{s,l}} \left(Y^l_{s,j+1,k} J^l_{s,j+1,k}\right)^2$$

can be computed in the same way and is equal to

$$\mathrm{E}\left[X^{O(2)}_{s,j+1} | I_{s,j}\right] = d^{(2)}_j X^{O(2)}_{s,j}(1 - q_{j+1}) + b_j N_{s,j} m^{(2)}_s (1 - q_{j+1}). \tag{3.36}$$

3.4.3 Expected values conditional on previous year - closed losses

The expected value of closed losses conditional on previous year can be determined as the difference of the expected values of overall and open losses,

$$\mathrm{E}\left[X^C_{s,j+1} | I_{s,j}\right] = \mathrm{E}\left[X_{s,j+1} | I_{s,j}\right] - \mathrm{E}\left[X^O_{s,j+1} | I_{s,j}\right].$$

This leads to

$$\mathrm{E}\left[X^C_{s,j+1} | I_{s,j}\right] = X^C_{s,j} + q_{j+1} d_j X^O_{s,j} + b_j N_{s,j} m_s q_{j+1}, \tag{3.37}$$

or equally

$$\mathrm{E}\left[X^C_{s,j+1} | I_{s,j}\right] = X_{s,j} + (q_{j+1} d_j - 1) X^O_{s,j} + b_j N_{s,j} m_s q_{j+1}. \tag{3.38}$$

In other words, the expected value of closed losses conditional on previous year is equal to the closed losses in development year j plus the losses that are in development year j open inclusive their development times the probability that they will be closed at development year $j + 1$ plus the mean value of losses that occur in the triangle in development year $j + 1$ times the probability that they will be closed in that year.

For the expected value of

$$X^{C(2)}_{s,j+1} = \sum_{l=0}^{j+1} \sum_{k=1}^{\tilde{N}_{s,l}} \left(Y^l_{s,j+1,k} \left(1 - J^l_{s,j+1,k}\right)\right)^2,$$

we get in a similar way the expression

$$\mathrm{E}\left[X_{s,j+1}^{C(2)}|I_{s,j}\right] = X_{s,j}^{C(2)} + q_{j+1}d_j^{(2)}X_{s,j}^{O(2)} + b_j N_{s,j}m_s^{(2)}q_{j+1}. \tag{3.39}$$

The second moments of overall, open and closed losses can be computed in a similar way. The expressions and the derivations are given in Appendix A.

3.5 Loss reserve

3.5.1 Loss reserve – best estimate

Under loss reserve we understand the undiscounted total estimated ultimated cost to an insurance undertaking of settling all claims arising from events which have occurred up to the end of the financial year, whether reported or not, less amount already paid in respect of such claims. Best estimate of loss reserve (i.e. loss reserve without any risk margin) can be expressed as

$$\mathrm{Res}_t = \sum_{j=0}^{r-1} \mathrm{E}\left[X_{t-j,\infty} - X_{t-j,j}|I_{t-j,j}\right] = \sum_{j=0}^{r-1} \mathrm{E}\left[X_{t-j,r} - X_{t-j,j}|I_{t-j,j}\right]. \tag{3.40}$$

We need to express the conditional expected value of the ultimate losses, $E[X_{t-j,r}|I_{t-j,j}]$, for which we also need to express $E[X_{t-j,r}^{O}|I_{t-j,j}]$.

3.5.2 Expected value of $X_{t-j,r}^{O}$ conditional on $I_{t-j,j}$

Expected value of ultimate open losses conditional on the information in calender year t can be, due to $I_{t-j,j} \subset I_{t-j,r-1}$, expressed as

$$\mathrm{E}\left[X_{t-j,r}^{O}|I_{t-j,j}\right] = \mathrm{E}\left[\mathrm{E}\left[X_{t-j,r}^{O}|I_{t-j,r-1}\right]|I_{t-j,j}\right].$$

Using the result for the conditional expected value of open losses conditional on previous development year (3.35) we can put in for $\mathrm{E}[X_{t-j,r}^{O}|I_{t-j,r-1}]$ and get

$$\mathrm{E}\left[d_{r-1}X_{t-j,r-1}^{O}(1-q_r) + b_{r-1}N_{t-j,r-1}m_{t-j}(1-q_r)\big|I_{t-j,j}\right],$$

which is equal to

$$d_{r-1}(1-q_r)\mathrm{E}\left[X_{t-j,r-1}^{O}|I_{t-j,j}\right] + b_{r-1}m_{t-j}(1-q_r)\mathrm{E}\left[N_{t-j,r-1}|I_{t-j,j}\right].$$

Putting in (3.11) for $\mathrm{E}[N_{t-j,r-1}|I_{t-j,j}]$ and denoting

$$A_{t-j,j,r-1} = b_{r-1}m_{t-j}\prod_{k=j}^{r-2}(1+b_k), \quad j = 0, \cdots, r-1, \tag{3.41}$$

where we again define $\prod_i^{i-1} := 1$ to cover the case of $j = r - 1$, we get

$$\mathrm{E}\left[X^O_{t-j,r}|I_{t-j,j}\right] = d_{r-1}(1-q_r)\mathrm{E}\left[X^O_{t-j,r-1}|I_{t-j,j}\right] + N_{t-j,j}A_{t-j,j,r-1}(1-q_r).$$

Then, again, we can put in for $\mathrm{E}\left[X^O_{t-j,r-1}|I_{t-j,j}\right]$ and after a further recursion the expression becomes

$$\mathrm{E}\left[X^O_{t-j,r}|I_{t-j,j}\right] = X^O_{t-j,j}D_{j,r-1}Q_{j+1,r} + N_{t-j,j}\tilde{Q}^{\beta}_{t-j,j+1,r}, \tag{3.42}$$

where

$$D_{i,j} = \prod_{h=i}^{j} d_h,\ i \le j, \quad D_{j+1,j} := 1,\ j = r-1, r, \tag{3.43}$$

are the cumulative development factors, and

$$\tilde{Q}^{\beta}_{t-j,j+1,r} = \sum_{h=j+1}^{r} D_{h,r-1}Q_{h,r}A_{t-j,j,h-1}. \tag{3.44}$$

The first part of (3.42) contains the development of losses that are at $I_{t-j,j}$ already known times the probability that they stay open till the end of development year r. The second part of (3.42) contains the expected value of losses that occur in the triangle in the later development years inclusive their development and multiplied by the probability that they stay open till the end of development year r. If we define $\sum_i^{i-1} := 0$ and $Q_{i+1,i} := 1$, then the expression holds even for the simple case of $j = r$.

3.5.3 Expected value of $X_{t-j,r}$ conditional on $I_{t-j,j}$

The conditional expected value of $X_{t-j,r}$ conditional on the information available at calender year t can be expressed as

$$\mathrm{E}\left[X_{t-j,r}|I_{t-j,j}\right] = \mathrm{E}\left[\mathrm{E}\left[X_{t-j,r}|I_{t-j,r-1}\right]|I_{t-j,j}\right].$$

Using the result for the conditional mean value of overall losses (3.30) we can put in for $\mathrm{E}[X_{t-j,r}|I_{t-j,r-1}]$ and get

$$\mathrm{E}\left[X_{t-j,r}|I_{t-j,j}\right] = \mathrm{E}\left[X_{t-j,r-1} + (d_{r-1}-1)X^O_{t-j,r-1} + b_{r-1}N_{t-j,r-1}m_{t-j}|I_{t-j,j}\right],$$

which is equal to

$$\mathrm{E}\left[X_{t-j,r-1}|I_{t-j,j}\right] + (d_{r-1}-1)\mathrm{E}\left[X^O_{t-j,r-1}|I_{t-j,j}\right] + b_{r-1}m_{t-j}\mathrm{E}\left[N_{t-j,r-1}|I_{t-j,j}\right].$$

Putting in (3.11) for $\mathrm{E}[N_{t-j,r-1}|I_{t-j,j}]$ and using the previous notation (3.41) we get

$$\mathrm{E}\left[X_{t-j,r}|I_{t-j,j}\right] = \mathrm{E}\left[X_{t-j,r-1}|I_{t-j,j}\right] + (d_{r-1}-1)\mathrm{E}\left[X^O_{t-j,r-1}|I_{t-j,j}\right] + A_{t-j,j,r-1}N_{t-j,j}.$$

This can be, putting in the expression (3.42) for $\mathrm{E}[X^O_{t-j,r-1}|I_{t-j,j}]$ developed above, expressed as

$$\mathrm{E}\left[X_{t-j,r-1}|I_{t-j,j}\right] + (d_{r-1}-1)\left(X^O_{t-j,j}D_{j,r-2}Q_{j+1,r-1} + N_{t-j,j}\tilde{Q}^{\beta}_{t-j,j+1,r-1}\right) +$$

$$+A_{t-j,j,r-1}N_{t-j,j}.$$

It easy to see that this can be rearranged as

$$\mathrm{E}\left[X_{t-j,r-1}\middle|I_{t-j,j}\right]+X^{O}_{t-j,j}(d_{r-1}-1)D_{j,r-2}Q_{j+1,r-1}+$$

$$+N_{t-j,j}\left[(d_{r-1}-1)\tilde{Q}^{\beta}_{t-j,j+1,r-1}+A_{t-j,j,r-1}\right].$$

Then we can again put in for $\mathrm{E}\left[X_{t-j,r-1}\middle|I_{t-j,j}\right]$ and after a recursion we get

$$\mathrm{E}\left[X_{t-j,r}\middle|I_{t-j,j}\right]=X_{t-j,j}+X^{O}_{t-j,j}\tilde{Q}^{D}_{t-j,j,r-1}+N_{t-j,j}\tilde{Q}^{A}_{t-j,j,r-1}, \tag{3.45}$$

where

$$\tilde{Q}^{D}_{t-j,j,r-1}=\sum_{h=j}^{r-1}(d_h-1)D_{j,h-1}Q_{j+1,h}, \tag{3.46}$$

and

$$\tilde{Q}^{A}_{t-j,j,r-1}=\sum_{h=j+1}^{r-1}(d_h-1)\tilde{Q}^{\beta}_{t-j,j+1,h}+\sum_{h=j}^{r-1}A_{t-j,j,h}. \tag{3.47}$$

The first part of (3.45) are the losses at $I_{t-j,j}$, the second part contains the development of the losses that are at $I_{t-j,j}$ open till development year r or till they are closed and the third part contains the losses that occur in the triangle in later development years after the development year j inclusive their development until development year r or till they are closed.

With these results the best estimate loss reserve can be according to (3.40) expressed as

$$\mathrm{Res}_t=\sum_{j=0}^{r-1}\left[X^{O}_{t-j,j}\tilde{Q}^{D}_{t-j,j,r-1}+N_{t-j,j}\tilde{Q}^{A}_{t-j,j,r-1}\right]. \tag{3.48}$$

3.5.4 Loss reserve with risk margin

Loss reserve with risk margin can be expressed as

$$\mathrm{Res}_t=\sum_{j=0}^{r-1}\mathrm{E}\left[X_{t-j,\infty}-X_{t-j,j}\middle|I_{t-j,j}\right]+z_\alpha\sqrt{\mathrm{Var}\left[\sum_{j=0}^{r-1}(X_{t-j,\infty}-X_{t-j,j})\middle|I_{t-j,j}\right]}, \tag{3.49}$$

where z_α is quantile of e.g. normal distribution. If we use the so called quantile approach for risk margin, we can interpret the formula as NP1-approximation of the $\alpha-$quantile of the distribution of loss reserve. If we instead define risk margin as the margin that market participants would require for bearing risk (third part of the current exit value according to IASB, cost of capital approach) we can interpret the parameter z_α simply as a parameter expressing the level of prudence, which can be calibrated according to the market. Generally, we will assume that z_α is a chosen parameter of the model.

As we have already expressed the first part of the expression, we have to develop the formula for the second part. Due to the independence of the rows in the loss triangle we have

$$\mathrm{Var}\left[\sum_{j=0}^{r-1}(X_{t-j,\infty}-X_{t-j,j})\Big|I_{t-j,j}\right]=\sum_{j=0}^{r-1}\mathrm{Var}\left[X_{t-j,\infty}-X_{t-j,j}|I_{t-j,j}\right],$$

and for the variance in the sum we have

$$\mathrm{Var}\left[X_{t-j,\infty}-X_{t-j,j}\big|I_{t-j,j}\right]=\mathrm{Var}\left[X_{t-j,\infty}\big|I_{t-j,j}\right]=\mathrm{Var}\left[X_{t-j,r}\big|I_{t-j,j}\right].$$

Thus, loss reserve with risk margin has the form

$$\mathrm{Res}_t=\sum_{j=0}^{r-1}\mathrm{E}\left[X_{t-j,r}-X_{t-j,j}\big|I_{t-j,j}\right]+z_\alpha\sqrt{\sum_{j=0}^{r-1}\mathrm{Var}\left[X_{t-j,r}\big|I_{t-j,j}\right]}.$$

After putting in the previous result for the first part, this becomes

$$\mathrm{Res}_t=\sum_{j=0}^{r-1}\left[X^O_{t-j,j}\tilde{Q}^D_{t-j,j,r-1}+N_{t-j,j}\tilde{Q}^A_{t-j,j,r-1}\right]+z_\alpha\sqrt{\sum_{j=0}^{r-1}\mathrm{Var}[X_{t-j,r}|I_{t-j,j}]}.$$

The variance $\mathrm{Var}[X_{t-j,r}|I_{t-j,j}]$ can be expressed as

$$\mathrm{Var}\left[X_{t-j,r}\big|I_{t-j,j}\right]=\mathrm{E}\left[X^2_{t-j,r}\big|I_{t-j,j}\right]-\left(\mathrm{E}\left[X_{t-j,r}\big|I_{t-j,j}\right]\right)^2.$$

The expression for $\mathrm{E}[X_{t-j,r}|I_{t-j,j}]$ is given in (3.45). After a bit tedious calculation we get for $\mathrm{E}[X^2_{t-j,r}|I_{t-j,j}]$ the expression

$$\begin{aligned}\mathrm{E}\left[X^2_{t-j,r}\big|I_{t-j,j}\right] &= X^2_{t-j,j}+\tilde{H}^{(1)}_{t-j,j,r-1}N_{t-j,j}X^O_{t-j,j}+\tilde{H}^{(2)}_{t-j,j,r-1}N^2_{t-j,j}+\\ &\quad+\tilde{H}^{(3)}_{t-j,j,r-1}N_{t-j,j}+\tilde{H}^{(4)}_{t-j,j,r-1}X^{O(2)}_{t-j,j}+\tilde{H}^{(5)}_{t-j,j,r-1}(X^O_{t-j,j})^2+\\ &\quad+\tilde{H}^{(6)}_{t-j,j,r-1}X_{t-j,j}X^O_{t-j,j}+\tilde{H}^{(7)}_{t-j,j,r-1}N_{t-j,j}X_{t-j,j},\end{aligned}\tag{3.50}$$

where the coefficients $\tilde{H}^{(i)}_{t-j,j,r-1}$, $i=1,\ldots,7$, are composed only from parameters of the model. The expressions for the coefficients and the derivation are given in Appendix B.

With the NP2-approximation we would get for the $\alpha-$quantile of the distribution of loss reserve the relation

$$\mathrm{E}\left[\mathrm{RES}_t\right]+z_\alpha\sqrt{\mathrm{Var}\left[\mathrm{RES}_t\right]}+\frac{\mathrm{Thr}\left[\mathrm{RES}_t\right]}{6\mathrm{Var}\left[\mathrm{RES}_t\right]}\left(z^2_\alpha-1\right),$$

where

$$\mathrm{RES}_t=X_{t-j,\infty}-X_{t-j,j}\big|I_{t-j,j}$$

is the random variable denoting loss reserve distribution.

3.6 Parameter estimates

There are a lot of parameters that have to be estimated in the model based on individual loss information. In the simpler multiplicative model (2.2) that is based on annual losses only the parameters c_j and $c_j^{(2)}$ have to be estimated, which is $2r$ parameters. In the model based on individual losses there are the parameters b_j, λ (if we don't assume the dependence on s), d_j, μ_s, σ_s, q_j and $d_j^{(2)}$ that have to be computed, which gives $6r + 1$ parameters. If we use curves with three parameters to estimate the development factors b_j, d_j, the closing probabilities q_j, and parameters μ_s and σ_s, then we can reduce this to $r + 16$ parameters. Similarly the number of parameters in the model (2.2) can be reduced with a three-parameter curve for c_j from $2r$ to $r + 3$.

3.6.1 Mean number of losses occurring in the first development year

In the following text we assume that parameter λ_s doesn't depend on s. In practice the dependence on the accident year s can be modeled through a deterministic trend.

It is easy to show that the estimate of the number of losses in the 0-th development year by means of the moment method,

$$\hat{\lambda} = \frac{1}{r+1} \sum_{s=t-r}^{t} N_{s,0}, \tag{3.51}$$

is equal to the maximal likelihood estimate. We show that this estimate is the best unbiased linear estimate of λ, i.e. an unbiased linear estimate with the lowest variance among all unbiased linear estimates.

Proposition 1 *Estimate*

$$\hat{\lambda} = \frac{1}{r+1} \sum_{s=t-r}^{t} N_{s,0} \tag{3.52}$$

is the best unbiased linear estimate of λ.

Proof
We have to show that the estimate $\hat{\lambda}$ is unbiased and has among all estimates of the form

$$\tilde{\lambda} = \sum_{s=t-r}^{t} w_s N_{s,0}, \quad \sum_{s=t-r}^{t} w_s = 1,$$

the lowest variance.
First, we show that the estimate is unbiased. We compute the expected value and get

$$\mathrm{E}\left[\hat{\lambda}\right] = \mathrm{E}\left[\frac{1}{r+1} \sum_{s=t-r}^{t} N_{s,0}\right] = \frac{1}{r+1} \sum_{s=t-r}^{t} \mathrm{E}N_{s,0} = \frac{1}{r+1}(r+1)\lambda = \lambda.$$

To compute the variance of $\tilde{\lambda}$ we first express the second moment $\mathrm{E}[\tilde{\lambda}]^2$ and get

$$\mathrm{E}\left[\tilde{\lambda}\right]^2 = \mathrm{E}\left[\sum_{s=t-r}^{t} w_s N_{s,0}\right]^2 = \mathrm{E}\left[\sum_{s=t-r}^{t} w_s^2 N_{s,0}^2 + \sum_{s\neq s'} w_s w_{s'} N_{s,0} N_{s',0}\right].$$

Due to the independence of $N_{s,0}$ and $N_{s',0}$ for $s \neq s'$ we get

$$\mathrm{E}\left[\tilde{\lambda}\right]^2 = \sum_{s=t-r}^{t} w_s^2 \mathrm{E}N_{s,0}^2 + \sum_{s\neq s'} w_s w_{s'} \mathrm{E}N_{s,0}\mathrm{E}N_{s',0}.$$

Putting in for $\mathrm{E}N_{s,0}$ and $\mathrm{E}N_{s,0}^2$ gives

$$\mathrm{E}\left[\tilde{\lambda}\right]^2 = \sum_{s=t-r}^{t} w_s^2(\lambda+\lambda^2) + \sum_{s\neq s'} w_s w_{s'}\lambda^2 = \lambda^2\left(\sum_{s=t-r}^{t} w_s\right)^2 + \lambda\sum_{s=t-r}^{t} w_s^2 = \lambda^2 + \lambda\sum_{s=t-r}^{t} w_s^2.$$

For the variance of $\tilde{\lambda}$ it then follows

$$\mathrm{Var}\left[\tilde{\lambda}\right] = \mathrm{E}\left[\tilde{\lambda}\right]^2 - \left(\mathrm{E}\left[\tilde{\lambda}\right]\right)^2 = \lambda^2 + \lambda\sum_{s=t-r}^{t} w_s^2 - \lambda^2 = \lambda\sum_{s=t-r}^{t} w_s^2.$$

Since $\sum_{s=t-r}^{t} w_s = 1$, the weights w_s are not independent. In order to look for the minimum we have to substitute one of the weights, e.g. w_t, with

$$w_t = 1 - \sum_{s=t-r}^{t-1} w_s.$$

Putting this into the formula for the variance gives

$$\mathrm{Var}\left[\tilde{\lambda}\right] = \lambda\left(\sum_{s=t-r}^{t-1} w_s^2 + \left(1 - \sum_{s=t-r}^{t-1} w_s\right)^2\right) =$$

$$= \lambda\left(\sum_{s=t-r}^{t-1} w_s^2 + 1 - 2\sum_{s=t-r}^{t-1} w_s + \left(\sum_{s=t-r}^{t-1} w_s\right)^2\right).$$

To find the minimum we compute the derivation with respect to $w_{s'}$, $t-r \leq s' \leq t-1$ and get

$$\frac{\partial \mathrm{Var}\left[\tilde{\lambda}\right]}{\partial w_{s'}} = \lambda\left(2w_{s'} - 2 + 2\sum_{s=t+1-r}^{t-1} w_s\right) = 2\lambda(w_{s'} - w_t).$$

Putting the derivation equal to 0 we get the point in which $\mathrm{Var}(\tilde{\lambda})$ has its extreme,

$$w_{s'} = w_t, \quad s' = t-r, \cdots, t-1.$$

From the condition $\sum_{s=t-r}^{t} w_s = 1$ follows that $\mathrm{Var}(\tilde{\lambda})$ attains its extreme for the weights

$$w_s = \frac{1}{r+1}, \quad s = t-r, \cdots, t.$$

To verify that we have found a minimum we compute the second derivation, which gives

$$2\lambda + 2\lambda = 4\lambda \geq 0.$$

Thus, we've verified that $\mathrm{Var}(\tilde{\lambda})$ is at the point $w_s = 1/(r+1), \quad s = t-r, \cdots, t$ convex and has a minimum at this point.

□

3.6.2 Development factors for loss numbers

In the following text we show some properties of the simple chain ladder estimate of parameters b_j. From the expression

$$\mathrm{E}\left[\tilde{N}_{s,j+1} \middle| N_{s,j}\right] = b_j N_{s,j}$$

follows

$$\mathrm{E}\tilde{N}_{s,j+1} = \mathrm{E}\left[\mathrm{E}\left[\tilde{N}_{s,j+1} \middle| N_{s,j}\right]\right] = \mathrm{E}\left[b_j N_{s,j}\right] = b_j \mathrm{E}N_{s,j}.$$

Thus, for the parameters b_j holds

$$b_j = \frac{\mathrm{E}\tilde{N}_{s,j+1}}{\mathrm{E}N_{s,j}}.$$

A candidate for the estimate of parameters b_j would then be

$$\hat{b}_j = \frac{\frac{1}{r-j}\sum_{s=t-r}^{t-j} \tilde{N}_{s,j+1}}{\frac{1}{r-j}\sum_{s=t-r}^{t-j} N_{s,j}} = \frac{\sum_{s=t-r}^{t-j} \tilde{N}_{s,j+1}}{\sum_{s=t-r}^{t-j} N_{s,j}}, \quad j = 0, ..., r-1,$$

which is nothing else than the chain ladder estimate. This estimate can be transformed in the form

$$\hat{b}_j = \sum_{s=t-r}^{t-j} \left(\frac{1}{\sum_{\tilde{s}=t-r}^{t-j} N_{\tilde{s},j}}\right) \tilde{N}_{s,j+1} = \sum_{s=t-r}^{t-j} \left(\frac{N_{s,j}}{\sum_{\tilde{s}=t-r}^{t-j} N_{\tilde{s},j}}\right) \frac{\tilde{N}_{s,j+1}}{N_{s,j}},$$

where

$$\frac{N_{s,j}}{\sum_{\tilde{s}=t-r}^{t-j} N_{\tilde{s},j}}$$

are weights satisfying

$$\sum_{s=t-r}^{t-j} \left(\frac{N_{s,j}}{\sum_{\tilde{s}=t-r}^{t-j} N_{\tilde{s},j}}\right) = 1.$$

The following proposition summarizes the properties of the estimate $\hat{b}_j$.

Proposition 2 *Estimate*

$$\hat{b}_j = \sum_{s=t-r}^{t-j} \left(\frac{N_{s,j}}{\sum_{\tilde{s}=t-r}^{t-j} N_{\tilde{s},j}} \right) \frac{\tilde{N}_{s,j+1}}{N_{s,j}} = \frac{\sum_{s=t-r}^{t-j} \tilde{N}_{s,j+1}}{\sum_{s=t-r}^{t-j} N_{s,j}} \tag{3.53}$$

is an unbiased estimate of the parameters b_j. Moreover, estimate $\hat{b}_j$ has the lowest variance conditional on $\{N_{s,j}, s = t-r, ..., t-j\}$ among all unbiased linear estimates of the form

$$\tilde{b}_j = \sum_{s=t-r}^{t-j} w_s \frac{\tilde{N}_{s,j+1}}{N_{s,j}}, \quad \sum_{s=t-r}^{t-j} w_s = 1,$$

where w_s are weights independent of $\tilde{N}_{s,j+1}$.

Proof

First we prove the unbiasedness of the estimate $\hat{b}_j$.We compute the conditional expected value

$$\mathrm{E}\left[\hat{b}_j \middle| N_{s,j}, s = t-r, ..., t-j\right] = \mathrm{E}\left[\frac{\sum_{s=t-r}^{t-j} \tilde{N}_{s,j+1}}{\sum_{s=t-r}^{t-j} N_{s,j}} \middle| N_{s,j}, s = t-r, ..., t-j \right],$$

and get

$$\mathrm{E}\left[\hat{b}_j \middle| N_{s,j}, s = t-r, ..., t-j\right] = \frac{1}{\sum_{s=t-r}^{t-j} N_{s,j}} \sum_{s=t-r}^{t-j} \mathrm{E}\left[\tilde{N}_{s,j+1} \middle| N_{s,j}\right] =$$

$$= \frac{1}{\sum_{s=t-r}^{t-j} N_{s,j}} \sum_{s=t-r}^{t-j} b_j N_{s,j} = b_j.$$

For the unconditional expected value it then follows

$$\mathrm{E}\left[\hat{b}_j\right] = \mathrm{E}\left[E\left[\hat{b}_j \middle| N_{s,j}, s = t-r, ..., t-j\right]\right] = \mathrm{E}\left[b_j\right] = b_j.$$

Thus, $\hat{b}_j$ is an unbiased estimate of parameter b_j. Further, we show that $\hat{b}_j$ has among all unbiased estimates of the type

$$\tilde{b}_j = \sum_{s=t-r}^{t-j} w_s \frac{\tilde{N}_{s,j+1}}{N_{s,j}}, \quad \sum_{s=t-r}^{t-j} w_s = 1,$$

where w_s are independent of $\tilde{N}_{s,j+1}$, the lowest variance conditional on $\{N_{s,j}, s = t-r, ..., t-j\}$. In other words,

$$\mathrm{Var}\left[\hat{b}_j \middle| N_{s,j}, s = t-r, ..., t-j\right] \leq \mathrm{Var}\left[\tilde{b}_j \middle| N_{s,j}, s = t-r, ..., t-j\right].$$

Due to the unbiasedness of the estimates it is sufficient to show that

$$\mathrm{E}\left[\hat{b}_j^2 \middle| N_{s,j}, s = t-r, ..., t-j\right] \leq \mathrm{E}\left[\tilde{b}_j^2 \middle| N_{s,j}, s = t-r, ..., t-j\right].$$

Using the independence of $N_{s,j}$ and $N_{\tilde{s},j}$ for $\tilde{s} \neq s$ we can write

$$\mathrm{E}\left[\tilde{b}_j^2 \middle| N_{s,j}, s = t-r, ..., t-j\right] = \mathrm{E}\left[\left(\sum_{s=t-r}^{t-j} w_s \frac{\tilde{N}_{s,j+1}}{N_{s,j}}\right)^2 \middle| N_{s,j}, s = t-r, ..., t-j\right] =$$

$$= \mathrm{E}\left[\sum_{s=t-r}^{t-j} w_s^2 \frac{\tilde{N}_{s,j+1}^2}{N_{s,j}^2} + \sum_{s \neq \tilde{s}} w_s w_{\tilde{s}} \frac{\tilde{N}_{s,j+1}\tilde{N}_{\tilde{s},j+1}}{N_{s,j}N_{\tilde{s},j}} \middle| N_{s,j}, s = t-r, ..., t-j\right] =$$

$$= \sum_{s=t-r}^{t-j} w_s^2 \frac{\mathrm{E}\left[\tilde{N}_{s,j+1}^2 \middle| N_{s,j}\right]}{N_{s,j}^2} + \sum_{s \neq \tilde{s}} w_s w_{\tilde{s}} \frac{\mathrm{E}\left[\tilde{N}_{s,j+1} \middle| N_{s,j}\right] \mathrm{E}\left[\tilde{N}_{\tilde{s},j+1} \middle| N_{\tilde{s},j}\right]}{N_{s,j}N_{\tilde{s},j}} =$$

$$= \sum_{s=t-r}^{t-j} w_s^2 \frac{1}{N_{s,j}^2}\left(b_j N_{s,j} + b_j^2 N_{s,j}^2\right) + \sum_{s \neq \tilde{s}} w_s w_{\tilde{s}} b_j^2 =$$

$$= b_j \sum_{s=t-r}^{t-j} w_s^2 \frac{1}{N_{s,j}} + b_j^2 \sum_{s=t-r}^{t-j} w_s^2 + b_j^2 \sum_{s \neq \tilde{s}} w_s w_{\tilde{s}} = b_j \sum_{s=t-r}^{t-j} w_s^2 \frac{1}{N_{s,j}} + b_j^2 \left(\sum_{s=t-r}^{t-j} w_s\right)^2,$$

and due to $\sum_{s=t-r}^{t-j} w_s = 1$, we obtain

$$\mathrm{E}\left[\tilde{b}_j^2 \middle| N_{s,j}, s = t-r, ..., t-j\right] = b_j \sum_{s=t-r}^{t-j} w_s^2 \frac{1}{N_{s,j}} + b_j^2.$$

We have to show that this expression attains its maximum for the weights

$$w_s = \frac{N_{s,j}}{\sum_{\tilde{s}=t-r}^{t-j} N_{\tilde{s},j}}. \tag{3.54}$$

Since $b_j > 0$, it is sufficient to show that the expression

$$\sum_{s=t-r}^{t-j} w_s^2 \frac{1}{N_{s,j}} \tag{3.55}$$

attains its maximum for the weights (3.54).
Again, the parameters w_s are not independent due to $\sum_{s=t-r}^{t-j} w_s = 1$. We have to use the substitution

$$w_{t-j} = 1 - \sum_{s=t-r}^{t-j-1} w_s.$$

Putting in for w_{t-j} gives

$$\sum_{s=t-r}^{t-j} w_s^2 \frac{1}{N_{s,j}} = \sum_{s=t-r}^{t-j-1} w_s^2 \frac{1}{N_{s,j}} + w_{t-j}^2 \frac{1}{N_{t-j,j}} =$$

$$= \sum_{s=t-r}^{t-j-1} w_s^2 \frac{1}{N_{s,j}} + \left(1 - \sum_{s=t-r}^{t-j-1} w_s\right)^2 \frac{1}{N_{t-j,j}} =$$

$$= \sum_{s=t-r}^{t-j-1} w_s^2 \frac{1}{N_{s,j}} + \left(1 - 2\sum_{s=t-r}^{t-j-1} w_s + \left(\sum_{s=t-r}^{t-j-1} w_s\right)^2\right) \frac{1}{N_{t-j,j}}.$$

Deriving the expression with respect to w_{s_0}, $t-r \leq s_0 \leq t-j-1$, gives

$$2w_{s_0} \frac{1}{N_{s_0,j}} - 2\frac{1}{N_{t-j,j}} + 2\frac{1}{N_{t-j,j}} \sum_{s=t-r}^{t-j-1} w_s.$$

It is easy to see that this is equal to

$$2\left(\frac{w_{s_0}}{N_{s_0,j}} - \frac{w_{t-j}}{N_{t-j,j}}\right).$$

Putting this equal to 0 we get the point, in which the expression (3.55) has its extreme,

$$w_{s_0} = \frac{w_{t-j}}{N_{t-j,j}} N_{s_0,j}.$$

Summing up over all $s_0 = t-j, ..., t-r$ we see that

$$1 = \sum_{s_0=t-r}^{t-j} w_{s_0} = \frac{w_{t-j}}{N_{t-j,j}} \sum_{s_0=t-r}^{t-j} N_{s_0,j},$$

which gives

$$w_{t-j} = \frac{N_{t-j,j}}{\sum_{s_0=t-r}^{t-j} N_{s_0,j}}.$$

After similar considerations for all other $s = t-r, \cdots, t-j-1$, we get

$$w_s = \frac{N_{s,j}}{\sum_{s_0=t-r}^{t-j} N_{s_0,j}}.$$

Computing the second derivation of the expression (3.55) with respect to s_0 gives

$$\frac{2}{N_{s_0,j}} + \frac{2}{N_{t-j,j}} \geq 0.$$

Thus, we have verified that the extreme we've found is a minimum. In other words, we have verified that

$$\text{Var}\left[\tilde{b}_j \middle| N_{s,j}, s = t-r, ..., t-j\right]$$

attains at the point

$$w_s = \frac{N_{s,j}}{\sum_{s_0=t-r}^{t-j} N_{s_0,j}}, s = t-r, ..., t-j,$$

its minimum.

□

3.6.3 Moments of individual large losses occurring in the triangle for the first time

The assumption of the model for the losses occurring in the triangle for the first time in development year $l \leq j$ has the form

$$\mathrm{E}Y^{l}_{t-j,l,k} = m_{t-j},$$

for $j = 0, \cdots, r,\ k = 1, \cdots, \tilde{N}_{t-j,l}$. Since the distribution of $Y^{l}_{t-j,l,k}$ is generally not specified, we use for m_{t-j} the estimates by the moment method,

$$\hat{m}_{t-j} = \frac{\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{t-j,l}} Y^{l}_{t-j,l,k}}{\sum_{l=0}^{j} \tilde{N}_{t-j,l}}. \tag{3.56}$$

If we would like to include the dependence on the development year l, we can use instead of $\hat{m}_{t-j}$ the estimates

$$\hat{m}_{t-j,l} = \frac{1}{\tilde{N}_{t-j,l}} \sum_{k=1}^{\tilde{N}_{t-j,l}} Y^{l}_{t-j,l,k}. \tag{3.57}$$

Similarly, for the parameter s^2_{t-j} we use the moment estimates

$$\hat{s}^2_{t-j} = \frac{\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{t-j,l}} \left(Y^{l}_{t-j,l,k} - \hat{m}_{t-j}\right)^2}{\sum_{l=0}^{j} \tilde{N}_{t-j,l} - 1}. \tag{3.58}$$

3.6.4 Development factors of individual large losses

Taking only losses that are open at the end of the development year j we assume the multiplicative model

$$Y^{l}_{s,j+1,k} = Y^{l}_{s,j,k}\left(1 + (d_j - 1)\tilde{E}^{l}_{s,j+1,k}\right). \tag{3.59}$$

For the expected values we obtain

$$\mathrm{E}Y^{l}_{s,j+1,k} = d_j \mathrm{E}Y^{l}_{s,j,k},$$

which leads us to the chain ladder estimates

$$\hat{d}_j = \frac{\sum_{s=t-r}^{t-j} \sum_{l=0}^{j} \sum_{\substack{k=1 \\ \& J^{l}_{s,j,k}=1}}^{\tilde{N}_{s,l}} Y^{l}_{s,j+1,k}}{\sum_{s=t-r}^{t-j} \sum_{l=0}^{j} \sum_{\substack{k=1 \\ \& J^{l}_{s,j,k}=1}}^{\tilde{N}_{s,l}} Y^{l}_{s,j,k}}. \tag{3.60}$$

This can be expressed also as

$$\hat{d}_j = \sum_{s=t-r}^{t-j} \sum_{l=0}^{j} \sum_{\substack{k=1 \\ \& J^{l}_{s,j,k}=1}}^{\tilde{N}_{s,l}} \frac{Y^{l}_{s,j+1,k}}{Y^{l}_{s,j,k}} \left(\frac{Y^{l}_{s,j,k}}{\sum_{s=t-r}^{t-j} \sum_{l=0}^{j} \sum_{\substack{k=1 \\ \& J^{l}_{s,j,k}=1}}^{\tilde{N}_{s,l}} Y^{l}_{s,j,k}} \right),$$

which is a weighted mean of the individual ratios $Y^l_{s,j+1,k}/Y^l_{s,j,k}$. As the individual factors will be rather volatile we can in practice analyze the averaged factors,

$$\hat{d}_{s,j} = \frac{\sum_{l=0}^{j} \sum_{\substack{k=1 \\ \& J^l_{s,j,k}=1}}^{\tilde{N}_{s,l}} Y^l_{s,j+1,k}}{\sum_{l=0}^{j} \sum_{\substack{k=1 \\ \& J^l_{s,j,k}=1}}^{\tilde{N}_{s,l}} Y^l_{s,j,k}}.$$

We can then choose which sort of mean (arithmetic, geometric), which weights (equal, based on loss amounts, or more weight for last years) are the most convenient estimates of d_j. We can also decide to exclude some outlier factors or all factors prior to some year in which we know there has been a change in the payment pattern.

For long tail lines of business it can take even thirty or more years till the losses are fully developed. As we rarely have such a long data history, we need to estimate the development factors beyond the triangle. For this we can use the curves usually used for tail development factors in the reserving of non-life insurance companies, in particular

Exponential curve	$d_j - 1$	$=$	$\exp(a + bj)$,
Weibull curve	d_j	$=$	$1/(1 - \exp(-aj^b))$,
Power curve	d_j	$=$	a^{b^j},
Inverse Power curve (Sherman curve)	$d_j - 1$	$=$	$a(j + c)^b$.

In case of the exponential and the inverse power curve we fit $d_j - 1$ instead of d_j as these two curves are used for incremental development factors. More detailed introduction of tail development factors can be found e.g. in [26].

3.6.5 Development factors of small losses

An unbiased estimate of parameters c_j can be obtained by transforming the model

$$X^{sm}_{s,j+1} = X^{sm}_{s,j}\left(1 + (c_j - 1)E_{s,j+1}\right) \tag{3.61}$$

in the form

$$\ln\left(\frac{X^{sm}_{s,j+1} - X^{sm}_{s,j}}{X^{sm}_{s,j}}\right) = \ln(c_j - 1) + \ln E_{s,j+1},$$

which is, if we use the notation

$$y_{s,j+1} = \ln\left(\frac{X^{sm}_{s,j+1} - X^{sm}_{s,j}}{X^{sm}_{s,j}}\right), \quad x = 1, \quad \delta_j = \ln(c_j - 1), \quad e_{s,j+1} = \ln E_{s,j+1},$$

the form of linear regression,

$$y_{s,j+1} = \delta_j x + e_{s,j+1}, \quad \mathrm{E}[e_{s,j+1}] = 0.$$

An unbiased estimate of δ_j is then, according to the theory of linear regression,

$$\hat{\delta}_j = \frac{\sum x y_{s,j+1}}{\sum x^2} = \frac{\sum y_{s,j+1}}{r - j} = \ln\left(\prod_{s=t-r}^{t-j}\left(\frac{X^{sm}_{s,j+1} - X^{sm}_{s,j}}{X^{sm}_{s,j}}\right)^{1/(r-j)}\right),$$

which can be transformed back to obtain the estimate of $\hat{c}_j$,

$$\hat{c}_j = 1 + \exp(\hat{\delta}_j) = 1 + \prod_{s=t-r}^{t-j} \left(\frac{X^{sm}_{s,j+1} - X^{sm}_{s,j}}{X^{sm}_{s,j}} \right)^{1/(r-j)}.$$

In other words, a natural estimate of the incremental development factors $c_j - 1$ in the model (3.61) is the geometrical mean of the relative increments $\frac{X^{sm}_{s,j+1} - X^{sm}_{s,j}}{X^{sm}_{s,j}}$, $s = t-r, ..., t-j$. Another possibility to get an unbiased estimate of c_j is to compute the chain ladder estimates

$$\hat{c}_j = \frac{\sum_{s=t-r}^{t-j} X^{sm}_{s,j+1}}{\sum_{s=t-r}^{t-j} X^{sm}_{s,j}}.$$

3.6.6 Variances of error terms of development of small and individual large losses

For small losses we use directly the model (3.61) introduced in [29]. From the model we get for the error terms the relation

$$E_{s,j+1} = \frac{X^{sm}_{s,j+1} / X^{sm}_{s,j} - 1}{c_j - 1}, \quad s = t-r, \ldots, t-j.$$

For the variance we then have the estimate

$$\hat{s}^2_{E^{sm}} = \frac{1}{r-j} \sum_{s=t-r}^{t-j} \left(E_{s,j+1} - 1 \right)^2.$$

Similarly, for large losses we start from the model for loss severities,

$$Y^l_{s,j+1,k} = Y^l_{s,j,k} \left(1 + (d_j - 1) J^l_{s,j,k} \tilde{E}^l_{s,j+1,k} \right),$$

and get for the error terms

$$J^l_{s,j,k} \tilde{E}^l_{s,j+1,k} = \frac{Y^l_{s,j+1,k} / Y^l_{s,j,k} - 1}{d_j - 1},$$

for $s = t-r, \ldots, t-j$, $, l = 0, \ldots, j$, $, k = 1, \ldots, \tilde{N}_{s,j}$. The estimate of the variance of the error terms of large loss severities, needed for $d^{(2)}_j$, is then

$$\hat{s}^2_{E_j} = \frac{1}{n-1} \sum_{s=t-r}^{t-j} \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,j}} J^l_{s,j,k} \left(\tilde{E}^l_{s,j+1,k} - 1 \right)^2,$$

where n is the corresponding number of open large losses

$$n = \sum_{s=t-r}^{t-j} \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,j}} J^l_{s,j,k}.$$

3.6.7 Probabilities that a large loss will be closed

For the probabilities q_j that a large loss will be closed in development year j we use the empirical probabilities

$$\hat{q}_j = \frac{\sum_{s=t-r}^{t-j} \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} J^l_{s,j+1,k} \; \& J^l_{s,j,k}=1}{\sum_{s=t-r}^{t-j} \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} J^l_{s,j,k}}. \tag{3.62}$$

Chapter 4

Reinsurance

In this chapter we derive the expressions for loss reserve and the corresponding moments for ultimate net losses for the most important non-proportional reinsurance types. We end with the framework for optimization of reinsurance in respect to net result and required risk capital.

First, we give a brief overview of the main functions and types of reinsurance. A short introduction is given for example in [35]. A very deep description of reinsurance can be found in [36] and [25]. Then we derive the results for XL-reinsurance and SL-reinsurance. At the end we introduce a concept how reinsurance can be optimized with respect to risk capital and net economic result of the company.

4.1 Functions of reinsurance

Functions of reinsurance can be divided into the two groups of microeconomic and macroeconomic functions.

Microeconomic functions of reinsurance

One of the most important and, in fact, the original function of reinsurance is the **transfer of risk** to the reinsurer. In its original form it is the insurance (or underwriting) risk that is transferred. Insurance risk can be divided into

- Stochastic risk — the risk that arises from the stochasticity of losses — even if the expected value of losses is estimated correctly, the observed loss amount will differ from the estimate,
- Estimation error risk — the risk that we don't estimate the expected amount of future losses correctly, in particular the parameter risk, which arises due to the limited amount of observed losses that are used to estimate the parameters of the model, and
- Change risk — the risk that there will be an unpredictable change in the development of losses or that the trend will develop differently from the estimation.

Various types of reinsurance treaties transfer more strongly different types of insurance risk. There are also alternative reinsurance treaties that transfer other types of risk as well (financial risk, interest risk,..).

Another important function of reinsurance is **risk financing**. Reinsurance is one of the most important ways to reduce required risk capital (which is main concern of the first pillar of Solvency II, rating agencies, management,...). It allows the company to increase its underwriting capacity. Usually, the reinsurer has a larger and better diversified portfolio than the cedent and doesn't need so much capital for the same risk. This is typical e.g. for the natural catastrophe risk. Often, the transfer of the risk is cheaper for the cedent than holding its own risk capital. We discuss this topic deeper later in the section on insurance optimization.

An important function, that shouldn't be forgotten, is training, consulting and other **services of the reinsurer**. The reinsurer is working on a global market and, thus, he has more statistical data and more experience, especially with new products. Among the most often services of reinsurer used by the insurance companies we can mention the training of employees, portfolio analysis, software products, composition of reinsurance program or loss regulation.

Macroeconomic functions of reinsurance

Reinsurance enables private or small companies to undertake risks, that they wouldn't be able to bear alone. This gives place for innovation and economic development. Through reinsurance many risks can be quantified, which is important for many decisions on future projects. The most important function of reinsurance is the worldwide diversification, which enables to insure risks that are locally uninsurable. Due to the diversification the capital can be used in a more effective way.

In this work we concentrate on the risk transfer and risk financing function of reinsurance. We show how risk capital and risk adjusted profitability can be managed through reinsurance.

4.2 Main forms and types of reinsurance

Reinsurance treaties can be divided according to several aspects. One aspect is whether the reinsurer (or the cedent) is obliged to accept (or transfer) the risk. This divides reinsurance treaties into obligatory, facultative or facultative-obligatory. Another aspect is the art of the risk being transferred - there is classical (or traditional) reinsurance or alternative risk transfer. Even if we use traditional reinsurance, there are prospective treaties that reinsure future losses or retrospective treaties to reinsure the development of losses that have already happened or even been reported.

According to the system how the risks are reinsured reinsurance treaties can be divided into proportional and non-proportional treaties. As we need this later in this section we give a brief overview of the main types of reinsurance.

Proportional reinsurance - Quota Share

In a quota share treaty the reinsurer takes the same proportion α of each risk. Hence, for the individual losses ceded to the reinsurer holds

$$Y^{ri} = \alpha Y^{gr},$$

where Y^{gr} is the gross loss, Y^{ri} the loss paid by the reinsurer and α the proportion of the reinsurer. The premium is divided in the same way,

$$B^{ri} = \alpha B^{gr},$$

where B^{gr} is the gross premium (including cost loading) and B^{ri} the reinsurance premium. Additionally, the reinsurer usually pays to the cedent a reinsurance commission that is a proportion of B^{ri}, which should cover the acquisition costs of the cedent. Recently, the reinsurance commission is computed by the reinsurer independently of the real acquisition cost of the cedent. In an quota share treaty the reinsurer shares proportionally the change risk and the estimation error risk with the cedent.

Proportional reinsurance - Surplus

In a surplus treaty the proportion the reinsurer takes of each risk depends on the exposure of the risk, which is expressed through sum insured or probable maximum loss. Let's denote PML the probable maximum loss of a particular risk. Then the reinsurer takes 0 for losses bellow the so called line or priority C and

$$Y^{ri} = \alpha Y^{gr}, \quad \alpha = \frac{\min(\mathrm{PML} - C, Ck)}{S},$$

for losses above C, where k is the so called number of lines. Again, the reinsurer pays to the cedent reinsurance commission. Surplus treaty is a good instrument to homogenize a portfolio as the reinsurer takes a greater part of larger risks and nothing for risks bellow the level C.

Non-proportional reinsurance - Excess of Loss (XL)

In an excess of loss (XL) treaty the reinsurer takes from each loss the part exceeding the priority a up to some limit L,

$$Y^{ri} = \min\left(\max(Y^{gr} - a, 0), L\right).$$

XL-reinsurance is a good protection against large losses. There is also an XL-reinsurance on event basis, which takes as a basis events instead of losses. This is important for natural catastrophe risk where many small losses occur within one event.

Often, if the reinsurer pays a part of some loss, the cedent has to pay the so called reinstatement premium to 'renew' the reinsurance coverage. The reinstatement premium, B^{rei}, is usually defined as a proportion of the original reinsurance premium in dependence of the part of the coverage that has been used

$$B^{rei} = \frac{\min(Y^{gr} - a, L)}{L - a} B^{ri}.$$

Non-proportional reinsurance - Stop Loss (SL)

A stop loss (SL) treaty works in a similar way as an XL-treaty, but on an annual basis. The reinsurer covers the part of the annual losses exceeding a predefined priority $\bar{a}$,

$$X^{ri} = \min\left(\max(X^{gr} - \bar{a}, 0), L\right),$$

where X^{gr} is the annual amount of gross losses and X^{ri} the amount of the losses paid by reinsurer. Again, a payment of reinstatement premium in case the coverage is used can be negotiated.

Reinsurance program

The reinsurance treaties mentioned above can be combined in many ways and there are many aspects that an insurer has to consider when he is creating the reinsurance program for the next year. One of the most important issues is the portfolio structure. If the portfolio is exposed to large losses the cedent will probably choose an XL-treaty. If the portfolio is homogeneous but exposed to parameter or change risk, than a proportional treaty will give a better protection. We give an overview of the most important factors that play role by the optimization of the reinsurance program. The objective factors are the amount of capital, underwriting capacity, the line of business, portfolio composition and diversification, types of risks the cedent is exposed to, and the business policy of the cedent. The subjective factors are risk appetite, balance sheet management, type of responsibility (central vs decentral), quality of reinsurer, and the continuity of the business relation to the reinsurer.

In the section on optimization of reinsurance we concentrate on the aspect of risk transfer and risk capital as the most important criterion on the reinsurance. In the following sections we derive expressions for loss reserve and the corresponding required moments for the most important non-proportional reinsurance types. We will denote net losses with a tilde.

4.3 XL-reinsurance

In this section we derive expressions for loss reserve and moments of the involved random variables for net losses after XL-reinsurance. Net losses after XL-reinsurance are defined as

$$\tilde{Y}^l_{s,j,k} = \min(Y^l_{s,j,k}, a_s), \tag{4.1}$$

where a_s is the priority of the XL-reinsurance, which depends generally on the accident year. For simplicity we assume unlimited reinsurance cover. For limited cover with limit L_s we have to expand the model to

$$\tilde{Y}^l_{s,j,k} = Y^l_{s,j,k} - \min(Y^l_{s,j,k}, a_s + L_s) + \min(Y^l_{s,j,k}, a_s). \tag{4.2}$$

Number of losses remains for net losses the same, so we get for the net cumulative paid losses after XL-reinsurance the expression

$$\tilde{X}^{XL}_{s,j} = \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} \tilde{Y}^l_{s,j,k}.$$

In the further text we will often use the definition of the reinsurance relief effect (see 'Entlastunseffekt' in ([27])).

Definition 1 Reinsurance relief effect *is defined as*

$$r_X = \frac{\mathrm{E}\tilde{X}}{\mathrm{E}X}, \tag{4.3}$$

where X is a random variable denoting gross individual losses (XL-reinsurance, quota share, surplus) or annual losses (SL-reinsurance, quota share) and $\tilde{X}$ denotes the corresponding net losses.

For unlimited XL-reinsurance we then have

$$r_X(a) = \frac{\mathrm{E}[\min(X, a)]}{\mathrm{E}X}, \tag{4.4}$$

where a is the priority of the XL-reinsurance. It's easy to show the following properties of XL-reinsurance relief effect (see [27]).

Proposition 3 *XL-reinsurance relief effect $r_X(a)$ has the following properties*

1. $r_X(0) = 0,\ r_X(v) = 1,$
2. $r'_X(a) = (1 - F(a))/\mathrm{E}X,$
3. $r'_X(a) > 0$ *for* $a < v,$
4. $r'_X(0) = 1/\mathrm{E}X,\ r'_X(v) = 0,$
5. r_X *is concave function of* a, *i.e.* $r''_X(a) \leq 0,$

where $v = \sup\{x | F(x) < 1\}$ and F is distribution function of X.

For higher moments the definition can be generalized in the following way (with the same notation).

Definition 2 Reinsurance relief effect of n-th power, $n = 2, 3, \cdots,$ *is defined as*

$$r_X^{(n)} = \frac{\mathrm{E}\tilde{X}^n}{\mathrm{E}X^n}. \tag{4.5}$$

For XL-reinsurance we then have

$$r_X^{(n)}(a) = \frac{\mathrm{E}[\min(X, a)]^n}{\mathrm{E}X^n}, \tag{4.6}$$

where a is the priority of the XL-reinsurance.

It is easy to show that for Pareto-distributed variable X with parameters α and θ we have

$$\mathrm{E}\left[\min\left(X, a\right)\right]^n = \frac{\alpha\theta^n}{\alpha - n} - \frac{n\theta^\alpha}{(\alpha - n)x^{\alpha - n}}, \quad x > \theta. \tag{4.7}$$

For log-normal distributed variable X we have for the expected value $\mathrm{E}\left[\min\left(X, a\right)\right]^n$, the following proposition.

Proposition 4 *Let the variable X have a log-normal distribution with parameters μ and σ. Then for $n = 1, 2, 3, \ldots$ it holds*

$$\mathrm{E}\left[\min\left(X, a\right)\right]^n = \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right)\Phi\left(\frac{\log(a) - \mu - n\sigma^2}{\sigma}\right) + a^n\left(1 - \Phi\left(\frac{\log(a) - \mu}{\sigma}\right)\right), \tag{4.8}$$

where Φ is the distribution function of the standard normal distribution.

Proof
Let's write

$$\mathrm{E}\left[\min\left(X, a\right)\right]^n = \int_{-\infty}^{a} x^n \frac{1}{\sqrt{2\pi}x\sigma} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}} dx + a^n(1 - F(a)),$$

where F is distribution function of log-normal distribution, in other words,

$$F(a) = \Phi\left(\frac{\log(a) - \mu}{\sigma}\right).$$

Hence it is sufficient to show that the integral in the first part of the expression is equal to

$$\exp\left(n\mu + \frac{n^2\sigma^2}{2}\right)\Phi\left(\frac{\log(a) - \mu - n\sigma^2}{\sigma}\right).$$

To do this, we use the substitution

$$y = \frac{\log(x) - \mu - n\sigma^2}{\sigma}.$$

Thus we get

$$x = \exp\left(\sigma y + \mu + n\sigma^2\right),$$

and

$$dx = \sigma \exp\left(\sigma y + \mu + n\sigma^2\right).$$

With this substitution and after an easy rearrangement we get

$$\int_{-\infty}^{a} x^n \frac{1}{\sqrt{2\pi}x\sigma} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}} dx = e^{n\mu + \frac{n^2\sigma^2}{2}} \int_{-\infty}^{\frac{\log(a)-\mu-n\sigma^2}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy,$$

which proves the proposition as the last integrand is the density of the standard normal distribution.

□

Figure 4.1 shows XL-reinsurance relief effect for the case of Pareto distribution with values of parameter α between 2 and 3. We can see that with decreasing value of the parameter α the functions converge more slowly to 1.

In the following subsection we develop formulas for net loss reserve after an XL-reinsurance.

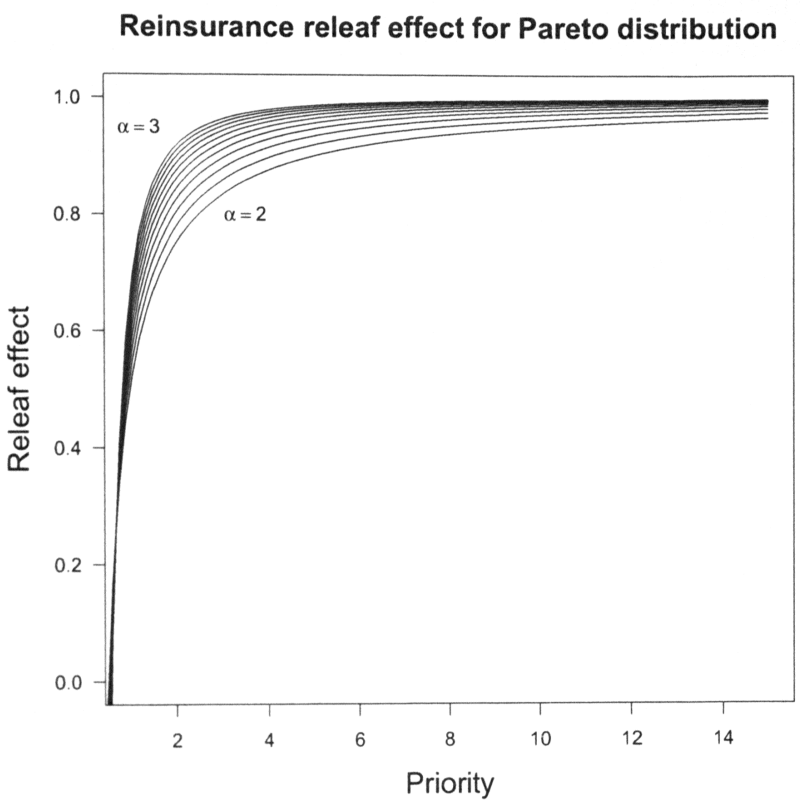


Figure 4.1: Reinsurance relief effect of Pareto distribution for various values of parameter α

4.3.1 Loss reserve – best estimate

Best estimate of net loss reserve after XL-reinsurance has the form

$$\mathrm{Res}_t^{XL} = \sum_{j=0}^{r-1} \mathrm{E}\left[\tilde{X}_{t-j,r}^{XL} - \tilde{X}_{t-j,j}^{XL} \middle| I_{t-j,j}\right], \tag{4.9}$$

where

$$\tilde{X}_{t-j,i}^{XL} = \sum_{l=0}^{i} \sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}_{t-j,i,k}^{l}$$

are cumulative annual net paid losses after XL-reinsurance from accident year $t - j$ at development year i. In case of an unlimited XL-reinsurance the individual net losses are defined as

$$\tilde{Y}_{s,j,k}^{l} = \min(Y_{s,j,k}^{l}, a_s).$$

As for losses that are at $I_{t-j,j}$ already closed holds

$$\tilde{Y}_{t-j,r,k}^{l} = \tilde{Y}_{t-j,j,k}^{l},$$

because these losses won't change any more (we don't assume that a closed loss can be reopened), loss reserve for these losses will be 0 and thus we will concentrate in the further computation of the net loss reserve on losses that are at $I_{t-j,j}$ open.

To express $\mathrm{E}[\tilde{X}^{XL}_{t-j,r}|I_{t-j,j}]$ we split $\tilde{X}^{XL}_{t-j,r}$ into the part with losses that have already had payment up to the development year j and losses that will have the first payment in the development year $j+1$ or later and write

$$\mathrm{E}\left[\tilde{X}^{XL}_{t-j,r}\middle|I_{t-j,j}\right] = \mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\tilde{Y}^{l}_{t-j,r,k}\middle|I_{t-j,j}\right]+\mathrm{E}\left[\sum_{l=j+1}^{r}\sum_{k=1}^{\tilde{N}_{t-j,l}}\tilde{Y}^{l}_{t-j,r,k}\middle|I_{t-j,j}\right] = R^1_t + R^2_t. \tag{4.10}$$

In case of an unlimited XL-reinsurance R^1_t is equal to

$$R^1_t = \mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\min\left(Y^{l}_{t-j,r,k}, a_{t-j}\right)\middle|I_{t-j,j}\right].$$

Using the multiplicative model we get

$$R^1_t = \mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\min\left(Y^{l}_{t-j,j,k}\prod_{i=j}^{r-1}\left(1+(d_i-1)J^{l}_{t-j,i,k}\tilde{E}^{l}_{t-j,i+1,k}\right), a_{t-j}\right)\middle|I_{t-j,j}\right].$$

As the variables $\tilde{N}_{t-j,l}, l = 0,\ldots,j$ are under $I_{t-j,j}$ known, we can put the expected value into the sum and write

$$R^1_t = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\mathrm{E}\left[\min\left(Y^{l}_{t-j,j,k}\prod_{i=j}^{r-1}\left(1+(d_i-1)J^{l}_{t-j,i,k}\tilde{E}^{l}_{t-j,i+1,k}\right), a_{t-j}\right)\middle|I_{t-j,j}\right],$$

which can be expressed as

$$R^1_t = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}K^{a}_{j,r-1}(Y^{l}_{t-j,j,k}),$$

where

$$K^{a}_{j,r-1}(Y^{l}_{t-j,j,k}) = \mathrm{E}\left[\min\left(Y^{l}_{t-j,j,k}\prod_{i=j}^{r-1}\left(1+(d_i-1)J^{l}_{t-j,i,k}\tilde{E}^{l}_{t-j,i+1,k}\right), a_{t-j}\right)\middle|I_{t-j,j}\right]. \tag{4.11}$$

Dividing the minimum through $Y^{l}_{t-j,j,k} > 0$ we get

$$K^{a}_{j,r-1}(Y^{l}_{t-j,j,k}) = Y^{l}_{t-j,j,k}\mathrm{E}\left[\min\left(\prod_{i=j}^{r-1}\left(1+(d_i-1)J^{l}_{t-j,i,k}\tilde{E}^{l}_{t-j,i+1,k}\right), \frac{a_{t-j}}{Y^{l}_{t-j,j,k}}\right)\middle|I_{t-j,j}\right].$$

For losses that are at $I_{t-j,j}$ closed holds

$$K^{a}_{j,r-1}(Y^{l}_{t-j,j,k}) = \tilde{Y}^{l}_{t-j,j,k}.$$

For losses that are at $I_{t-j,j}$ open we denote the multiplicative development

$$W^l_{t-j,j,r,k} = \prod_{i=j}^{r-1} \left(1 + (d_i - 1) J^l_{t-j,i,k} \tilde{E}^l_{t-j,i+1,k}\right), \tag{4.12}$$

and get

$$K^a_{j,r-1}(Y^l_{t-j,j,k}) = Y^l_{t-j,j,k} \mathrm{E}\left[\min\left(W^l_{t-j,j,r,k}, \frac{a_{t-j}}{Y^l_{t-j,j,k}}\right) \middle| I_{t-j,j}, J^l_{t-j,j,k} = 1\right].$$

Using the definition of reinsurance relief effect this can be expressed as

$$K^a_{j,r-1}(Y^l_{t-j,j,k}) = Y^l_{t-j,j,k} \mathrm{E}\left[W^l_{t-j,j,r,k} | J^l_{t-j,j,k} = 1\right] r_{W^l_{t-j,j,r,k}|J^l_{t-j,j,k}=1}\left(\frac{a_{t-j}}{Y^l_{t-j,j,k}}\right).$$

Note that for the variable $W^l_{t-j,j,r,k}$ we have, due to $d_i \geq 1$, $J^l_{t-j,i,k} \geq 0$ and $\tilde{E}^l_{t-j,i+1,k} \geq 0$, the inequality

$$W^l_{t-j,j,r,k} \geq 1.$$

In order to estimate the reinsurance relief effect of $W^l_{t-j,j,r,k}$ we can approximate its distribution. One possibility would be to approximate the distribution of $W^l_{t-j,j,r,k}$ by Pareto distribution. Another possibility would be to shift the variable to 0 by subtracting 1 and approximate this shifted variable by log-normal distribution. In this case we would use for $K^a_{j,r-1}(Y^l_{t-j,j,k})$ the form

$$Y^l_{t-j,j,k}\left(\mathrm{E}\left[W^l_{t-j,j,r,k} - 1 | J^l_{t-j,j,k} = 1\right] r_{W^l_{t-j,j,r,k}-1|J^l_{t-j,j,k}=1}\left(\frac{a_{t-j}}{Y^l_{t-j,j,k}} - 1\right) + 1\right).$$

A goodness of fit test and a graphical representation should help us to decide, which approximation is the most appropriate. In both cases we need, in order to use the approximation, to express the conditional mean value $\mathrm{E}[W^l_{t-j,j,k} | J^l_{t-j,j,k} = 1]$, $l \leq j$ and in the case of log-normal approximation the variance or the second moment as well.

For the first two conditional moments we get

$$\mathrm{E}\left[W^l_{t-j,j,r,k} | J^l_{t-j,j,k} = 1\right] = \sum_{h=j+1}^{r} D_{j,h-1} Q_{j+1,h-1} q_h,$$

and

$$\mathrm{E}\left[(W^l_{t-j,j,r,k})^2 | J^l_{t-j,j,k} = 1\right] = \sum_{h=j+1}^{r} D^{(2)}_{j,h-1} Q_{j+1,h-1} q_h,$$

where

$$D^{(2)}_{i,j} = d^{(2)}_i \cdots d^{(2)}_j,$$

and $\sum_i^{i-1} := 0$, $q_r := 1$, $Q_{i+1,i} := 1$, so that the relation holds even for $j = r - 1$. The derivation is given in Appendix C.

The second term in (4.10), R_t^2, contains losses that will occur in the loss triangle in future. The expression is an expected value of a compound distribution. Due to the independence of $Y^l_{t-j,l,k}$, and $\tilde{E}^l_{t-j,i+1,k}$, $i = l, \ldots, r-1$ and hence $Y^l_{t-j,r,k}$ on $I_{t-j,j}$, we can write

$$R_t^2 = \mathrm{E}\left[\sum_{l=j+1}^{r}\sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}^l_{t-j,r,k}\middle| I_{t-j,j}\right] = \sum_{l=j+1}^{r} \mathrm{E}\left[\tilde{N}_{t-j,l}|N_{t-j,j}\right]\mathrm{E}\left[\min\left(Y^l_{t-j,r,k}, a_{t-j}\right)\right].$$

Putting in for $\mathrm{E}[\tilde{N}_{t-j,l}|N_{t-j,j}]$ we obtain

$$R_t^2 = N_{t-j,j}\sum_{l=j+1}^{r} b_{l-1}\prod_{i=j}^{l-2}(1+b_i)\mathrm{E}\left[\min\left(Y^l_{t-j,r,k}, a_{t-j}\right)\right]. \tag{4.13}$$

Using the reinsurance relief effect, the expected value $\mathrm{E}[\min(Y^l_{t-j,r,k}, a_{t-j})]$, $l \geq j+1$ can be expressed as

$$\mathrm{E}\left[\min\left(Y^l_{t-j,r,k}, a_{t-j}\right)\right] = \mathrm{E}\left[Y^l_{t-j,r,k}\right] r_{Y^l_{t-j,r,k}}(a_{t-j}).$$

The term $r_{Y^l_{t-j,r,k}}(a_{t-j})$ can be approximated e.g. by lognormal distribution. To be able to do this, we need again to express the first two moments. For the first moment we have

$$\mathrm{E}\left[Y^l_{t-j,r,k}\right] = \mathrm{E}\left[Y^l_{t-j,l,k}\prod_{i=l}^{r-1}\left(1+(d_i-1)J^l_{t-j,i,k}\tilde{E}^l_{t-j,i+1,k}\right)\right],$$

where if we define $\prod_r^{r-1} \ldots := 1$, the formula holds even for $l = r$. Due to the independence of $Y^l_{t-j,l,k}$ and $J^l_{t-j,i,k}$ for $i = l, \ldots, r-1$, this can be expressed as

$$\mathrm{E}\left[Y^l_{t-j,l,k}\right]\mathrm{E}\left[\prod_{i=l}^{r-1}\left(1+(d_i-1)J^l_{t-j,i,k}\tilde{E}^l_{t-j,i+1,k}\right)\right].$$

Similar to the case of R_t^1 we define

$$W^l_{t-j,l,r,k} = \prod_{i=l}^{r-1}\left(1+(d_i-1)J^l_{t-j,i,k}\tilde{E}^l_{t-j,i+1,k}\right) \tag{4.14}$$

the multiplicative development, and get

$$\mathrm{E}\left[Y^l_{t-j,r,k}\right] = m_{t-j,l}\mathrm{E}\left[W^l_{t-j,l,r,k}\right].$$

In a similar way for the second moment holds

$$\mathrm{E}\left[Y^l_{t-j,r,k}\right]^2 = m^{(2)}_{t-j,l}\mathrm{E}\left[W^l_{t-j,l,r,k}\right]^2.$$

In the previous text we have already derived the formulas for $\mathrm{E}[W^l_{t-j,j,r,k}]^i$, $i = 1, 2$ for $l \leq j$. In a similar way we can derive also the expression for $\mathrm{E}[W^l_{t-j,l,r,k}]$, $l > j$, and get

$$\mathrm{E}\left[W^l_{t-j,l,r,k}\right] = \sum_{h=l}^{r} D_{l,h-1}Q_{l,h-1}q_h,$$

where we define $q_r := 1, \quad Q_{l,l-1} := 1, \quad D_{l,l-1} := 1$. Note, that there is a slight difference in the first index of $Q_{l,h-1}$, being l and not $l+1$ as we could expect based on the case of $W^l_{t-j,j,r,k}$. This is due to the condition in the case of $W^l_{t-j,j,r,k}$. Similarly, for the second moment holds

$$\mathrm{E}\left[W^l_{t-j,l,r,k}\right]^2 = \sum_{h=l}^{r} D^{(2)}_{l,h-1} Q_{l,h-1} q_h.$$

To summarize, we have for R^2_t the relation

$$R^2_t = N_{t-j,j} \sum_{l=j+1}^{r} b_{l-1} \prod_{i=j}^{l-2} (1+b_i) m_{t-j,l} \mathrm{E}\left[W^l_{t-j,l,r,k}\right] r_{Y^l_{t-j,r,k}}(a_{t-j}), \tag{4.15}$$

which can be, with the notation used in previous chapter, expressed as

$$R^2_t = N_{t-j,j} \sum_{l=j+1}^{r} A_{t-j,j,l-1} \mathrm{E}\left[W^l_{t-j,l,r,k}\right] r_{Y^l_{t-j,r,k}}(a_{t-j}). \tag{4.16}$$

In the following text we denote

$$v^a_{t-j,j,l-1,r} = A_{t-j,j,l-1} \mathrm{E}\left[W^l_{t-j,l,r,k}\right] r_{Y^l_{t-j,r,k}}(a_{t-j}). \tag{4.17}$$

The coefficient $v^a_{t-j,j,l-1,r}$ is composed only from the parameters of the model. With this notation we can express the reserve built by the insurance company at time t as

$$\mathrm{Res}^{XL}_t = \sum_{j=0}^{r-1} \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{t-j,l}} \left[K^a_{j,r-1}(Y^l_{t-j,j,k}) - \tilde{Y}^l_{t-j,j,k}\right] + \sum_{j=0}^{r-1} N_{t-j,j} \sum_{l=j+1}^{r} v^a_{t-j,j,l-1,r}. \tag{4.18}$$

Note that for losses closed at $I_{t-j,j}$ we have $K^a_{j,r-1}(Y^l_{t-j,j,k}) = \tilde{Y}^l_{t-j,j,k}$, so that these terms in the sum are equal to 0.

For an XL-reinsurance with limit L the individual net losses are according to (4.2) defined as

$$\tilde{Y}^l_{s,j,k} = Y^l_{s,j,k} - \min(Y^l_{s,j,k}, a_s + L_s) + \min(Y^l_{s,j,k}, a_s).$$

It can be seen easily that in the expression for R^1_t we would have to replace

$$r_{W^l_{t-j,j,r,k}|J^l_{t-j,j,k}=1}\left(\frac{a_{t-j}}{Y^l_{t-j,r,k}}\right)$$

with

$$r_{W^l_{t-j,j,r,k}|J^l_{t-j,j,k}=1}\left(\frac{a_{t-j}}{Y^l_{t-j,r,k}}\right) + 1 - r_{W^l_{t-j,j,r,k}|J^l_{t-j,j,k}=1}\left(\frac{a_{t-j}+L_{t-j}}{Y^l_{t-j,r,k}}\right),$$

and in the expression for R^2_t we would have to replace

$$r_{Y^l_{t-j,r,k}}(a_{t-j})$$

with

$$r_{Y^l_{t-j,r,k}}(a_{t-j}) + 1 - r_{Y^l_{t-j,r,k}}(a_{t-j}+L_{t-j}).$$

As the reinsurance relief effect $r_{Y^l_{t-j,r,k}}(a_{t-j}+L_{t-j})$ converges to 1 when the limit L_{t-j} goes to infinity, the formulas converge to the unlimited case.

4.3.2 Comparison to the formula for gross loss reserve

In this chapter we derived formula for net loss reserve after XL-reinsurance. In the limit case with infinite priority the reinsurance relief effect converges to 1 and with the notation

$$X^O_{t-j,j} = \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{t-j,l}} Y^l_{t-j,j,k} J^l_{t-j,j,k},$$

and putting in for $\mathrm{E}[W^l_{t-j,j,r,k} | J^l_{t-j,j,k} = 1]$ and $\mathrm{E}[W^l_{t-j,l,r,k}]$ the formula (4.18) becomes

$$\mathrm{Res}^{XL}_t = \sum_{j=0}^{r-1} X^O_{t-j,j} \left(\sum_{h=j+1}^{r} D_{j,h-1} Q_{j+1,h-1} q_h - 1 \right) +$$

$$+ \sum_{j=0}^{r-1} N_{t-j,j} \sum_{l=j+1}^{r} \left(A_{t-j,j,l-1} \sum_{h=l}^{r} D_{l,h-1} Q_{l,h-1} q_h \right), \tag{4.19}$$

which is in this case the formula for gross loss reserve. We have already derived a formula for gross loss reserve in the previous chapter. The formula for gross loss reserve was

$$\mathrm{Res}_t = X^O_{t-j,j} \tilde{Q}^D_{t-j,j,r-1} + N_{t-j,j} \tilde{Q}^A_{t-j,j,r-1}, \tag{4.20}$$

where

$$\tilde{Q}^D_{t-j,j,r-1} = \sum_{h=j}^{r-1} (d_h - 1) D_{j,h-1} Q_{j+1,h}, \tag{4.21}$$

$$\tilde{Q}^A_{t-j,j,r-1} = \sum_{h=j+1}^{r-1} (d_h - 1) \tilde{Q}^\beta_{t-j,j+1,h} + \sum_{h=j}^{r-1} A_{t-j,j,h}, \tag{4.22}$$

and

$$\tilde{Q}^\beta_{t-j,j+1,h} = \sum_{l=j+1}^{h} D_{l,h-1} Q_{l,h} A_{t-j,j,l-1}. \tag{4.23}$$

In the following text we show that these two expressions for gross loss reserve are equal. First we compare the terms containing $X^O_{t-j,j}$. In the model for gross losses the factor is, due to the fact that for the cumulative development factors holds $D_{j,h} = d_h D_{j,h-1}$, equal to

$$\tilde{Q}^D_{t-j,j,r-1} = \sum_{h=j}^{r-1} (d_h - 1) D_{j,h-1} Q_{j+1,h} = \sum_{h=j}^{r-1} D_{j,h} Q_{j+1,h} - \sum_{h=j}^{r-1} D_{j,h-1} Q_{j+1,h}. \tag{4.24}$$

Shifting the index in the first sum we get

$$\tilde{Q}^D_{t-j,j,r-1} = \sum_{h=j+1}^{r} D_{j,h-1} Q_{j+1,h-1} - \sum_{h=j}^{r-1} D_{j,h-1} Q_{j+1,h}.$$

If we put the sums together for $h = j+1, ..., r-1$, and write the terms for $h = j$ and $h = r$ separately, the expression becomes

$$\tilde{Q}^{D}_{t-j,j,r-1} = \sum_{h=j+1}^{r-1} D_{j,h-1}\left(Q_{j+1,h-1} - Q_{j+1,h}\right) + D_{j,r-1}Q_{j+1,r-1} - D_{j,j-1}Q_{j+1,j}. \quad (4.25)$$

It's easy to see that

$$Q_{j+1,h-1} - Q_{j+1,h} = Q_{j+1,h-1}q_h.$$

Additionally, we have defined $D_{j,j-1} = 1$ and $Q_{j+1,j} = 1$. Putting all this into (4.25) we get

$$\tilde{Q}^{D}_{t-j,j,r-1} = \sum_{h=j+1}^{r-1} D_{j,h-1}Q_{j+1,h-1}q_h + D_{j,r-1}Q_{j+1,r-1} - 1.$$

This can be, with the previous definition $q_r = 1$, rewritten as

$$\sum_{h=j+1}^{r} D_{j,h-1}Q_{j+1,h-1}q_h - 1,$$

which is equal to the expression in the term with $X^{O}_{t-j,j}$ in (4.19).
To compare the terms containing $N_{t-j,j}$, we use similar consideration to the previous case. Putting (4.23) into (4.22) we get

$$\tilde{Q}^{A}_{t-j,j,r-1} = \sum_{h=j+1}^{r-1}(d_h - 1)\sum_{l=j+1}^{h} D_{l,h-1}Q_{l,h}A_{t-j,j,l-1} + \sum_{h=j}^{r-1} A_{t-j,j,h},$$

which is equal to (again using $D_{j,h} = d_h D_{j,h-1}$)

$$\sum_{h=j+1}^{r-1}\left(\sum_{l=j+1}^{h} D_{l,h}Q_{l,h}A_{t-j,j,l-1} - \sum_{l=j+1}^{h} D_{l,h-1}Q_{l,h}A_{t-j,j,l-1}\right) + \sum_{l=j}^{r-1} A_{t-j,j,h}.$$

Switching the sums in the last part and shifting the index in the last sum we get

$$\sum_{l=j+1}^{r} A_{t-j,j,l-1}\left(\sum_{h=l}^{r-1} D_{l,h}Q_{l,h} - \sum_{h=l}^{r-1} D_{l,h-1}Q_{l,h}\right) + \sum_{h=j+1}^{r} A_{t-j,j,h-1}.$$

Shifting the index in the first inner sum gives

$$\sum_{l=j+1}^{r} A_{t-j,j,l-1}\left(\sum_{h=l+1}^{r} D_{l,h-1}Q_{l,h-1} - \sum_{h=l}^{r-1} D_{l,h-1}Q_{l,h} + 1\right).$$

Now we put the sums in the subtraction for $h = l+1, ..., r-1$ together and let the terms with $h = l$ and $h = r$ stay separately and get

$$\sum_{h=j+1}^{r} A_{t-j,j,l-1}\left(D_{l,r-1}Q_{l,r-1} + \sum_{h=l+1}^{r-1} D_{l,h-1}(Q_{l,h-1} - Q_{l,h}) - D_{l,l-1}Q_{l,l} + 1\right).$$

It's easy to see that $D_{l,l-1}Q_{l,l}$ is equal to $1 - q_l$, because $D_{l,l-1} = 1$. Moreover, $Q_{l,h-1} - Q_{l,h} = Q_{l,h-1}q_h$. The expression then becomes

$$\sum_{h=j+1}^{r} A_{t-j,j,l-1}\left(D_{l,r-1}Q_{l,r-1} + \sum_{h=l+1}^{r-1} D_{l,h-1}Q_{l,h-1}q_h + q_l\right).$$

With the denotation from the previous sections ($q_r = 1, \ldots$), this can be rewritten as

$$\sum_{h=j+1}^{r} A_{t-j,j,l-1}\left(\sum_{h=l}^{r} D_{l,h-1}Q_{l,h-1}q_h\right),$$

which is equal to the expression in the term by $N_{t-j,j}$ in (4.19).

4.3.3 Loss reserve with risk margin

Net loss reserve with risk margin has, due to the independence of the accident years, the form

$$\mathrm{Res}_t^{XL} = \sum_{j=0}^{r-1} \mathrm{E}\left[\tilde{X}_{t-j,r}^{XL} - \tilde{X}_{t-j,j}^{XL} \middle| I_{t-j,j}\right] + z_\alpha \sqrt{\sum_{j=0}^{r-1} \mathrm{Var}\left[\tilde{X}_{t-j,r}^{XL} \middle| I_{t-j,j}\right]}. \tag{4.26}$$

The first part has been already derived in the formula for best estimate of net loss reserve. For the conditional variance of $\tilde{X}_{t-j,r}^{XL}$ we have

$$\mathrm{Var}\left[\tilde{X}_{t-j,r}^{XL} \middle| I_{t-j,j}\right] = \mathrm{E}\left[\left(\tilde{X}_{t-j,r}^{XL}\right)^2 \middle| I_{t-j,j}\right] - \left(\mathrm{E}\left[\tilde{X}_{t-j,r}^{XL} \middle| I_{t-j,j}\right]\right)^2.$$

As the variance of losses that are at $I_{t-j,j}$ closed is 0, we compute the second moment only for losses that are at $I_{t-j,j}$ open. The expression for $\mathrm{E}[(\tilde{X}_{t-j,r}^{XL})^2 | I_{t-j,j}]$ is developed in Appendix D. After putting the results together and some simplifying we get

$$\mathrm{Var}\left[\tilde{X}_{t-j,r}^{XL} \middle| I_{t-j,j}\right] = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} K_{j,r-1}^{a(2)}(Y_{t-j,j,k}^{l}) - \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} \left(K_{j,r-1}^{a}(Y_{t-j,j,k}^{l})\right)^2 +$$

$$+ N_{t-j,j}\sum_{l=j+1}^{r} v_{t-j,j,l-1,r}^{a(2)} - N_{t-j,j}^2 \sum_{l=j+1}^{r}\left(v_{t-j,j,l-1,r}^{a}\right)^2,$$

where we take only the terms with losses that are open at $I_{t-j,j}$.

4.4 SL-reinsurance

With SL-reinsurance the overall paid net losses have the form

$$\tilde{X}_{s,j}^{SL} = \min\left(X_{s,j}, \bar{a}_s\right) = \min\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y_{s,j,k}^{l}, \bar{a}_s\right), \tag{4.27}$$

where $\bar{a}_s$ is the priority of the SL-reinsurance, which depends generally on the accident year. In order to distinguish the priority of a SL-reinsurance from the priority of an XL-reinsurance we denote the priority of the SL-reinsurance with a bar. If we have a reinsurance program with an XL-treaty followed by a SL-reinsurance then net losses would have the form

$$\tilde{X}_{s,j}^{XL,SL} = \min\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} \min\left(Y_{s,j,k}^{l}, a_s\right), \bar{a}_s\right). \tag{4.28}$$

4.4.1 Loss reserve – best estimate

Best estimate of net loss reserve after SL-reinsurance has the form

$$\text{Res}_t^{SL} = \sum_{j=0}^{r-1} \text{E}\left[\tilde{X}_{t-j,r}^{SL} - \tilde{X}_{t-j,j}^{SL} | I_{t-j,j}\right],$$

where

$$\tilde{X}_{t-j,i}^{SL} = \min\left(X_{t-j,i}, \bar{a}_{t-j}\right), \quad i = 0, \ldots, r.$$

The conditional expected value $\text{E}[\tilde{X}_{t-j,r}^{SL} | I_{t-j,j}]$ is equal to

$$\text{E}\left[\tilde{X}_{t-j,r}^{SL} | I_{t-j,j}\right] = \text{E}\left[\min\left(X_{t-j,r}, \bar{a}_{t-j}\right) | I_{t-j,j}\right].$$

Using reinsurance relief effect this can be expressed as

$$\text{E}\left[\tilde{X}_{t-j,r}^{SL} | I_{t-j,j}\right] = \text{E}\left[X_{t-j,r} | I_{t-j,j}\right] r_{X_{t-j,r}|I_{t-j,j}}(\bar{a}_{t-j}).$$

We have already derived the expressions for the conditional moments of $X_{t-j,r}$. We will need the moments also when approximating $r_{X_{t-j,r}|I_{t-j,j}}(\bar{a}_{t-j})$. Putting in for $\text{E}\left[X_{t-j,r}|I_{t-j,j}\right]$ the expression developed in the previous chapter we get

$$\text{E}\left[\tilde{X}_{t-j,r}^{SL} | I_{t-j,j}\right] = \left(X_{t-j,j} + X_{t-j,j}^{O}\tilde{Q}_{t-j,j,r-1}^{D} + N_{t-j,j}\tilde{Q}_{t-j,j,r-1}^{A}\right) r_{X_{t-j,r}|I_{t-j,j}}(\bar{a}_{t-j}),$$

where $\tilde{Q}_{t-j,j,r-1}^{D}$ and $\tilde{Q}_{t-j,j,r-1}^{A}$ are defined in (4.24) and (4.22). With these results net loss reserve after SL-reinsurance has the form

$$\text{Res}_t^{SL} = \sum_{j=0}^{r-1}\left[\left(X_{t-j,j} + X_{t-j,j}^{O}\tilde{Q}_{t-j,j,r-1}^{D} + N_{t-j,j}\tilde{Q}_{t-j,j,r-1}^{A}\right) r_{X_{t-j,r}|I_{t-j,j}}(\bar{a}_{t-j}) - \tilde{X}_{t-j,j}^{SL}\right]. \tag{4.29}$$

If we subtract on both sides the losses that are at $I_{t-j,j}$ already closed (for these losses no loss reserve is needed), we get

$$\text{E}\left[\tilde{X}_{t-j,r}^{SL} - X_{t-j,j}^{C} | I_{t-j,j}\right] = \text{E}\left[\min\left(X_{t-j,r} - X_{t-j,j}^{C}, \bar{a}_{t-j} - X_{t-j,j}^{C}\right) \middle| I_{t-j,j}\right].$$

We denote with $\bar{a}_{t-j}^{O}$ the priority after subtracting the losses closed at $I_{t-j,j}$

$$\bar{a}_{t-j,j}^{O} = \bar{a}_{t-j} - X_{t-j,j}^{C},$$

and use, again, the reinsurance relief effect to get

$$\mathrm{E}\left[X_{t-j,r} - X^C_{t-j,j} | I_{t-j,j}\right] r_{X_{t-j,r}-X^C_{t-j,j}|I_{t-j,j}}(\bar{a}^O_{t-j,j}).$$

The conditional moments of $X_{t-j,r} - X^C_{t-j,j}$ can be expressed very easily with the results that we already have. For the expected value we get

$$\mathrm{E}\left[X_{t-j,r} - X^C_{t-j,j} | I_{t-j,j}\right] = X^O_{t-j,j}(1 + \tilde{Q}^D_{t-j,j,r-1}) + N_{t-j,j}\tilde{Q}^A_{t-j,j,r-1}.$$

For the term $\tilde{X}^{SL}_{t-j,j}$ in the reserve we get after subtracting losses that are at $I_{t-j,j}$ closed the expression

$$\min(X_{t-j,j} - X^C_{t-j,j}, \bar{a}^O_{t-j,j}),$$

which is equal to

$$\min(X^O_{t-j,j}, \bar{a}^O_{t-j,j}).$$

Thus, the net loss reserve after SL-reinsurance can be expressed alternatively to (4.29) as

$$\mathrm{Res}^{SL}_t = \sum_{j=0}^{r-1}\Big[\left(X^O_{t-j,j}(1 + \tilde{Q}^D_{t-j,j,r-1}) + N_{t-j,j}\tilde{Q}^A_{t-j,j,r-1}\right) r_{X_{t-j,r}-X^C_{t-j,j}|I_{t-j,j}}(\bar{a}^O_{t-j,j}) -$$

$$- \min\left(X^O_{t-j,j}, \bar{a}^O_{t-j,j}\right)\Big].$$

4.4.2 Loss reserve with risk margin

Similar to the previous section, loss reserve with risk margin has, due to the independence of the accident years, the form

$$\mathrm{Res}^{SL}_t = \sum_{j=0}^{r-1} \mathrm{E}\left[\tilde{X}^{SL}_{t-j,r} - \tilde{X}^{SL}_{t-j,j} | I_{t-j,j}\right] + z_\alpha \sqrt{\sum_{j=0}^{r-1} \mathrm{Var}\left[\tilde{X}^{SL}_{t-j,r} | I_{t-j,j}\right]}. \tag{4.30}$$

Again, the variance in (4.30) can be expressed as

$$\mathrm{Var}\left[\tilde{X}^{SL}_{t-j,r} | I_{t-j,j}\right] = \mathrm{E}\left[\left(\tilde{X}^{SL}_{t-j,r}\right)^2 | I_{t-j,j}\right] - \left(\mathrm{E}\left[\tilde{X}^{SL}_{t-j,r} | I_{t-j,j}\right]\right)^2.$$

We have already derived the expression for the second term. For the first term we use, in a similar way, the reinsurance relief effect of the second power and get

$$\mathrm{E}\left[\left(\tilde{X}^{SL}_{t-j,r}\right)^2 | I_{t-j,j}\right] = \mathrm{E}\left[X^2_{t-j,r} | I_{t-j,j}\right] r^{(2)}_{X_{t-j,r}|I_{t-j,j}}(\bar{a}_{t-j}).$$

4.5 Risk capital and reinsurance optimization

To optimize reinsurance there are two criteria that can be optimized. First, we want to minimize risk capital (and thus capital costs), second we want to maximize profit. As normally reinsurance causes costs, because the reinsurer loads his marge into the reinsurance premium, there won't be a unique reinsurance program that optimizes both criteria. There will be some sort of efficient frontier. One way how to define a unique optimization criterion is to set the cost of capital and define economic value added. In the following text, we assume all variables at time $t+1$ to be the expected values as the company can only compute with expected values for $t+1$ when optimizing reinsurance structure at time t.

4.5.1 Risk capital

Let's denote

B_t	earned risk premium in the year t at the end of the period (determ.),
u_0	initial allocated capital,
u_t	capital allocated in the year t for the next period,
$F = 1 + i$,	where i is the interest rate of the financial means,
$G = 1 + (1-\gamma)i$	where γ is the mean time of loss payments during the year,
Res_t	loss reserve at the end of the year t, and
Pmt_t	payments for losses in the year t.

With this notation the risk reserve (the free allocated capital of the company) at the end of the year $t+1$ is equal to

$$U_{t+1} = u_t F + B_{t+1} + F\mathrm{Res}_t - \mathrm{Res}_{t+1} - G\mathrm{Pmt}_{t+1}. \tag{4.31}$$

In other words, risk reserve at time $t+1$ consists of the allocated capital from the previous period incl. interest yield, premiums in the year $t+1$, loss reserve built in the previous year t incl. interest yield minus the reserve created in the year $t+1$ and payments for losses in the year $t+1$ (incl. the corresponding interest yield). For the payments in the year $t+1$ we have

$$\mathrm{Pmt}_{t+1} = X_{t+1,0} + \sum_{j=1}^{r-1} \left(X_{t+1-j,j} - X_{t+1-j,j-1}\right) +$$

$$+ \sum_{l=0}^{r-1} \sum_{k=1}^{\tilde{N}_{t+1-r,r-1}} Y^l_{t+1-r,r-1,k} J^l_{t+1-r,r-1,k} \tilde{E}^l_{t+1-r,r,k} (d_{r-1} - 1).$$

The first term contains paid losses from the last accident year $t+1$, the second term contains incremental payments for losses from previous accident years and the last term contains incremental payments for losses from the accident year $t+1-r$.

For the one-period model we determine the required risk capital u_0 from the requirement $P(U_1 < 0) = \varepsilon$. Applying NP2-approximation we get

$$0 = \mathrm{E}U_1 - z_{1-\varepsilon}\sqrt{\mathrm{E}\overline{U}_1^2} + \frac{\mathrm{E}\overline{U}_1^3}{6\mathrm{E}\overline{U}_1^2}(z_{1-\varepsilon}^2 - 1). \tag{4.32}$$

The expected values in (4.32) can be expressed from (4.31) and the results developed in previous sections. For net risk capital we have to replace the gross values with the net values.

Reinsurance premium

Usually, when the cedent makes the decision on the reinsurance structure for the next year, he has to request the information on reinsurance premium B_{t+1} for various alternatives. As this can normally be done only for a limited number of alternatives, it's important to have some estimation of the reinsurance premium at this time.

Both premium and reinsurance premium consist of the part covering future expected losses and a loading that covers the costs and the uncertainty connected with the future losses. In the following definition we summarize the most common principles for the computation of (reinsurance) premium.

Definition 3 Premium principles. *Let X be random variable expressing annual loss and B be the premium to cover the loss X. Following principles are used to compute B,*

Expected value principle	$B = (1+\beta)X,$
Standard deviation principle	$B = \mathrm{E}X + \beta\sigma(X),$
Variance principle	$B = \mathrm{E}X + \beta\mathrm{Var}(X),$
Modified variance principle	$B = \mathrm{E}X + \beta\mathrm{Var}(X)/\mathrm{E}X,$
Mixed principle	$B = \mathrm{E}X + \alpha\sigma(X) + \beta\mathrm{Var}(X).$

For proportional treaties the reinsurance premium is defined as a proportion of the gross premium. However, the reinsurer pays a part of the reinsurance premium back as a reinsurance commission. This commission should originally cover a part of the cedent's acquisition costs. However, usually the reinsurer calculates his premium with one of the principles mentioned above and then calibrates the commission so that after paying the commission he ends with the calculated premium.

4.5.2 Economic Value Added (EVA)

Most companies define economic value added (EVA) as the expected economic result minus the required capital multiplied by cost of capital rate,

$$\mathrm{EVA}_{t+1} = \mathrm{Result}_{t+1} - u_t \mathrm{CoC}. \tag{4.33}$$

Cost of capital should express the interest that investors expect from other investment with comparable risk. Cost of capital, CoC, can be split in risk free interest rate and risk margin

$$\mathrm{CoC} = i + \Delta i_{\mathrm{CoC}}, \tag{4.34}$$

where Δi_{CoC} is risk margin required by investors when investing in the company. In other words, a company generates EVA if it generates return on the allocated capital that is higher than an

alternative investment with comparable risk. We don't assume dependence of Δi_{CoC} on u_t. In the model setting developed in this work the result can be expressed as

$$\text{Result}_{t+1} = U_{t+1} - u_t = iu_t + B_{t+1} + F\text{Res}_t - \text{Res}_{t+1} - G\text{Pmt}_{t+1}. \tag{4.35}$$

From this it can be seen easily, that

$$\text{Result}_{t+1} = iu_t + \text{Result}^{UW}_{t+1}, \tag{4.36}$$

where

$$\text{Result}^{UW}_{t+1} = B_{t+1} + F\text{Res}_t - \text{Res}_{t+1} - G\text{Pmt}_{t+1} \tag{4.37}$$

is the underwriting result in year $t+1$. Combining these equations we get for EVA the expression

$$\text{EVA}_{t+1} = iu_t + \text{Result}^{UW}_{t+1} - u_t\text{CoC}, \tag{4.38}$$

which is equal to

$$\text{EVA}_{t+1} = \text{Result}^{UW}_{t+1} - u_t\Delta i_{\text{CoC}}. \tag{4.39}$$

In other words, economic value added is equal to the underwriting result less capital costs above the risk free interest rate.

4.5.3 Return on Equity

Another important key profit indicator in an insurance company is return on equity (ROE). It is defined as economic result divided by the amount of capital,

$$\text{ROE}_{t+1} = \frac{\text{Result}_{t+1}}{u_t}. \tag{4.40}$$

This can be equally expressed as

$$\text{ROE}_{t+1} = \frac{U_{t+1}}{u_t} - 1,$$

or

$$\text{ROE}_{t+1} = \frac{\text{Result}^{UW}_{t+1}}{u_t} + i. \tag{4.41}$$

Using ROE, EVA can be expressed from (4.39) as

$$\text{EVA}_{t+1} = u_t\left(\frac{\text{Result}^{UW}_{t+1}}{u_t} + i - i\right) - u_t\Delta i_{\text{CoC}} = u_t\left(\text{ROE}_{t+1} - i\right) - u_t\Delta i_{\text{CoC}},$$

which gives

$$\text{EVA}_{t+1} = u_t\left(\text{ROE}_{t+1} - \text{CoC}\right). \tag{4.42}$$

Having this, it is easy to see that to generate a positive EVA,

$$\text{EVA}_{t+1} > 0,$$

is equal to generate ROE that is higher than cost of capital rate,

$$\text{ROE}_{t+1} > \text{CoC}.$$

4.5.4 Reinsurance Economic Value Added (RI-EVA)

In order to express the economic value that is added through reinsurance, we can use the concept of reinsurance value added,

$$\text{RIEVA}_{t+1} = \text{EVA}_{t+1}^{net} - \text{EVA}_{t+1}^{gr}, \tag{4.43}$$

where *net* and *gr* denote the net resp. the gross values. It is easy to see that reinsurance economic value added can be alternatively expressed as

$$\text{RIEVA}_{t+1} = \text{Result}_{t+1}^{(UW)net} - \text{Result}_{t+1}^{(UW)gr} + \Delta i_{\text{CoC}} \left(u_t^{gr} - u_t^{net}\right). \tag{4.44}$$

Denoting

$$\Delta u_t = u_t^{gr} - u_t^{net} \tag{4.45}$$

the capital released through reinsurance, and

$$\text{Result}_{t+1}^{(UW)ri} = \text{Result}_{t+1}^{(UW)gr} - \text{Result}_{t+1}^{(UW)net} \tag{4.46}$$

the underwriting result of the reinsurer, we get

$$\text{RIEVA}_{t+1} = -\text{Result}_{t+1}^{(UW)ri} + \Delta i_{\text{CoC}} \Delta u_t. \tag{4.47}$$

In other words, reinsurance economic value added is equal to the capital cost above i saved through reinsurance less the underwriting result of the reinsurer. Reinsurer's result can be equivalently expressed as

$$\text{Result}_{t+1}^{(UW)ri} = B_{t+1}^{ri} + F\text{Res}_t^{ri} - \text{Res}_{t+1}^{ri} - G\text{Pmt}_{t+1}^{ri}, \tag{4.48}$$

where

$$B_{t+1}^{ri} = B_{t+1}^{gr} - B_{t+1}^{net},\ \text{Res}_t^{ri} = \text{Res}_t^{gr} - \text{Res}_t^{net} \text{ and } \text{Pmt}_{t+1}^{ri} = \text{Pmt}_{t+1}^{gr} - \text{Pmt}_{t+1}^{net}.$$

Reinsurance economic value added can be expressed in terms of ROE. From (4.44) we get

$$\text{RIEVA}_{t+1} = u_t^{net} \left(\frac{\text{Result}_{t+1}^{(UW)net}}{u_t^{net}} + i - i \right) - u_t^{gr} \left(\frac{\text{Result}_{t+1}^{(UW)gr}}{u_t^{net}} + i - i \right) + \Delta i_{\text{CoC}} \Delta u_t.$$

Adding and subtracting Δi_{CoC} in both brackets and simplifying the expression gives

$$\text{RIEVA}_{t+1} = u_t^{net} \left(\text{ROE}_{t+1}^{net} - \text{CoC}\right) - u_t^{gr} \left(\text{ROE}_{t+1}^{gr} - \text{CoC}\right). \tag{4.49}$$

Thus, it is easy to see that a positive value of reinsurance economic value added,

$$\text{RIEVA}_{t+1} > 0,$$

is generated if and only if

$$u_t^{net} \left(\text{ROE}_{t+1}^{net} - \text{CoC}\right) > u_t^{gr} \left(\text{ROE}_{t+1}^{gr} - \text{CoC}\right).$$

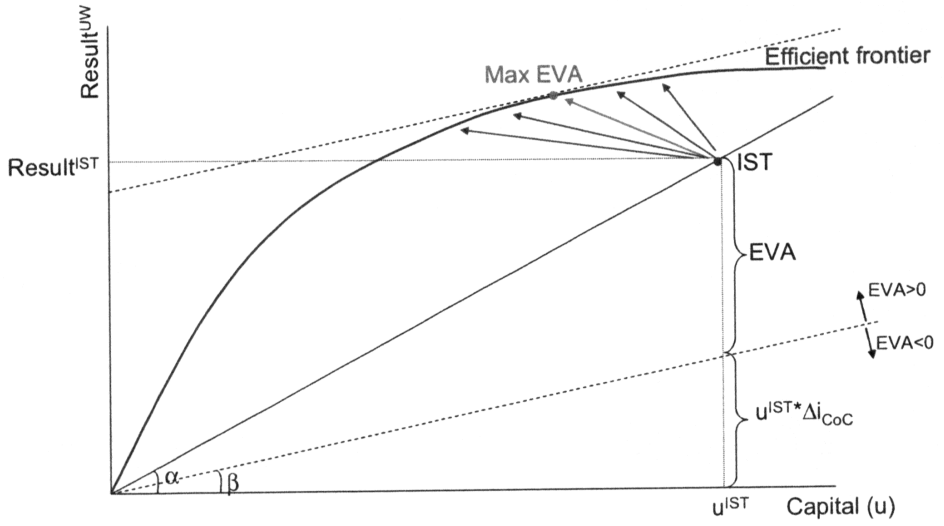


Figure 4.2: Efficient frontier

4.5.5 Efficient frontier

In the following text we assume that all variables are net of reinsurance. We can plot underwriting result and risk capital of all possible reinsurance structures in one plot as in Figure 4.2. In this plot we are able to reject reinsurance structures that are not Pareto optimal, i.e. there is another reinsurance structure giving the same underwriting result with less required capital or giving higher underwriting result with the same required capital. All Pareto optimal reinsurance structures lie on a curve called efficient frontier.

If we plot a line from the point $(0, 0)$ through the recent reinsurance point (IST), we can interpret the angle α, using (4.41), in the following way,

$$\tan\gamma = \frac{\text{Result}_{t+1}^{UW}}{u_t} = \text{ROE}_{t+1} - i.$$

This means that maximizing ROE is equal to maximizing the angle α.
EVA can be found in Figure 4.2 very simply. It holds

$$\text{EVA}_{t+1} = \text{Result}_{t+1}^{UW} - u\Delta i_{\text{CoC}},$$

hence the underwriting result can be split into two parts

$$\text{Result}_{t+1}^{UW} = \text{EVA}_{t+1} + u\Delta i_{\text{CoC}}. \tag{4.50}$$

Putting a line through the points $(0, 0)$ and $(u, u\Delta i_{\text{CoC}})$ we can interpret the tangens of the angle β as Δi_{CoC}. We already know that generating a positive value of EVA is equivalent to ROE $>$ CoC, which is if and only if $\tan\alpha > \tan\beta$. Maximizing EVA then leads to a single point on the efficient frontier.

Chapter 5

Application in general liability

In this chapter we show how the model developed in chapters 3 and 4 can be used in practice. We use anonymized data on individual losses from general liability reinsurance from the years 1997 – 2006. The losses contain allocated loss adjustment expenses but do not contain unallocated loss adjustment expenses. First, the model is parametrized, then distribution of loss reserve is simulated and deterministic results for loss reserve with an excess of loss reinsurance are given. We split the losses into small and large losses with a threshold of 25,000.

5.1 Model parametrization

5.1.1 Development factors of small losses

For the estimation of the development factors of small losses we use the cumulative triangle of paid small losses given in Table 5.1.

Table 5.1: Triangle of paid small losses (in Tsd. Eur)

AY	0	1	2	3	4	5	6	7	8	9
1997	7,484	13,521	15,010	15,833	16,298	16,440	16,600	16,687	16,694	16,749
1998	7,556	13,831	15,529	16,456	16,927	17,048	17,212	17,315	17,396	
1999	8,725	15,212	17,600	18,499	18,973	19,302	19,497	19,629		
2000	6,977	14,954	17,016	18,020	18,599	18,911	19,114			
2001	6,420	12,577	14,335	15,239	15,605	15,715				
2002	6,539	12,569	14,036	14,805	15,230					
2003	5,885	11,369	12,976	13,563						
2004	5,592	11,047	12,570							
2005	5,654	11,094								
2006	6,084									

From the triangle of paid small losses we compute the multiplicative individual development factors as

$$c_{s,j} = \frac{X^{sm}_{s,j+1}}{X^{sm}_{s,j}}, \quad j = 0, \ldots, r-1, \; s = t-r, \ldots, t-j,$$

where $X^{sm}_{s,j}$ are cumulative small losses from accident year s in development year j. The values are given in the Table 5.2.

In practice, it is usual to compute the overall development factors in several ways and to choose the most convenient values based on experience. Usually arithmetical mean, geometrical mean, mean with just 3–5 last years (if the older data is thought to be less appropriate e.g. due to changes in the processes), weighted mean and so on, are computed. It is possible as well to exclude outliers from the analysis. In Table 5.3 we show arithmetical and geometrical mean, mean based on last 4 years, mean without outliers and weighted mean as computed by the chain ladder method. In the last line the values that have been chosen to describe the reality in the most appropriate way are given. In this case we took the mean based on the last 4 years, because there have been larger changes in claims process 4-5 years ago.

In general liability it is typical that development periods are up to 30 or 35 years long. We have to estimate tail factors otherwise the reserve would be underestimated. As described in section on parameter estimates, we have fitted Weibull, power, exponential and Sherman curve. Figure 5.1 shows the fitted curves. All the curves have been fitted to the development factors starting with the second value.

The fitting resulted in the following sums of squared errors:

Weibull curve	0.000024,
Power curve	0.000055,
Exponential curve	0.000063,
Sherman curve	0.000011.

We use the empirical observation of the first value. For all the following values we've chosen to take the Sherman curve to smooth the values and to extrapolate the tail.

To estimate $X_{t+1,0}$ we have fitted a curve through the first column of the triangle of small losses. We observe some sort of cyclical behavior. The best fit was achieved with a polynomial of the fourth power

$$X^{sm}_{t_1+1997,0} = -2,611t_1^4 + 78,817t_1^3 - 761,190t_1^2 + 2,374,735t_1 + 5,711,548,$$

where $t_1 = t - 1997$. For the accident year $t = 2007$ we get $X^{sm}_{2007,0} = 6,394,109$. The R^2 of

Table 5.2: Individual paid development factors for small losses

AY	0	1	2	3	4	5	6	7	8
1997	1.8067	1.1101	1.0548	1.0294	1.0087	1.0097	1.0053	1.0004	1.0033
1998	1.8304	1.1228	1.0597	1.0286	1.0071	1.0096	1.0060	1.0047	
1999	1.7435	1.1569	1.0511	1.0256	1.0173	1.0101	1.0068		
2000	2.1431	1.1379	1.0590	1.0321	1.0168	1.0107			
2001	1.9588	1.1398	1.0630	1.0240	1.0070				
2002	1.9221	1.1167	1.0548	1.0287					
2003	1.9317	1.1414	1.0452						
2004	1.9756	1.1378							
2005	1.9619								

Table 5.3: Paid development factors for small losses

AY	0	1	2	3	4	5	6	7	8
arith. mean	1.9193	1.1329	1.0554	1.0281	1.0114	1.0100	1.0060	1.0026	1.0033
geom. mean	1.9162	1.1328	1.0554	1.0281	1.0114	1.0100	1.0060	1.0026	1.0033
last 4 years	1.9478	1.1339	1.0555	1.0276	1.0121	1.0100	1.0060	1.0026	1.0033
without outl.	1.9125	1.1327	1.0559	1.0273	1.0114	1.0100	1.0060	1.0026	1.0033
chain ladder	1.9097	1.1332	1.0555	1.0281	1.0117	1.0101	1.0061	1.0026	1.0033
chosen values	1.9478	1.1339	1.0555	1.0276	1.0121	1.0100	1.0060	1.0026	1.0033

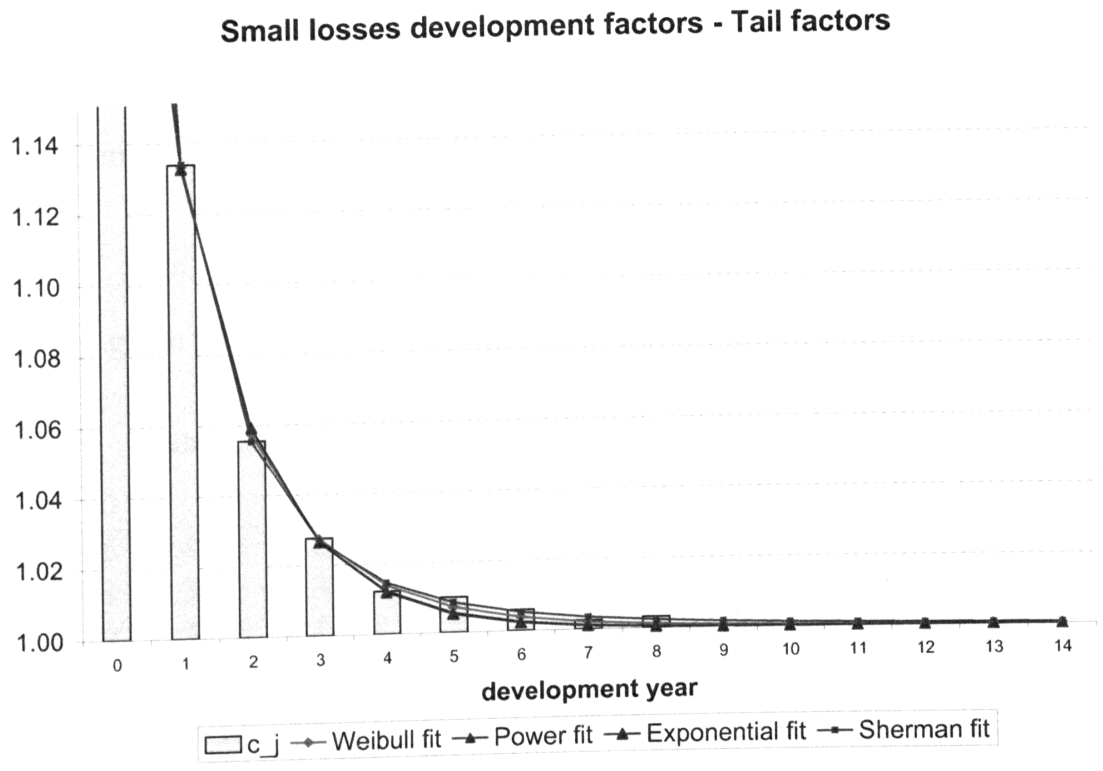


Figure 5.1: Tail fitting of development factors for small losses

the regression is 0.8514. The standard deviation from the trend is 408,685. These values would be needed if we would compute risk capital.

5.1.2 Development factors of large loss numbers and the expected number of large losses in the first development year

For the estimation of the development factors of large loss numbers we use the cumulative triangle of the number of large losses given in Table 5.4.

Table 5.4: Triangle of large losses numbers

AY	0	1	2	3	4	5	6	7	8	9
1997	7	24	35	42	49	59	62	64	70	71
1998	14	33	44	58	71	83	92	95	99	
1999	21	45	61	72	83	90	94	98		
2000	12	48	69	78	92	101	109			
2001	14	49	61	74	83	93				
2002	6	44	61	69	77					
2003	8	25	40	59						
2004	6	20	38							
2005	10	42								
2006	5									

From the cumulative large loss numbers we compute the individual development factors $b_{s,j}$ as

$$b_{s,j} = \frac{\tilde{N}_{s,j+1}}{N_{s,j}}, \quad j = 0, \ldots, r-1, \; s = t-r, \ldots, t-j.$$

The values are given in Table 5.5.

Table 5.5: Individual development factors for large loss numbers

AY	0	1	2	3	4	5	6	7	8
1997	2.4286	0.4583	0.2000	0.1667	0.2041	0.0508	0.0323	0.0938	0.0143
1998	1.3571	0.3333	0.3182	0.2241	0.1690	0.1084	0.0326	0.0421	
1999	1.1429	0.3556	0.1803	0.1528	0.0843	0.0444	0.0426		
2000	3.0000	0.4375	0.1304	0.1795	0.0978	0.0792			
2001	2.5000	0.2449	0.2131	0.1216	0.1205				
2002	6.3333	0.3864	0.1311	0.1159					
2003	2.1250	0.6000	0.4750						
2004	2.3333	0.9000							
2005	3.2000								

Again, we compute several types of mean and chose the most appropriate one, which is the arithmetical mean in this case. The values are shown in Table 5.6.

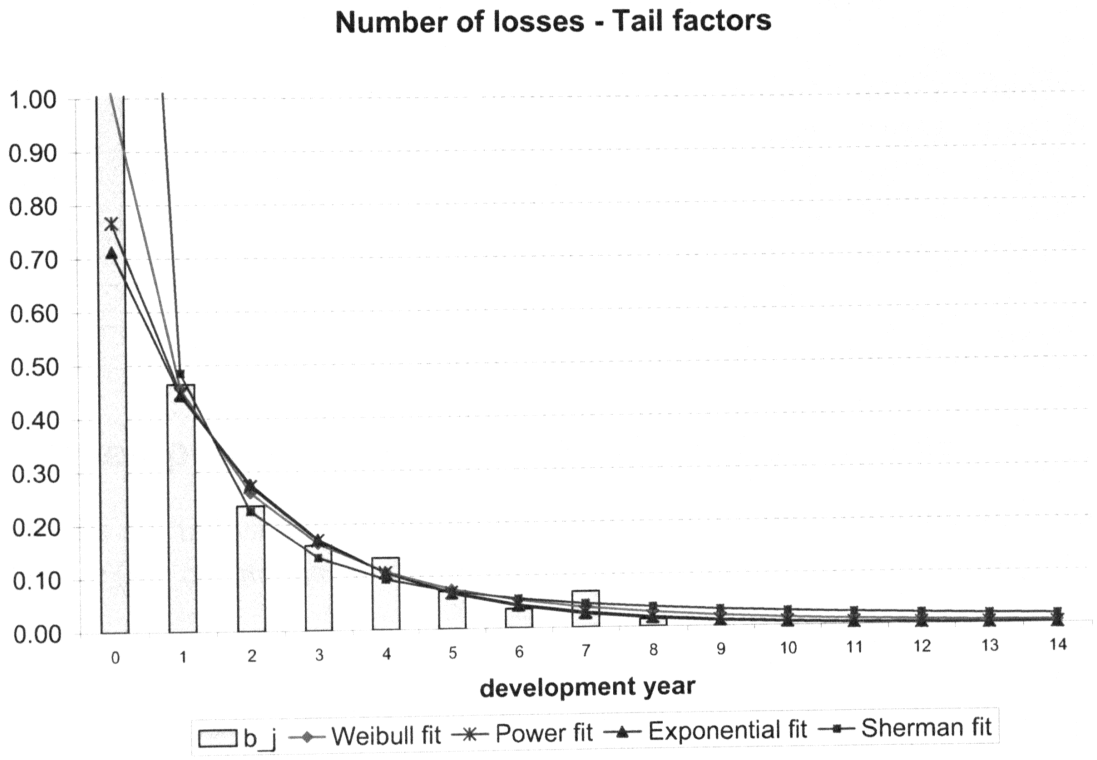


Figure 5.2: Tail fitting of development factors for number of losses

As in the case of small losses we fit four curves to the chosen values of development factors, these are given in Figure 5.2. The fits resulted in the following sums of square errors:

Weibull curve	0.002775,
Power curve	0.004143,
Exponential curve	0.004849,
Sherman curve	0.004409.

For the first three values we've chosen to take directly the empirical values. For all other values we've chosen to take the Weibull curve, which gives the best fit. We use the fit to smooth the values and to extrapolate the tail. If we don't smooth the values we would have the eighth value larger that the sixth value which doesn't seem to be reasonable.

To estimate the mean number of losses in the first development year, λ, we analyze the first column of the triangle of loss numbers. We can see that the values are rather volatile and there is no clearly visible trend. That's why we use the arithmetical mean to estimate parameter λ, which gives the value of 10.3.

Table 5.6: Development factors for large loss numbers

AY	0	1	2	3	4	5	6	7	8
arith. mean	2.8270	0.4645	0.2355	0.1601	0.1351	0.0707	0.0358	0.0679	0.0143
geom. mean	2.3995	0.4321	0.2130	0.1561	0.1279	0.0664	0.0355	0.0628	0.0143
last 4 years	3.4979	0.5328	0.2374	0.1425	0.1179	0.0707	0.0358	0.0679	0.0143
without outl.	2.3179	0.4285	0.2355	0.1601	0.1351	0.0707	0.0358	0.0679	0.0143
chain ladder	2.3673	0.4201	0.2183	0.1578	0.1270	0.0721	0.0363	0.0629	0.0143
chosen values	2.8270	0.4645	0.2355	0.1601	0.1351	0.0707	0.0358	0.0679	0.0143

5.1.3 Development factors of individual large losses

We compute the individual development factors $d_{s,j}$ of individual large losses as a chain ladder estimate (see chapter on parameter estimates),

$$\hat{d}_{s,j} = \frac{\sum_{l=0}^{j} \sum_{\substack{k=1 \\ \& J^l_{s,j,k}=1}}^{\tilde{N}_{t-j,l}} Y^l_{s,j+1,k}}{\sum_{l=0}^{j} \sum_{\substack{k=1 \\ \& J^l_{s,j,k}=1}}^{\tilde{N}_{t-j,l}} Y^l_{s,j,k}}, \quad j = 0, \ldots, r-1, \; s = t-r, \ldots, t-j.$$

Another possibility, which seems to give more stable results, is to compute the development factors as a mean of all individual large loss development factors for all individual losses. There are two outlier individual development factors, one in the development year 5 with value of 10.69 and second in the development year 6 with a value of 5.13. We remove these two individual development factors because they would distort the estimate of the mean. However, we don't remove these two observations when computing the error terms $\tilde{E}^l_{s,j,k}$. We get the estimates shown in Table 5.7.

Table 5.7: Development factors for large loss severities

AY	0	1	2	3	4	5	6	7	8
d_j estimates	1.2789	1.1569	1.1341	1.0734	1.1211	1.1181	1.1285	1.1040	1.0581

It can be seen that there is more risk connected with the estimation of parameters than if we are working with annual cumulative values. Again, to reduce the number of parameters, to smooth the values and to get the estimates of the tail factors we fit several curves as shown in Figure 5.3. The fits give the following sums of square residuals:

Weibull curve	0.006886,
Power curve	0.660818,
Exponential curve	0.012337,
Sherman curve	0.006145.

As the fits are not as good as in the case of small losses, we just smooth the value in the third development year with the Weibull curve and the value in the sixth development year with a

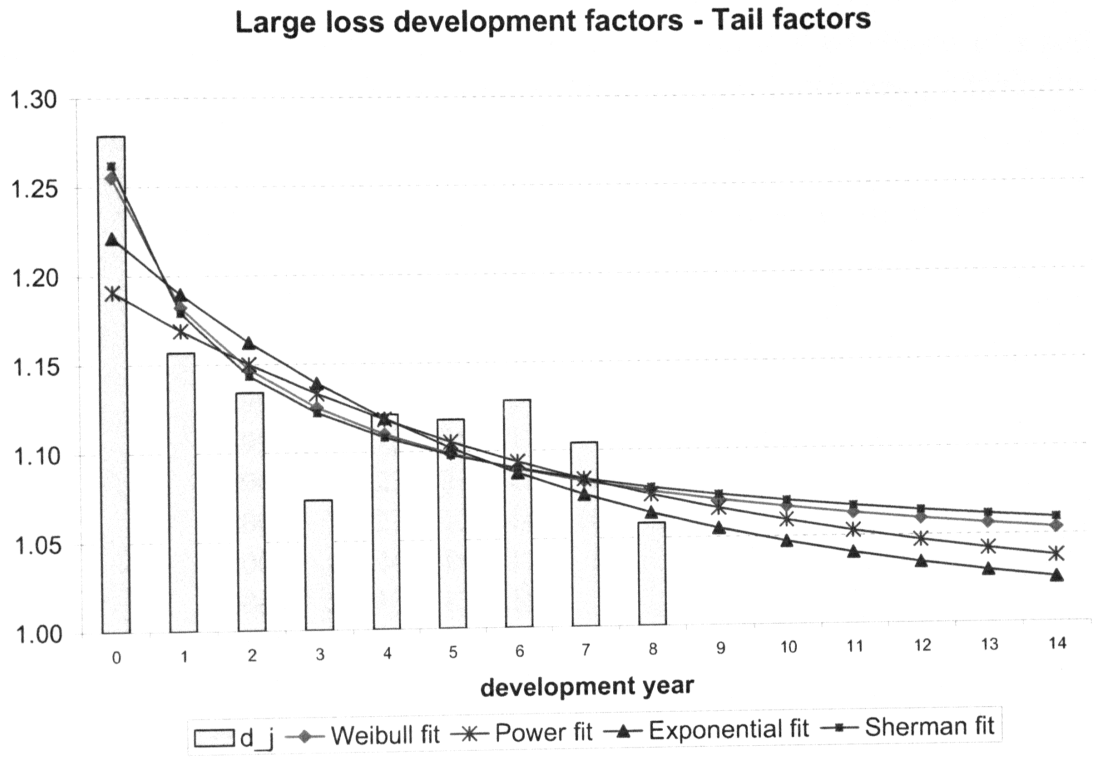


Figure 5.3: Tail fitting of development factors for large losses severities

mean of the two neighboring years. To estimate the tail the power curve seems to be the most reasonable choice even if this curve doesn't give the best fit in the bulk of the data. The high value of R^2 is in this case mainly due to the bad fit in the first value (power curve gives the worst fit for this value with a large residual). The other curves seem to give tails that are too short or too long.

5.1.4 The moments of individual large losses occurring in the triangle for the first time

The parameters of the distribution of $Y^l_{t,l,k}$ are estimated with the moment method. As we don't see any significant trend in dependence of the accident or development years, we compute a simple mean and standard deviation of the losses occurring in the triangle for the first time. We get the mean of 53,218 and the standard deviation of 44,488. With the assumption of log-normal distribution we get the following values of the parameters,

$$\sigma = \sqrt{\log(t) - 2\log(m)} = 0.7280,$$

and

$$\mu = \log(m) - \sigma^2/2 = 10.6172,$$

where $m = \mathrm{E}Y$ and $t = \mathrm{E}Y^2$.

5.1.5 Probabilities of a large loss being closed

To compute the probabilities that an individual large loss will be closed in a particular development year we need the number of losses closed in a particular year, which are shown in Table 5.8.

Table 5.8: Number of losses closed in a particular development year

AY	0	1	2	3	4	5	6	7	8	9
1997	2	9	8	6	11	10	4	4	4	1
1998	3	13	10	21	13	10	8	9	13	
1999	7	13	20	17	14	4	6	5		
2000	4	23	21	12	15	6	7			
2001	7	27	13	14	10	9				
2002	1	24	19	11	9					
2003	2	15	17	16						
2004	3	8	16							
2005	3	28								
2006	1									

Additionally, we need the number of open losses in all development years. These are given in Table 5.9.

Table 5.9: Number of open losses

AY	0	1	2	3	4	5	6	7	8	9
1997	7	22	23	22	23	22	15	12	16	13
1998	14	30	28	32	24	22	21	17	12	
1999	21	38	40	32	26	19	19	17		
2000	12	44	42	29	32	25	28			
2001	14	42	27	27	22	22				
2002	6	43	36	25	22					
2003	8	23	23	25						
2004	6	17	27							
2005	10	39								
2006	5									

Empirical probability of large loss from accident year s to be closed in development year j is then computed from these two triangles. The values are given in Table 5.10.

It is easy to see that in the accident year 1998 in the development year 8 there were 13 claims closed but only 12 claims were open at the beginning of that development year. It is because one claim has been reopened and closed in this development year. In our model we don't assume that a claim can be reopened. Fortunately, this is the only case in the data that a loss has been reopened. If there were systematically claims reopened one would have to include this possibility into the model.

Table 5.10: Probabilities of large losses to be closed in a particular development year

AY	0	1	2	3	4	5	6	7	8	9
1997	0.29	0.41	0.35	0.27	0.48	0.45	0.27	0.33	0.25	0.08
1998	0.21	0.43	0.36	0.66	0.54	0.45	0.38	0.53	1.08	
1999	0.33	0.34	0.50	0.53	0.54	0.21	0.32	0.29		
2000	0.33	0.52	0.50	0.41	0.47	0.24	0.25			
2001	0.50	0.64	0.48	0.52	0.45	0.41				
2002	0.17	0.56	0.53	0.44	0.41					
2003	0.25	0.65	0.74	0.64						
2004	0.50	0.47	0.59							
2005	0.30	0.72								
2006	0.20									

We compared three alternative methods to compute the overall probabilities q_j and chose the simple mean. As the last values are based on too few observations, we assume $q_j = q_5 = 0.35$ for $j \geq 6$.

AY	0	1	2	3	4	5	6	7	8	9
arith. mean	0.31	0.53	0.51	0.50	0.48	0.35	0.30	0.39	0.67	0.08
weighted mean	0.32	0.54	0.50	0.51	0.48	0.35	0.30	0.39	0.61	0.08
mean without l. and sm.	0.33	0.53	0.49	0.51	0.49	0.37	0.29	0.31		
chosen values	0.31	0.53	0.51	0.50	0.48	0.35	0.35	0.35	0.35	0.35

Table 5.11: Probabilities of large loss to be closed

5.1.6 Analysis of the error terms

Error terms of the development of small losses

In the Monte Carlo simulation of the reserve in the next section we generate the error terms directly from the empirical distribution. The error terms o f small losses have an empirical mean of 0.9939, which is near the assumed value of 1, and a standard deviation of 0.2307.

Error terms of the development of large losses

Figures 5.4 and 5.5 show the values of the error terms $\tilde{E}^{l}_{s,j,k}$ in dependence on the accident and development year. We don't see any significant trend and assume the same distribution for all error terms in the following subsection.

The histogram of the error terms is shown in Figure 5.6. There are many observation with the value of 0, which is the case if a loss is open but there is no payment in that particular development year. If we would like to use a parametric distribution for the simulation of the error terms we would have to create a distribution that has a positive weight at zero. If we look at the histogram of the logarithms of error terms 5.7 we see that, conditional that the value is positive, the distribution of the logarithms is rather symetric. One possibility would be to use for the conditional distribution e.g. the lognormal distribution.

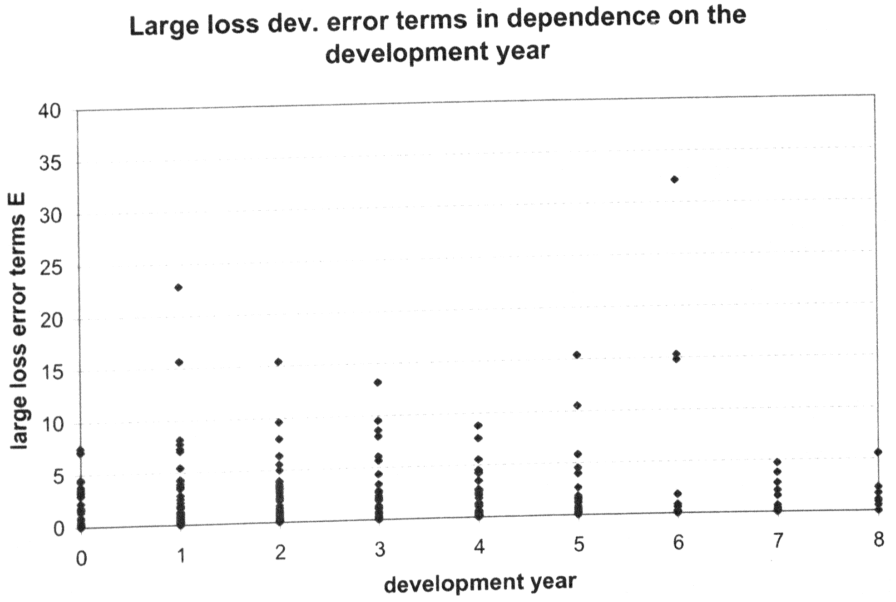


Figure 5.4: Individual loss error terms in dependence on development year

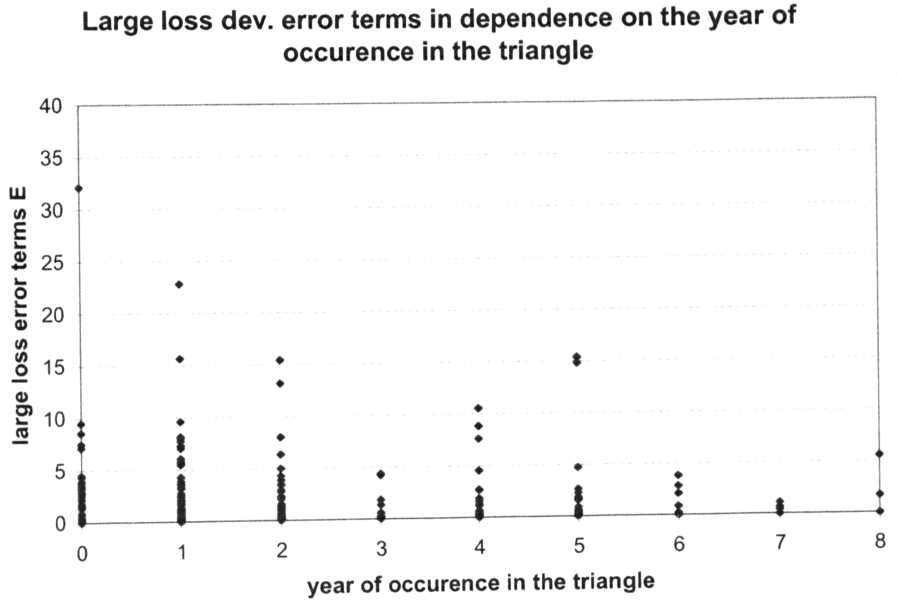


Figure 5.5: Individual loss error terms in dependence on accident year

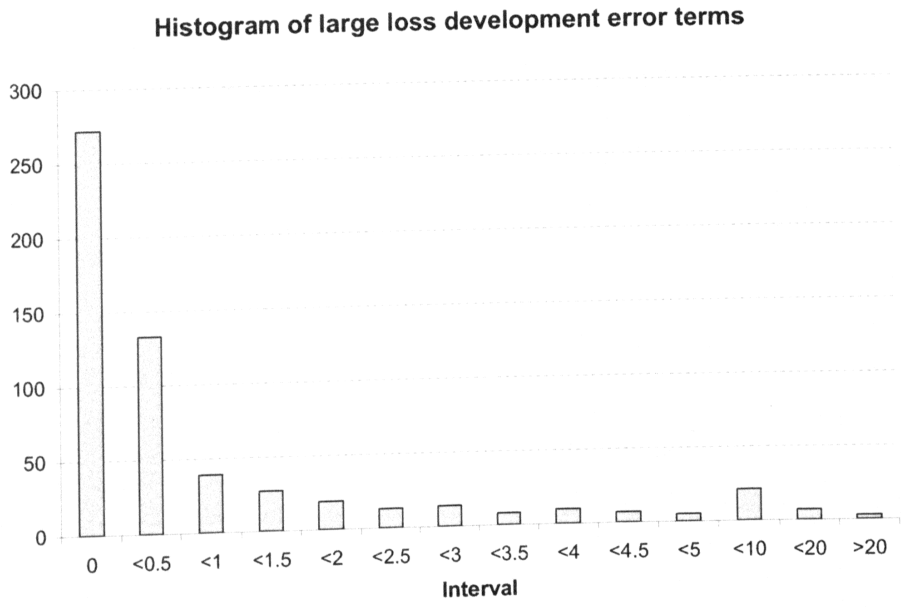


Figure 5.6: Histogram of individual loss error terms

In the Monte Carlo simulation of the reserve in the next section we generate the error terms

directly from the empirical distribution. The empirical mean is equal to 1.1942, which is significantly higher than the assumed 1. This is due to the outlier observation. We can see from the histogram that there are some rather high values. If we remove the highest observation we get a mean value slightly bellow 1. The empirical standard deviation is equal to 4.3273, which is much higher than for the error terms of small losses. This is not surprising, because there are many values at 0 and some rather high observations.

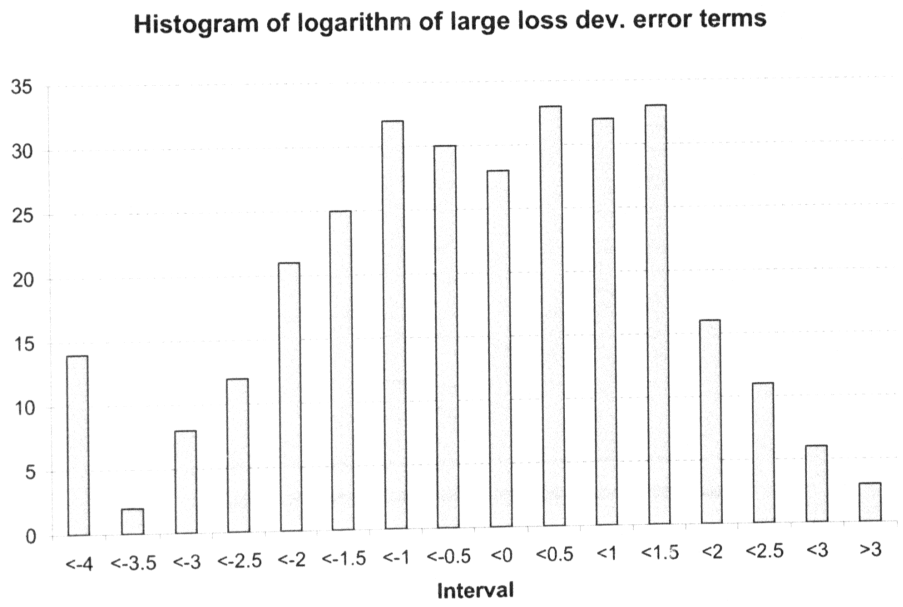


Figure 5.7: Histogram of logarithms of individual loss error terms

5.2 Loss reserve calculation

We calculated loss reserve first with Monte Carlo simulation directly from the definition of our model. We made the additional assumption of lognormal distribution of the variables $Y^l_{t-j,l,k}$ (first payments). The error terms of the development of small and large losses were simulated directly from the empirical distribution.

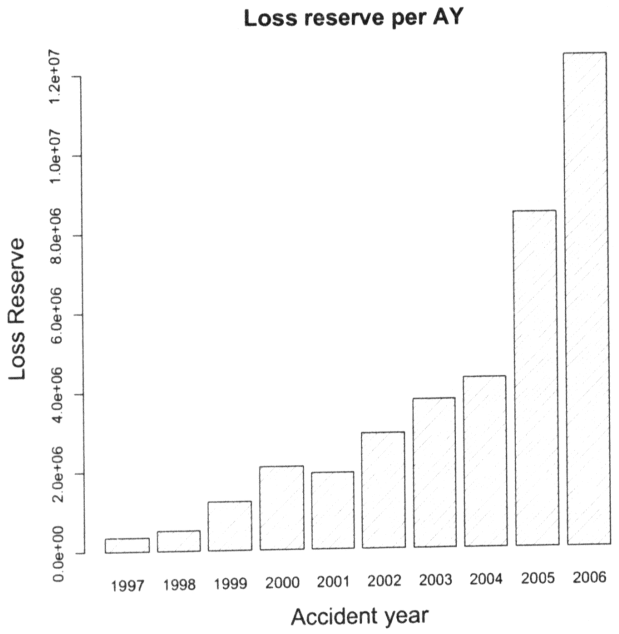


Figure 5.8: Loss reserves for individual accidents years

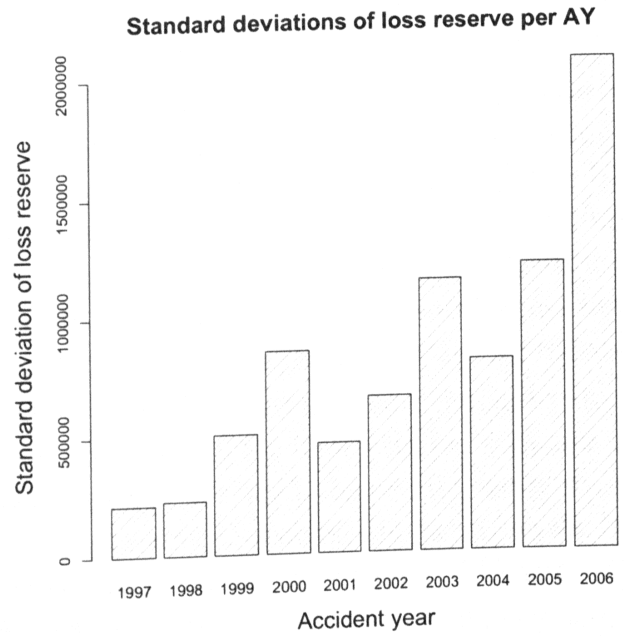


Figure 5.9: Standard deviations of loss reserves for individual accidents years

As we didn't see any significant trends over the accident or development years we use the

same distribution for all error terms. We computed 1000 simulations. All simulation work has been done in the statistical package R[1].

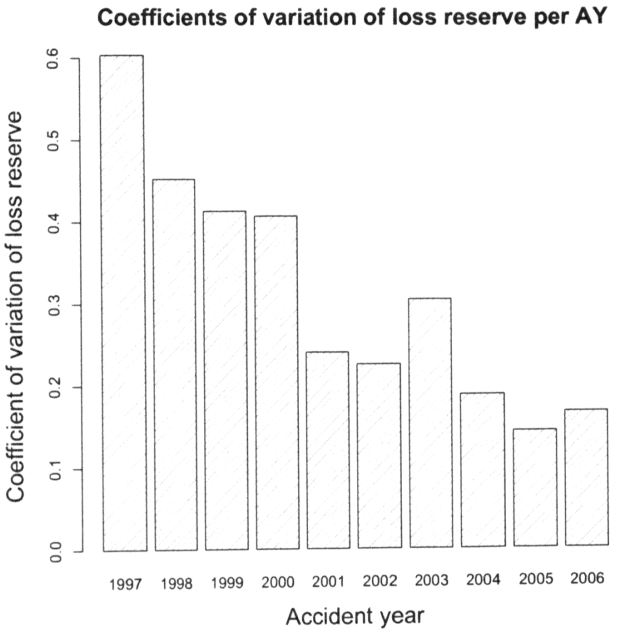


Figure 5.10: Coefficients of variation of loss reserves for individual accidents years

Figure 5.8 shows the expected values of the simulated loss reserves for all accidents years. As expected, the reserves are increasing with increasing accident year. This is due to the fact that old losses are much more developed (paid) and hence need lower reserve. Only in the accident year 2000 we can see that the reserve is higher than the reserve for 2001. This is due the a particular large loss (the highest individual loss in the data) in this accident year. This was a cable way fire in a famous ski area in Austria causing many people to die in the the tunnel. Many Austrian insurance companies are affected by this event.

In a similar way we plot the standard deviation of the loss reserves of all accident years. These are shown in figure 5.9. Here we would expect increasing volatility of the reserves as more recent losses have more uncertainty included. Again, the increasing trend is interrupted by the accident year 2000 and in this case the accident year 2003 as well. We analyzed the individual losses from the accident year 2003 (as the volatility is not caused by number of losses) and found that in this accident year the second largest individual loss happened. The two largest losses are significantly larger than the remaining individual losses.

If we divide the standard deviation through the expected values we get the coefficients of variation. These are shown in figure 5.10. This relative measure of volatility is not dependent on the volume of the open losses. We see a decreasing trend over the accident years. This is, again, intuitive, as the old losses have less diversification than the bulk of the more recent losses. Again, the trend is discontinued in the accident years 2000 and 2003.

Figure 5.11 shows the whole histogram of the loss reserve distribution. We can see that the distribution is rather symmetrical with right tail slightly heavier that the left one. This is supported by the results of other stochastic methods that have been analyzed by the company. The

[1]R is a free version of the statistical software SPLUS. The R software can be downloaded from www.r-project.org/.

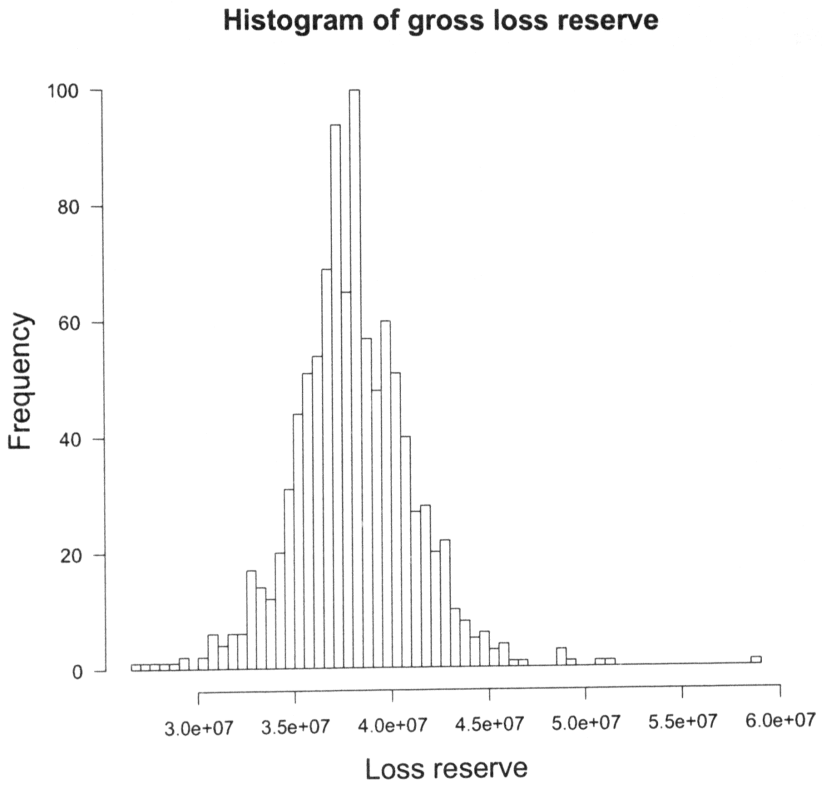


Figure 5.11: Histogram of loss reserve

rather symmetrical distribution can be a bit surprising as the assumptions contain a lognormal distribution for the first loss payments, Poisson distribution for number of paid losses in the first development year and an empirical distribution for the error terms of the development of both small and large losses, which is pretty heavy-tailed. That's why we analyze the particular accident year reserves as well.

Figures 5.12, 5.13 and 5.14 show the histograms of the distribution of loss reserves for all accident years. It can be seen clearly that the particular distributions have much heavier right tails than the overall reserve. In the model for overall reserve we assumed, similarly to the chain ladder method, independence of the accident years, which leads to diversification in the overall reserve.

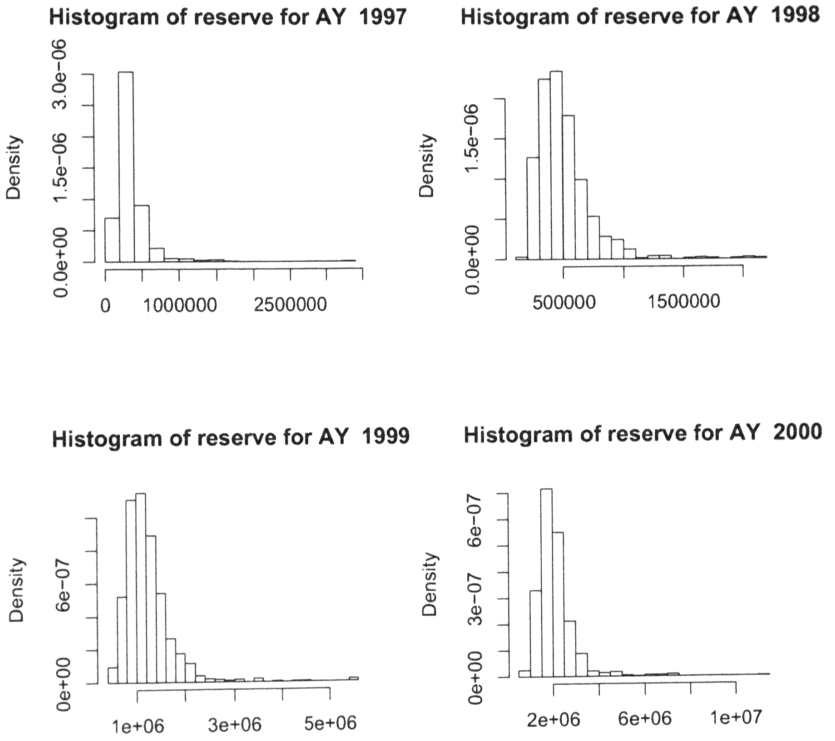


Figure 5.12: Histogram of loss reserves for individual accidents years 1997 – 2000

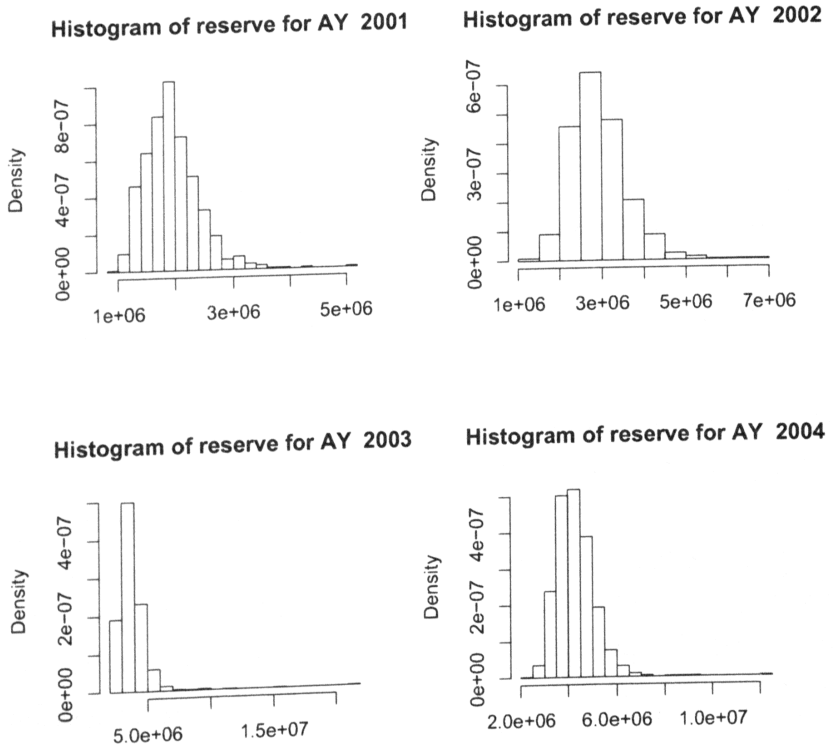


Figure 5.13: Histogram of loss reserves for individual accidents years 2001 – 2004

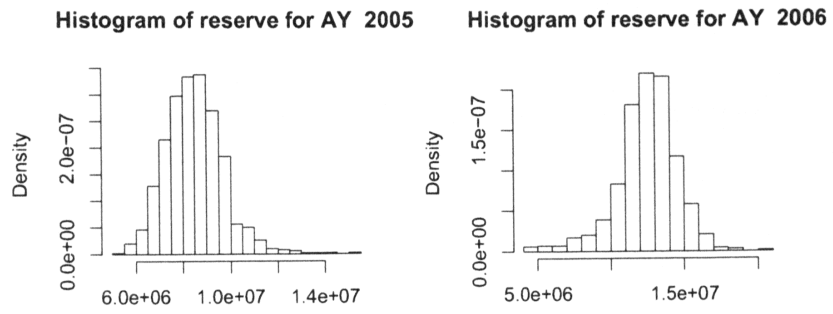


Figure 5.14: Histogram of loss reserves for individual accidents years 2005 – 2006

To illustrate the effect of reinsurance, we simulated loss reserve distribution for the case of an XL-reinsurance with a priority of 400,000. Figure 5.15 shows the density distributions for gross and net loss reserve. We can clearly see the effect of the XL-reinsurance which moves the whole distribution slightly to the left, but especially reduces the right tail of the distribution. The gross distribution is more fat tailed than the net distribution.

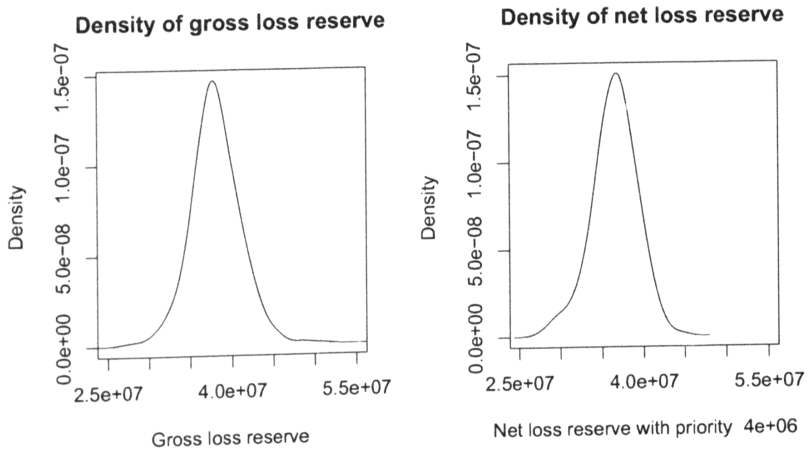


Figure 5.15: Gross and net density (XL-priority 400,000)

The same effect of the XL-reinsurance can be seen from the plot of gross and net distribution functions, that are shown in Figure 5.16. The distribution function of net loss reserve goes much faster to 1 than the distribution function of gross loss reserve.

To illustrate how the analytical formula, that has been developed in previous chapters, can be used, we computed analytically the value of net loss reserve for reinsurance program with various priorities. We plot these values in Figure 5.17 and see that these values are increasing with increasing priority having in limit for indefinite priority the value of gross loss reserve.

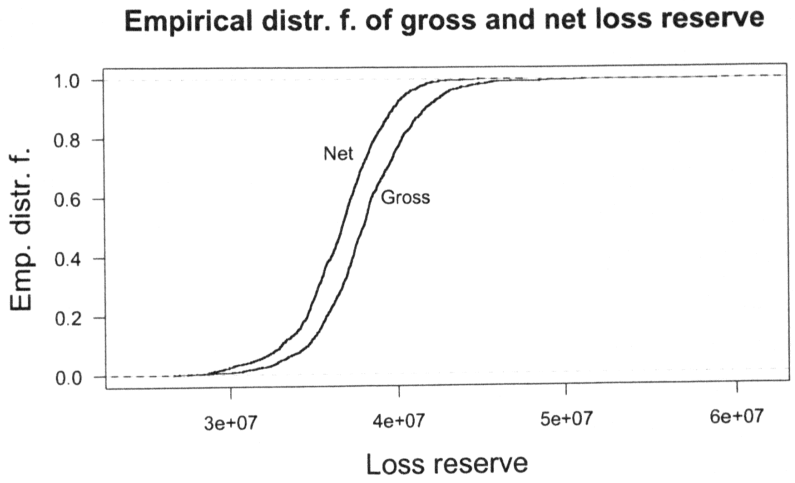


Figure 5.16: Gross and net distribution function (XL-priority 400,000)

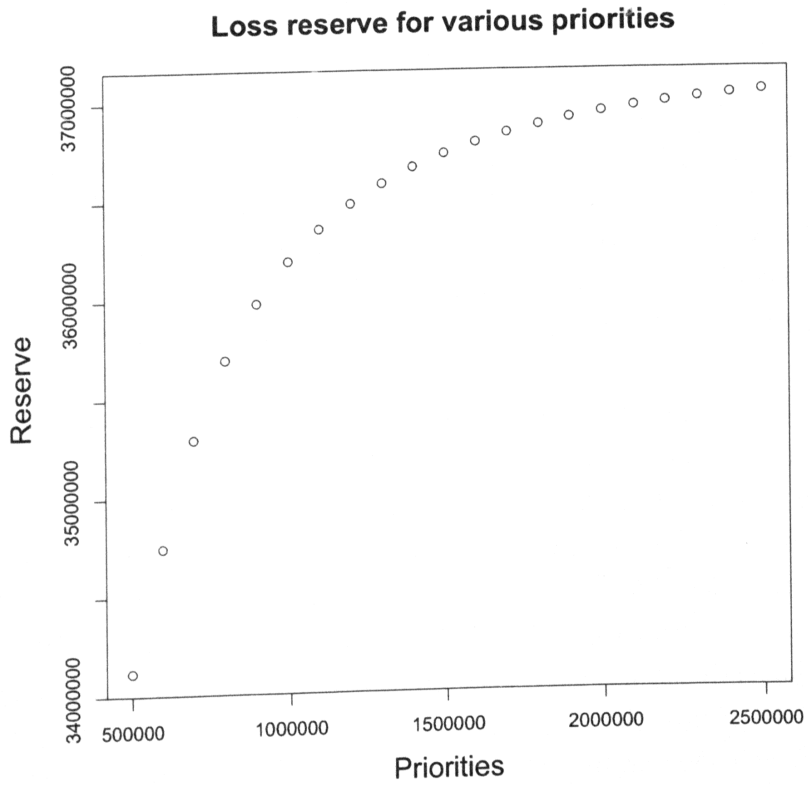


Figure 5.17: Loss reserves for various XL-priorities

Conclusion

With the new solvency framework being currently developed the uncertainty connected with insurance business becomes an important issue. In the P&C business especially the premium and reserve risk are of concern. Several models have already been developed to quantify the uncertainty in loss reserves. However, none of the models can cope with non-proportional reinsurance in a convenient way. In practice, reinsurance is one of the most powerful means to influence risk and risk capital of an insurance company. Inadequate consideration of non-proportional reinsurance within the solvency framework leads to decisions that are not optimal.

This work suggests a model for computing the distribution of loss reserve that enables

- more intensive usage of data (information on individual losses),
- explicit consideration of non-proportional reinsurance,
- deterministic computation of loss reserve in a spreadsheet,
- simulation of the entire loss reserve distribution,
- simulation of risk capital for reserve and premium risk in a total balance sheet view that can be extended to cope with further risk types (investment risk, reinsurance credit risk,...).

We are aware of the problems connected with the model. In particular the model requires rather detailed data information that doesn't have to be available in many cases. Further, the model uses a large number of parameters that have to be estimated. In some cases, the variation due to the parameter uncertainty can even outweight the variation from the pure model itself. Hence, the extension of the model to cope with parameter risk would be one of the important future research topics. Possible approach would be e.g. to use Bayes technique. This approach is used e.g. in [16] or [30].

The model developed in this work is constructed (similar to many other known models) for one line of business. As in practice we are concerned about the loss reserve hold by the company for the whole portfolio, we need to be able to model more lines of business and the dependencies between them. In practice the individual lines of business are often modeled by means of simulation and the resulting distributions are then combined in a way that they give prescribed values of correlation. There has to be done more work on this topic.

In this work the expressions for undiscounted loss reserves are derived. As in the long-tail business the interest rate itself is an important source of risk, the model can be extended to model the discounted loss reserves. The model contains a large number of assumptions as well. E.g. we don't assume that closed losses can be reopened, which could be a problem in some areas, where this can be typically the case.

Last but not least, in this work only the expressions for traditional reinsurance treaties are derived. In practice there can be special reinsurance programs containing multi-year or multi-line contracts that are not discussed in this work.

Appendix A

Second moments of cumulative paid losses conditional on previous year

A.1 Overall cumulative paid losses

To compute $\mathrm{E}[X_{s,j+1}^2|I_{s,j}]$ we divide $X_{s,j+1}$ into losses that have occurred in the triangle up to development year j and losses occurring in development year $j+1$ and write, using the independence of the individual losses,

$$\mathrm{E}\left[X_{s,j+1}^2\middle|I_{s,j}\right] = \mathrm{E}\left[\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y_{s,j+1,k}^{l}\right)^2\middle|I_{s,j}\right] + \mathrm{E}\left[\left(\sum_{k=1}^{\tilde{N}_{s,j+1}} Y_{s,j+1,k}^{j+1}\right)^2\middle|I_{s,j}\right] +$$

$$+2\,\mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y_{s,j+1,k}^{l}\middle|I_{s,j}\right]\mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y_{s,j+1,k}^{j+1}\middle|I_{s,j}\right] = A + B + C.$$

In the following text we compute the terms A, B, and C.
For the first term

$$A = \mathrm{E}\left[\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y_{s,j+1,k}^{l}\right)^2\middle|I_{s,j}\right],$$

we expand the sum and put the expectation into the sum and get

$$A = \mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}\sum_{l'=0}^{j}\sum_{k'=1}^{\tilde{N}_{s,l'}} Y_{s,j+1,k}^{l} Y_{s,j+1,k'}^{l'}\middle|I_{s,j}\right] = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}\sum_{l'=0}^{j}\sum_{k'=1}^{\tilde{N}_{s,l'}} \mathrm{E}\left[Y_{s,j+1,k}^{l} Y_{s,j+1,k'}^{l'}|I_{s,j}\right].$$

Using the multiplicative moment for individual losses gives

$$A = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}\sum_{l'=0}^{j}\sum_{k'=1}^{\tilde{N}_{s,l'}} Y_{s,j,k}^{l} Y_{s,j,k'}^{l'}\mathrm{E}\left[\left(1+(d_j-1)J_{s,j,k}^{l}\tilde{E}_{s,j+1,k}^{l}\right)\left(1+(d_j-1)J_{s,j,k'}^{l'}\tilde{E}_{s,j+1,k'}^{l'}\right)|I_{s,j}\right].$$

In order to use the independence of $\tilde{E}^l_{s,j+1,k}$ and $\tilde{E}^{l'}_{s,j+1,k'}$ for different l, l', k, k', we separate the term with $l = l' \ \& \ k = k'$ and write

$$A = \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} \sum_{l'=0}^{j} \sum_{\substack{k'=1 \\ (k,l)\neq(k',l')}}^{\tilde{N}_{s,l'}} Y^l_{s,j,k} Y^{l'}_{s,j,k'} \mathrm{E}\left[1 + (d_j - 1) J^l_{s,j,k} \tilde{E}^l_{s,j+1,k} \middle| I_{s,j}\right] \cdot$$

$$\cdot \mathrm{E}\left[1 + (d_j - 1) J^{l'}_{s,j,k'} \tilde{E}^{l'}_{s,j+1,k'} \middle| I_{s,j}\right] +$$

$$+ \sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} (Y^l_{s,j,k})^2 \mathrm{E}\left[1 + (d_j - 1) J^l_{s,j,k} \tilde{E}^l_{s,j+1,k} \middle| I_{s,j}\right]^2 = A_1 + A_2.$$

To compute the term A_1 we split the sum according to whether $Y^l_{s,j,k}$ and $Y^{l'}_{s,j,k'}$ are closed or open. For the case where both are closed, i.e. $J^l_{s,j,k} = 0 \ \& \ J^{l'}_{s,j,k'} = 0$, which we denote condition C_1, we get

$$\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} \sum_{l'=0}^{j} \sum_{\substack{k'=1 \\ (k,l)\neq(k',l')\&C_1}}^{\tilde{N}_{s,l'}} Y^l_{s,j,k} Y^{l'}_{s,j,k'},$$

which is with previous notation equal to

$$(X^C_{s,j})^2 - X^{C(2)}_{s,j}.$$

For the case where both are open, i.e. $J^l_{s,j,k} = 1 \ \& \ J^{l'}_{s,j,k'} = 1$, which we denote condition C_2, we have

$$\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} \sum_{l'=0}^{j} \sum_{\substack{k'=1 \\ (k,l)\neq(k',l')\&C_2}}^{\tilde{N}_{s,l'}} Y^l_{s,j,k} Y^{l'}_{s,j,k'} d_j^2,$$

which is with previous notation equal to

$$\left((X^O_{s,j})^2 - X^{O(2)}_{s,j}\right) d_j^2.$$

For the case where one of the losses $Y^l_{s,j,k}$ and $Y^{l'}_{s,j,k'}$ is closed and one open, i.e. $(J^l_{s,j,k} = 0 \ \& \ J^{l'}_{s,j,k'} = 1)$ or $(J^l_{s,j,k} = 1 \ \& \ J^{l'}_{s,j,k'} = 0)$, which we denote condition C_3, we get

$$\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} \sum_{l'=0}^{j} \sum_{\substack{k'=1 \\ (k,l)\neq(k',l')\&C_3}}^{\tilde{N}_{s,l'}} Y^l_{s,j,k} Y^{l'}_{s,j,k'} d_j,$$

which can be rewritten as

$$\left(X^2_{s,j} - (X^C_{s,j})^2 - (X^O_{s,j})^2\right) d_j.$$

To summarize, we get for the term A_1 the expression

$$A_1 = (X^C_{s,j})^2 - X^{C(2)}_{s,j} + \left((X^O_{s,j})^2 - X^{O(2)}_{s,j}\right) d_j^2 + \left(X^2_{s,j} - (X^C_{s,j})^2 - (X^O_{s,j})^2\right) d_j.$$

Similarly, for the term A_2 we split the sum according to whether $Y^l_{s,j,k}$ is closed ($J^l_{s,j,k} = 0$) or open ($J^l_{s,j,k} = 1$) and get

$$A_2 = \sum_{l=0}^{j} \sum_{\substack{k=1 \\ J^l_{s,j,k}=0}}^{\tilde{N}_{s,l}} (Y^l_{s,j,k})^2 + \sum_{l=0}^{j} \sum_{\substack{k=1 \\ J^l_{s,j,k}=1}}^{\tilde{N}_{s,l}} (Y^l_{s,j,k})^2 d_j^{(2)} = X^{C(2)}_{s,j} + X^{O(2)}_{s,j} d_j^{(2)}.$$

Adding up A_1 and A_2 we get

$$A = (X^C_{s,j})^2 + \left((X^O_{s,j})^2 - X^{O(2)}_{s,j}\right) d_j^2 + \left(X^2_{s,j} - (X^C_{s,j})^2 - (X^O_{s,j})^2\right) d_j + X^{O(2)}_{s,j} d_j^{(2)}.$$

Simplifying this expression and putting in $X^C_{s,j} = X_{s,j} - X^O_{s,j}$ and $X^{C(2)}_{s,j} = X^{(2)}_{s,j} - X^{O(2)}_{s,j}$ gives

$$A = X^2_{s,j} + (X^O_{s,j})^2 (d_j - 1)^2 + 2X_{s,j} X^O_{s,j} (d_j - 1) + X^{O(2)}_{s,j} (d_j^{(2)} - d_j^2).$$

The second term,

$$B = \mathrm{E}\left[\left(\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k}\right)^2 \middle| I_{s,j}\right],$$

can be expressed as

$$B = \mathrm{Var}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k} \middle| I_{s,j}\right] + \left(\mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k} \middle| I_{s,j}\right]\right)^2,$$

which is with the assumption of the conditional Poisson distribution and the previous results equal to

$$B = b_j N_{s,j} m_s^{(2)} + b_j^2 N^2_{s,j} m_s^2.$$

The third term,

$$C = 2\,\mathrm{E}\left[\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{s,l}} Y^l_{s,j+1,k} \middle| I_{s,j}\right] \mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k} \middle| I_{s,j}\right],$$

can be with previous results expressed as

$$C = 2\left(X_{s,j} + (d_j - 1) X^O_{s,j}\right) b_j N_{s,j} m_s.$$

Summarizing, we have

$$\mathrm{E}[X^2_{s,j+1} | I_{s,j}] = X^2_{s,j} + (X^O_{s,j})^2 (d_j - 1)^2 + 2X_{s,j} X^O_{s,j} (d_j - 1) + X^{O(2)}_{s,j} (d_j^{(2)} - d_j^2) +$$
$$+ b_j N_{s,j} m_s^{(2)} + b_j^2 N^2_{s,j} m_s^2 + 2X_{s,j} b_j N_{s,j} m_s + 2(d_j - 1) X^O_{s,j} b_j N_{s,j} m_s. \qquad \text{(A.1)}$$

A.2 Cumulative open paid losses

Similar to the computation for overall losses we express the second moment,

$$\mathrm{E}\left[(X^{O}_{s,j+1})^2 \middle| I_{s,j}\right] = \mathrm{E}\left[\left(\sum_{l=0}^{j+1}\sum_{k=1}^{\tilde{N}_{s,l}} Y^{l}_{s,j+1,k} J^{l}_{s,j+1,k}\right)^2 \middle| I_{s,j}\right],$$

using the independence of individual losses as

$$\mathrm{E}\left[\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y^{l}_{s,j+1,k} J^{l}_{s,j+1,k}\right)^2 \middle| I_{s,j}\right] + \mathrm{E}\left[\left(\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k} J^{j+1}_{s,j+1,k}\right)^2 \middle| I_{s,j}\right] +$$

$$+2\,\mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y^{l}_{s,j+1,k} J^{l}_{s,j+1,k} \middle| I_{s,j}\right] \mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y^{j+1}_{s,j+1,k} J^{j+1}_{s,j+1,k} \middle| I_{s,j}\right] = A + B + C.$$

In the following text we compute the terms A, B, and C.
For the first term,

$$A = \mathrm{E}\left[\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y^{l}_{s,j+1,k} J^{l}_{s,j+1,k}\right)^2 \middle| I_{s,j}\right],$$

we write

$$A = \mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}\sum_{l'=0}^{j}\sum_{k'=1}^{\tilde{N}_{s,l'}} Y^{l}_{s,j+1,k} Y^{l'}_{s,j+1,k'} J^{l}_{s,j+1,k} J^{l'}_{s,j+1,k'} \middle| I_{s,j}\right].$$

Putting the expected value into the sum gives

$$A = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}\sum_{l'=0}^{j}\sum_{k'=1}^{\tilde{N}_{s,l'}} \mathrm{E}\left[Y^{l}_{s,j+1,k} Y^{l'}_{s,j+1,k'} J^{l}_{s,j+1,k} J^{l'}_{s,j+1,k'} \middle| I_{s,j}\right].$$

Using the multiplicative model for individual losses this can be expressed as

$$A = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}\sum_{l'=0}^{j}\sum_{k'=1}^{\tilde{N}_{s,l'}} Y^{l}_{s,j,k} Y^{l'}_{s,j,k'} \cdot$$

$$\cdot\mathrm{E}\left[J^{l}_{s,j+1,k} J^{l'}_{s,j+1,k'} \left(1 + (d_j - 1) J^{l}_{s,j,k} \tilde{E}^{l}_{s,j+1,k}\right)\left(1 + (d_j - 1) J^{l'}_{s,j,k'} \tilde{E}^{l'}_{s,j+1,k'}\right) \middle| I_{s,j}\right].$$

Separating the terms with $l = l'$ and $k = k'$ and using the independence of $\tilde{E}^{l}_{s,j+1,k}$ and $\tilde{E}^{l'}_{s,j+1,k'}$ for different l, l', k, k' gives

$$A = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}\sum_{l'=0}^{j} \sum_{\substack{k'=1 \\ (k,l)\neq(k',l')}}^{\tilde{N}_{s,l'}} Y^{l}_{s,j,k} Y^{l'}_{s,j,k'} \mathrm{E}\left[J^{l}_{s,j+1,k} J^{l'}_{s,j+1,k'} \middle| I_{s,j}\right].$$

$$\cdot \mathrm{E}\left[1+(d_j-1)J^l_{s,j,k}\tilde{E}^l_{s,j+1,k}|I_{s,j}\right]\mathrm{E}\left[1+(d_j-1)J^{l'}_{s,j,k'}\tilde{E}^{l'}_{s,j+1,k'}|I_{s,j}\right]+$$

$$+\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}(Y^l_{s,j,k})^2\mathrm{E}\left[(J^l_{s,j,k})^2|I_{s,j}\right]\mathrm{E}\left[1+(d_j-1)J^l_{s,j,k}\tilde{E}^l_{s,j+1,k}|I_{s,j}\right]^2=A_1+A_2.$$

To compute the term A_1 we again split the sum according to whether $Y^l_{s,j,k}$ and $Y^{l'}_{s,j,k'}$ are closed or open. It is easy to see that the terms with $J^l_{s,j,k}=0$ or $J^{l'}_{s,j,k'}=0$ are zero. Thus, only the terms with $J^l_{s,j,k}=1$ & $J^{l'}_{s,j,k'}=1$ remain and we get

$$A_1=\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}\sum_{l'=0}^{j}\sum_{\substack{k'=1\\(k,l)\neq(k',l')}}^{\tilde{N}_{s,l'}}Y^l_{s,j,k}Y^{l'}_{s,j,k'}J^l_{s,j,k}J^{l'}_{s,j,k'}d_j^2(1-q_{j+1})^2,$$

which is with previous notation equal to

$$A_1=d_j^2(1-q_{j+1})^2\left((X^O_{s,j})^2-X^{O(2)}_{s,j}\right).$$

Similarly, for the term

$$A_2=\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}}(Y^l_{s,j,k})^2\mathrm{E}\left[(J^l_{s,j,k})^2|I_{s,j}\right]\mathrm{E}\left[1+(d_j-1)J^l_{s,j,k}\tilde{E}^l_{s,j+1,k}|I_{s,j}\right]^2,$$

we split the sum according to whether $Y^l_{s,j,k}$ is closed or open and see that the terms with $\{J^l_{s,j,k}=0\}$ are zero. Hence, we get

$$A_2=\sum_{l=0}^{j}\sum_{\substack{k'=1\\J^l_{s,j,k}=1}}^{\tilde{N}_{s,l}}(Y^l_{s,j,k})^2d_j^{(2)}\mathrm{E}\left[(J^l_{s,j+1,k})^2|J^l_{s,j,k}=1\right],$$

which is equal to

$$A_2=X^{O(2)}_{s,j}d_j^{(2)}(1-q_{j+1}).$$

Adding up A_1 and A_2 we get

$$A=d_j^2(1-q_{j+1})^2\left((X^O_{s,j})^2-X^{O(2)}_{s,j}\right)+X^{O(2)}_{s,j}d_j^{(2)}(1-q_{j+1}).$$

The term

$$B=\mathrm{E}\left[\left(\sum_{k=1}^{\tilde{N}_{s,j+1}}Y^{j+1}_{s,j+1,k}J^{j+1}_{s,j+1,k}\right)^2\middle|I_{s,j}\right],$$

can be expressed as

$$B=\mathrm{Var}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}}Y^{j+1}_{s,j+1,k}J^{j+1}_{s,j+1,k}\middle|I_{s,j}\right]+\left(\mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}}Y^{j+1}_{s,j+1,k}J^{j+1}_{s,j+1,k}\middle|I_{s,j}\right]\right)^2,$$

from which we get with the assumption of the Poisson distribution

$$B = b_j N_{s,j} m_s^{(2)} (1 - q_{j+1}) + b_j^2 N_{s,j}^2 m_s^2 (1 - q_{j+1})^2.$$

The term

$$C = 2\,\mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{s,l}} Y_{s,j+1,k}^{l} J_{s,j+1,k}^{l} \middle| I_{s,j}\right] \mathrm{E}\left[\sum_{k=1}^{\tilde{N}_{s,j+1}} Y_{s,j+1,k}^{j+1} J_{s,j+1,k}^{j+1} \middle| I_{s,j}\right]$$

can be with previous results expressed as

$$C = 2 d_j X_{s,j}^{O} b_j N_{s,j} m_s (1 - q_{j+1})^2.$$

Summarizing, we have

$$\mathrm{E}\left[(X_{s,j+1}^{O})^2 | I_{s,j}\right] = (X_{s,j}^{O})^2 (1 - q_{j+1})^2 d_j^2 + X_{s,j}^{O(2)} \left[(1 - q_{j+1}) d_j^{(2)} - (1 - q_{j+1})^2 d_j^2\right] +$$

$$+ b_j N_{s,j} m_s^{(2)} (1 - q_{j+1}) + b_j^2 N_{s,j}^2 m_s^2 (1 - q_{j+1})^2 + 2 d_j X_{s,j}^{O} b_j N_{s,j} m_s (1 - q_{j+1})^2. \quad \text{(A.2)}$$

A.3 Cumulative closed paid losses

The second conditional moment $\mathrm{E}[(X_{s,j+1}^{C})^2 | I_{s,j}]$ can be computed using the relation

$$X_{s,j+1}^{C} = X_{s,j+1} - X_{s,j+1}^{O},$$

as

$$\mathrm{E}\left[(X_{s,j+1}^{C})^2 | I_{s,j}\right] = \mathrm{E}\left[(X_{s,j+1} - X_{s,j+1}^{O})^2 | I_{s,j}\right],$$

which gives

$$\mathrm{E}\left[(X_{s,j+1}^{C})^2 | I_{s,j}\right] = \mathrm{E}\left[X_{s,j+1}^2 | I_{s,j}\right] - 2\mathrm{E}\left[X_{s,j+1} X_{s,j+1}^{O} | I_{s,j}\right] + \mathrm{E}\left[(X_{s,j+1}^{O})^2 | I_{s,j}\right]. \quad \text{(A.3)}$$

As we can't expect $X_{s,j+1}$ and $X_{s,j+1}^{O}$ to be independent, we have to split $X_{s,j+1}$ in the second expected value in (A.3) into open and closed losses and get for this term

$$\mathrm{E}\left[X_{s,j+1} X_{s,j+1}^{O} | I_{s,j}\right] = \mathrm{E}\left[(X_{s,j+1}^{O})^2 | I_{s,j}\right] + \mathrm{E}\left[X_{s,j+1}^{C} | I_{s,j}\right] \mathrm{E}\left[X_{s,j+1}^{O} | I_{s,j}\right].$$

Putting this into (A.3) gives

$$\mathrm{E}\left[(X_{s,j+1}^{C})^2 | I_{s,j}\right] = \mathrm{E}\left[X_{s,j+1}^2 | I_{s,j}\right] - \mathrm{E}\left[(X_{s,j+1}^{O})^2 | I_{s,j}\right] - 2\mathrm{E}\left[X_{s,j+1}^{C} | I_{s,j}\right] \mathrm{E}\left[X_{s,j+1}^{O} | I_{s,j}\right].$$

Appendix B

Second moments of cumulative paid losses conditional on information at time t

B.1 Cumulative ultimate open paid losses

Using the theorem on conditional expected values we have

$$\mathrm{E}\left[(X^{O}_{t-j,r})^2 | I_{t-j,j}\right] = \mathrm{E}\left[\mathrm{E}\left[(X^{O}_{t-j,r})^2 | I_{t-j,r-1}\right] | I_{t-j,j}\right].$$

Putting in the previous results for $\mathrm{E}[(X^{O}_{s,j+1})^2 | I_{s,j}]$ we get

$$\begin{aligned}\mathrm{E}\left[(X^{O}_{t-j,r})^2 | I_{t-j,j}\right] = \mathrm{E}\Big[& b_{r-1} m^{(2)}_{t-j} (1-q_r) N_{t-j,r-1} + b^2_{r-1} m^2_{t-j} (1-q_r)^2 N^2_{t-j,r-1} + \\ & + 2 d_{r-1} b_{r-1} m_{t-j} (1-q_r)^2 N_{t-j,r-1} X^{O}_{t-j,r-1} + d^2_{r-1} (1-q_r)^2 (X^{O}_{t-j,r-1})^2 + \\ & + \left((1-q_r) d^{(2)}_{r-1} - (1-q_r)^2 d^2_{r-1}\right) X^{O(2)}_{t-j,r-1} | I_{t-j,j}\Big].\end{aligned}$$

Using the independence of loss numbers and the loss severities we can write

$$\begin{aligned}\mathrm{E}\left[(X^{O}_{t-j,r})^2 | I_{t-j,j}\right] = & \, b_{r-1} m^{(2)}_{t-j} (1-q_r) \mathrm{E}\left[N_{t-j,r-1} | I_{t-j,j}\right] + \\ & + b^2_{r-1} m^2_{t-j} (1-q_r)^2 \mathrm{E}\left[N^2_{t-j,r-1} | I_{t-j,j}\right] + \\ & + 2 d_{r-1} b_{r-1} m_{t-j} (1-q_r)^2 \mathrm{E}\left[N_{t-j,r-1} | I_{t-j,j}\right] \mathrm{E}\left[X^{O}_{t-j,r-1} | I_{t-j,j}\right] + \\ & + d^2_{r-1} (1-q_r)^2 \mathrm{E}\left[(X^{O}_{t-j,r-1})^2 | I_{t-j,j}\right] + \left((1-q_r) d^{(2)}_{r-1} - (1-q_r)^2 d^2_{r-1}\right) \mathrm{E}\left[X^{O(2)}_{t-j,r-1} | I_{t-j,j}\right].\end{aligned}$$

For $\mathrm{E}[X^{O}_{t-j,r-1} | I_{t-j,j}]$ it easy to see that we get, in analogy to $\mathrm{E}[X_{t-j,r-1} | I_{t-j,j}]$, the relation

$$\mathrm{E}\left[X^{O}_{t-j,r-1} | I_{t-j,j}\right] = X^{(o)2}_{t-j,j} D^{(2)}_{j,r-2} Q_{j+1,r-1} + N_{t-j,j} \tilde{Q}^{\beta(2)}_{t-j,j+1,r-1},$$

where

$$\tilde{Q}^{\beta(2)}_{t-j,j+1,i} = \sum_{h=j+1}^{r} D^{(2)}_{h,r-1} Q_{h,r} A^{(2)}_{t-j,j,h-1}, \tag{B.1}$$

and

$$D^{(2)}_{j,i-1} = D_{i,j} = d^{(2)}_i \cdots d^{(2)}_j, \quad D_{j+1,j} := 1, \ j = r-1, r, \tag{B.2}$$

and

$$A^{(2)}_{t-j,j,i} = b_i m^{(2)}_{t-j} \prod_{k=j}^{i-1} (1 + b_k). \tag{B.3}$$

Putting in the expressions for the expected values $\mathrm{E}[N_{t-j,r-1}|I_{t-j,j}]$, $\mathrm{E}[X^O_{t-j,r-1}|I_{t-j,j}]$, $\mathrm{E}[N^2_{t-j,r-1}|I_{t-j,j}]$, and $\mathrm{E}[X^{O(2)}_{t-j,r-1}|I_{t-j,j}]$, and rearranging the expression we get

$$\mathrm{E}\left[(X^O_{t-j,r})^2 \middle| I_{t-j,j}\right] = d^2_{r-1}(1-q_r)^2 \mathrm{E}\left[(X^O_{t-j,r-1})^2 \middle| I_{t-j,j}\right] + N_{t-j,j} X^O_{t-j,j} H^{O(1)}_{t-j,j,r-1} +$$

$$+ N^2_{t-j,j} H^{O(2)}_{t-j,j,r-1} + N_{t-j,j} H^{O(3)}_{t-j,j,r-1} + X^{O(2)}_{t-j,j} H^{O(4)}_{t-j,j,r-1},$$

where

$$\begin{aligned}
H^{O(1)}_{t-j,j,i} &= 2d_i(1-q_{i+1})^2 D_{j,i-1} Q_{j+1,i} A_{t-j,j,i}, \\
H^{O(2)}_{t-j,j,i} &= (1-q_{i+1})^2 A^2_{t-j,j,i} + 2d_i(1-q_{i+1})^2 \tilde{Q}^{\beta}_{t-j,j+1,i} A_{t-j,j,i}, \\
H^{O(3)}_{t-j,j,i} &= (1-q_{i+1}) A^{(2)}_{t-j,j,i} + (1-q_{i+1})^2 b^2_{t-j,i} m^2_{t-j} \kappa_{j,i} + \\
&\quad + \left((1-q_{i+1}) d^{(2)}_i - (1-q_{i+1})^2 d^2_i\right) \tilde{Q}^{\beta(2)}_{t-j,j+1,i}, \\
H^{O(4)}_{t-j,j,i} &= \left((1-q_{i+1}) d^{(2)}_i - (1-q_{i+1})^2 d^2_i\right) D^{(2)}_{j,i-1} Q_{j+1,i}.
\end{aligned} \tag{B.4}$$

In order to use recursion we denote

$$H^O_{t-j,j,r-1} = N_{t-j,j} X^O_{t-j,j} H^{O(1)}_{t-j,j,r-1} + N^2_{t-j,j} H^{O(2)}_{t-j,j,r-1} +$$

$$+ N_{t-j,j} H^{O(3)}_{t-j,j,r-1} + X^{O(2)}_{t-j,j} H^{O(4)}_{t-j,j,r-1},$$

the part containing only random variables known at $I_{t-j,j}$, and write

$$\mathrm{E}\left[(X^O_{t-j,r})^2 \middle| I_{t-j,j}\right] = d^2_{r-1}(1-q_r)^2 \mathrm{E}\left[(X^O_{t-j,r-1})^2 \middle| I_{t-j,j}\right] + H^O_{t-j,j,r-1}.$$

Applying recursion we get

$$\mathrm{E}\left[(X^O_{t-j,r})^2 \middle| I_{t-j,j}\right] = d^2_{r-1}(1-q_r)^2 \Big[d^2_{r-2}(1-q_{r-1})^2 \Big[\ldots$$

$$\ldots \left[d^2_j (1-q_{j+1})^2 (X^O_{t-j,j})^2 + H^O_{t-j,j,j}\right] + \ldots \Big] + H^O_{t-j,j,r-2} \Big] + H^O_{t-j,j,r-1}.$$

Expanding the brackets the expression becomes

$$\mathrm{E}\left[(X^O_{t-j,r})^2|I_{t-j,j}\right] = D^2_{j,r-1}Q^2_{j+1,r}(X^O_{t-j,j})^2 + \sum_{h=j+1}^{r-1} D^2_{h,r-1}Q^2_{h+1,r}H^O_{t-j,j,h-1} + H^O_{t-j,j,r-1}.$$

Defining $Q_{r+1,r} := 1$ and $D_{r,r-1} := 1$ this can be simplified to

$$\mathrm{E}\left[(X^O_{t-j,r})^2|I_{t-j,j}\right] = D^2_{j,r-1}Q^2_{j+1,r}(X^O_{t-j,j})^2 + \sum_{h=j+1}^{r} D^2_{h,r-1}Q^2_{h+1,r}H^O_{t-j,j,h-1}.$$

Putting in for $H^O_{t-j,j,h-1}$ we get

$$\mathrm{E}\left[(X^O_{t-j,r})^2|I_{t-j,j}\right] = D^2_{j,r-1}Q^2_{j+1,r}(X^O_{t-j,j})^2 +$$

$$+ \sum_{h=j+1}^{r} D^2_{h,r-1}Q^2_{h+1,r}H^{O(1)}_{t-j,j,h-1}N_{t-j,j}X^O_{t-j,j} + \sum_{h=j+1}^{r} D^2_{h,r-1}Q^2_{h+1,r}H^{O(2)}_{t-j,j,h-1}N^2_{t-j,j} +$$

$$+ \sum_{h=j+1}^{r} D^2_{h,r-1}Q^2_{h+1,r}H^{O(3)}_{t-j,j,h-1}N_{t-j,j} + \sum_{h=j+1}^{r} D^2_{h,r-1}Q^2_{h+1,r}H^{O(4)}_{t-j,j,h-1}X^{O(2)}_{t-j,j}.$$

This can be expressed as

$$\mathrm{E}\left[(X^O_{t-j,r})^2|I_{t-j,j}\right] = D^2_{j,r-1}Q^2_{j+1,r}(X^O_{t-j,j})^2 + \tilde{H}^{O(1)}_{t-j,j,r}N_{t-j,j}X^O_{t-j,j} +$$

$$+\tilde{H}^{O(2)}_{t-j,j,r}N^2_{t-j,j} + \tilde{H}^{O(3)}_{t-j,j,r}N_{t-j,j} + \tilde{H}^{O(4)}_{t-j,j,r}X^{O(2)}_{t-j,j},$$

where

$$\tilde{H}^{O(i)}_{t-j,j,r} = \sum_{h=j+1}^{r} D^2_{h,r-1}Q^2_{h+1,r}H^{O(i)}_{t-j,j,h-1}, \quad i = 1, ..., 4. \tag{B.5}$$

Values $N_{t-j,j}$, $X^O_{t-j,j}$ and $X^{O(2)}_{t-j,j}$ are at time t known and coefficients $\tilde{H}^{O(i)}_{t-j,j,r}$ are composed only from the parameters of the model.

B.2 Cumulative overall ultimate paid losses

To express the conditional second moment $\mathrm{E}[X^2_{t-j,r}|I_{t-j,j}]$ we use the theorem on conditional expected values and write

$$\mathrm{E}\left[X^2_{t-j,r}|I_{t-j,j}\right] = \mathrm{E}\left[\mathrm{E}\left[X^2_{t-j,r}|I_{t-j,r-1}\right]|I_{t-j,j}\right],$$

Putting in the relation for $\mathrm{E}[X^2_{s,j+1}|I_{s,j}]$ gives

$$\mathrm{E}\left[X^2_{t-j,r}|I_{t-j,j}\right] = \mathrm{E}\Big[X^2_{t-j,r-1} + (X^O_{t-j,r-1})^2(d_{r-1}-1)^2 + 2X_{t-j,r-1}X^O_{t-j,r-1}(d_{r-1}-1) +$$

$$+(d_{r-1}^{(2)} - d_{r-1})X_{t-j,r-1}^{O(2)} + b_{t-j,r-1}m_{t-j}^{(2)}N_{t-j,r-1} + b_{t-j,r-1}^2 m_{t-j}^2 N_{t-j,r-1}^2 +$$

$$+2\, b_{t-j,r-1}m_{t-j}N_{t-j,r-1}X_{t-j,r-1} + 2(d_{r-1}-1)X_{t-j,r-1}^{O}b_{t-j,r-1}m_{t-j}N_{t-j,r-1}\big|I_{t-j,j}\Big].$$

Using the independence of loss numbers and the loss severities we can write

$$\mathrm{E}\left[X_{t-j,r}^2\big|I_{t-j,j}\right] = \mathrm{E}\big[X_{t-j,r-1}^2\big|I_{t-j,j}\big] + \mathrm{E}\big[X_{t-j,r-1}^{O}\big|I_{t-j,j}\big](d_{r-1}-1)^2 +$$

$$+2(d_{r-1}-1)\mathrm{E}\Big[\left(X_{t-j,r-1}^{C} + X_{t-j,r-1}^{O}\right)X_{t-j,r-1}^{O}\big|I_{t-j,j}\Big] + (d_{r-1}^{(2)} - d_{r-1})\mathrm{E}\big[X_{t-j,r-1}^{O(2)}\big|I_{t-j,j}\big] +$$

$$+b_{t-j,r-1}m_{t-j}^{(2)}\mathrm{E}\big[N_{t-j,r-1}\big|I_{t-j,j}\big] + b_{t-j,r-1}^2 m_{t-j}^2\mathrm{E}\big[N_{t-j,r-1}^2\big|I_{t-j,j}\big] +$$

$$+2b_{t-j,r-1}m_{t-j}\mathrm{E}\big[N_{t-j,r-1}\big|I_{t-j,j}\big]\mathrm{E}\big[X_{t-j,r-1}\big|I_{t-j,j}\big] +$$

$$+2(d_{r-1}-1)b_{t-j,r-1}m_{t-j}\mathrm{E}\big[N_{t-j,r-1}\big|I_{t-j,j}\big]\mathrm{E}\big[X_{t-j,r-1}^{O}\big|I_{t-j,j}\big].$$

Putting in the expressions for the expected values and simplifying and rearranging the expression we get

$$\mathrm{E}\left[X_{t-j,r}^2\big|I_{t-j,j}\right] = \mathrm{E}\left[X_{t-j,r-1}^2\big|I_{t-j,j}\right] + H_{t-j,j,r-1}^{(1)}N_{t-j,j}X_{t-j,j}^{O} + H_{t-j,j,r-1}^{(2)}N_{t-j,j}^2 +$$

$$+H_{t-j,j,r-1}^{(3)}N_{t-j,j} + H_{t-j,j,r-1}^{(4)}X_{t-j,j}^{O(2)} + H_{t-j,j,r-1}^{(5)}(X_{t-j,j}^{O})^2 +$$

$$+H_{t-j,j,r-1}^{(6)}X_{t-j,j}X_{t-j,j}^{O} + H_{t-j,j,r-1}^{(7)}N_{t-j,j}X_{t-j,j},$$

where

$$\begin{aligned}
H_{t-j,j,r-1}^{(1)} &= (d_{r-1}^2-1)H_{t-j,j,r-1}^{O(1)} + 2A_{t,j,j,r-1}\tilde{Q}_{t-j,j,r-2}^{D} + \\
&\quad +2(d_{r-1}-1)\tilde{Q}_{t-j,j,r-2}^{D}\tilde{Q}_{t-j,j+1,r-1}^{\beta} + \\
&\quad +2(d_{r-1}-1)D_{j,r-2}Q_{t-j,r-1}\big(\tilde{Q}_{t-j,j,r-2}^{A} - 2\tilde{Q}_{t-j,j+1,r-1}^{\beta} + A_{t,j,j,r-1},\big) \\
H_{t-j,j,r-1}^{(2)} &= (d_{r-1}^2-1)H_{t-j,j,r-1}^{O(2)} + 2(d_{r-1}-1)\tilde{Q}_{t-j,j,r-2}^{A}\tilde{Q}_{t-j,j+1,r-1}^{\beta} + \\
&\quad +A_{t,j,j,r-1}^2 - 2(d_{r-1}-1)\left(\tilde{Q}_{t-j,j+1,r-1}^{\beta}\right)^2 + 2A_{t,j,j,r-1}\tilde{Q}_{t-j,j,r-2}^{A} + \\
&\quad +2(d_{r-1}-1)A_{t,j,j,r-1}\tilde{Q}_{t-j,j+1,r-1}^{\beta}, \\
H_{t-j,j,r-1}^{(3)} &= (d_{r-1}^2-1)H_{t-j,j,r-1}^{O(3)} + A_{t,j,j,r-1}^{(2)} + b_{t-j,r-1}^2 m_{t-j}^2\kappa_{j,r-2} + \\
&\quad +(d_{r-1}^{(2)} - d_{r-1}^2)\tilde{Q}_{t-j,j+1,r-1}^{\beta(2)}, \\
H_{t-j,j,r-1}^{(4)} &= (d_{r-1}^2-1)H_{t-j,j,r-1}^{O(4)} + (d_{r-1}^{(2)} - d_{r-1}^2)D_{j,r-2}^{(2)}Q_{j+1,r-1}, \\
H_{t-j,j,r-1}^{(5)} &= (d_{r-1}-1)^2Q_{j+1,r-1}^2D_{j,r-2}^2 + 2(d_{r-1}-1)\tilde{Q}_{t-j,j,r-2}^{D}D_{j,r-2}Q_{t-j,r-1}, \\
H_{t-j,j,r-1}^{(6)} &= 2(d_{r-1}-1)D_{j,r-2}Q_{j+1,r-1}, \\
H_{t-j,j,r-1}^{(7)} &= 2(d_{r-1}-1)\tilde{Q}_{t-j,j+1,r-1}^{\beta} + 2A_{t,j,j,r-1}.
\end{aligned} \tag{B.6}$$

Again, in order to use recursion, we write

$$\mathrm{E}\left[X^2_{t-j,r}\middle|I_{t-j,j}\right] = \mathrm{E}\left[X^2_{t-j,r-1}\middle|I_{t-j,j}\right] + H_{t-j,j,r-1},$$

where

$$H_{t-j,j,r-1} = +H^{(1)}_{t-j,j,r-1}N_{t-j,j}X^O_{t-j,j} + H^{(2)}_{t-j,j,r-1}N^2_{t-j,j} + H^{(3)}_{t-j,j,r-1}N_{t-j,j} +$$

$$+H^{(4)}_{t-j,j,r-1}X^{O(2)}_{t-j,j} + H^{(5)}_{t-j,j,r-1}(X^O_{t-j,j})^2 + H^{(6)}_{t-j,j,r-1}X_{t-j,j}X^O_{t-j,j} + H^{(7)}_{t-j,j,r-1}N_{t-j,j}X_{t-j,j}.$$

Using recursion we get

$$E\left[X^2_{t-j,r}\middle|I_{t-j,j}\right] = X^2_{t-j,j} + \sum_{h=j+1}^{r} H_{t-j,j,h-1}.$$

Putting in for $H_{t-j,j,h-1}$ we get

$$\mathrm{E}\left[X^2_{t-j,r}\middle|I_{t-j,j}\right] = X^2_{t-j,j} + \sum_{h=j+1}^{r} H^{(1)}_{t-j,j,r-1}N_{t-j,j}X^O_{t-j,j} +$$

$$+ \sum_{h=j+1}^{r} H^{(2)}_{t-j,j,r-1}N^2_{t-j,j} + \sum_{h=j+1}^{r} H^{(3)}_{t-j,j,r-1}N_{t-j,j} + \sum_{h=j+1}^{r} H^{(4)}_{t-j,j,r-1}X^{O(2)}_{t-j,j} +$$

$$+ \sum_{h=j+1}^{r} H^{(5)}_{t-j,j,r-1}(X^O_{t-j,j})^2 + \sum_{h=j+1}^{r} H^{(6)}_{t-j,j,r-1}X_{t-j,j}X^O_{t-j,j} + \sum_{h=j+1}^{r} H^{(7)}_{t-j,j,r-1}N_{t-j,j}X_{t-j,j},$$

which can be expressed as

$$\mathrm{E}\left[X^2_{t-j,r}\middle|I_{t-j,j}\right] = X^2_{t-j,j} + \tilde{H}^{(1)}N_{t-j,j}X^O_{t-j,j} + \tilde{H}^{(2)}_{t-j,j,r-1}N^2_{t-j,j} +$$

$$+\tilde{H}^{(3)}_{t-j,j,r-1}N_{t-j,j} + \tilde{H}^{(4)}_{t-j,j,r-1}X^{O(2)}_{t-j,j} + \tilde{H}^{(5)}_{t-j,j,r-1}(X^O_{t-j,j})^2 +$$

$$+ \tilde{H}^{(6)}_{t-j,j,r-1}X_{t-j,j}X^O_{t-j,j} + \tilde{H}^{(7)}_{t-j,j,r-1}N_{t-j,j}X_{t-j,j}, \qquad \text{(B.7)}$$

where

$$\tilde{H}^{(i)}_{t-j,j,r-1} = \sum_{h=j+1}^{r} H^{(i)}_{t-j,j,r-1}, \quad i = 1, ..., 7. \qquad \text{(B.8)}$$

Appendix C

Moments of development of individual losses

Under the condition that the losses are at calender year t open the random variable

$$W^l_{t-j,j,k} = \prod_{i=j}^{r-1} \left(1 + (d_i - 1) J^l_{t-j,i,k} \tilde{E}^l_{t-j,i+1,k}\right)$$

can be expressed as

$$W^l_{t-j,j,k} = \left(1 + (d_j - 1)\tilde{E}^l_{t-j,j+1,k}\right) \prod_{i=j+1}^{r-1} \left(1 + (d_i - 1) J^l_{t-j,i,k} \tilde{E}^l_{t-j,i+1,k}\right),$$

which is equal to

$$W^l_{t-j,j,k} = \left(1 + (d_j - 1)\tilde{E}^l_{t-j,j+1,k}\right) W^l_{t-j,j+1,r,k}.$$

The conditional expected value can then be expressed as

$$\mathrm{E}\left[W^l_{t-j,j,k} \middle| J^l_{t-j,j,k} = 1\right] = \mathrm{E}\left[\left(1 + (d_j - 1)\tilde{E}^l_{t-j,j+1,k}\right) W^l_{t-j,j+1,r,k} \middle| J^l_{t-j,j,k} = 1\right].$$

Due to the independence of $\tilde{E}^l_{t-j,j+1,k}$ on $\tilde{E}^l_{t-j,i+1,k}$ and $J^l_{t-j,i,k}$ for $i = j+1, \ldots, r-1$, and hence on $W^l_{t-j,j+1,r,k}$ this is equal to

$$\mathrm{E}\left[\left(1 + (d_j - 1)\tilde{E}^l_{t-j,j+1,k}\right)\right] \mathrm{E}\left[W^l_{t-j,j+1,r,k} \middle| J^l_{t-j,j,k} = 1\right].$$

The first expected value is equal to d_j, so that we get

$$\mathrm{E}\left[W^l_{t-j,j,k} \middle| J^l_{t-j,j,k} = 1\right] = d_j \mathrm{E}\left[W^l_{t-j,j+1,r,k} \middle| J^l_{t-j,j,k} = 1\right].$$

The expected value $\mathrm{E}[W^l_{t-j,j+1,r,k} | J^l_{t-j,j,k} = 1]$ can be expressed as

$$P\left[J^l_{t-j,j+1,k} = 1 \middle| J^l_{t-j,j,k} = 1\right] \mathrm{E}\left[W^l_{t-j,j+1,r,k} \middle| J^l_{t-j,j+1,k} = 1\right] +$$

$$+P\left[J^l_{t-j,j+1,k}=0\middle|J^l_{t-j,j,k}=1\right]\mathrm{E}\left[W^l_{t-j,j+1,r,k}\middle|J^l_{t-j,j+1,k}=0\right],$$

which is, after similar consideration as for $W^l_{t-j,j,r,k}$, equal to

$$(1-q_{j+1})d_{j+1}\mathrm{E}\left[W^l_{t-j,j+2,r,k}\middle|J^l_{t-j,j+1,k}=1\right]+q_{j+1}.$$

Thus, we have

$$\mathrm{E}\left[W^l_{t-j,j,r,k}\middle|I_{t-j,j}\right]=d_j\left((1-q_{j+1})d_{j+1}\mathrm{E}\left[W^l_{t-j,j+2,r,k}\middle|J^l_{t-j,j+1,k}=1\right]+q_{j+1}\right).$$

If we expand the brackets and use recursion for the expected value we get

$$\begin{aligned}\mathrm{E}\left[W^l_{t-j,j,r,k}\middle|I_{t-j,j}\right] = {} & d_j\cdots d_{r-1}(1-q_{j+1})\cdots(1-q_{r-1})+\\ & +\ d_j\cdots d_{r-2}(1-q_{j+1})\cdots(1-q_{r-2})q_{r-1}+\\ & +\ d_j\cdots d_{r-3}(1-q_{j+1})\cdots(1-q_{r-3})q_{r-2}+\cdots+\\ & +\ d_jd_{j+1}(1-q_{j+1})q_{j+2}+d_jq_{j+1}.\end{aligned}$$

Using the notation from previous chapter this can be expressed as

$$\mathrm{E}\left[W^l_{t-j,j,r,k}\middle|I_{t-j,j}\right]=D_{j,r-1}Q_{j+1,r-1}+\sum_{h=j+2}^{r-1}D_{j,h-1}Q_{j+1,h-1}q_h+D_{j,j}q_{j+1},\ j<r-1,$$

and for $j=r-1$,

$$\mathrm{E}\left[W^l_{t-r+1,r-1,r,k}\middle|I_{t-r+1,r-1}\right]=d_{r-1}.$$

If we set

$$q_r:=1,\ Q_{j+1,j}:=1,\ j=0,\ldots,r-1,$$

and define $\sum_r^{r-1}:=0$, we can express this together as a simple sum,

$$\mathrm{E}\left[W^l_{t-j,j,r,k}\middle|I_{t-j,j}\right]=\sum_{h=j+1}^{r}D_{j,h-1}Q_{j+1,h-1}q_h.$$

In a similar way, we get for the second moment

$$\mathrm{E}\left[(W^l_{t-j,j,r,k})^2\middle|I_{t-j,j}\right]=\sum_{h=j+1}^{r}D^{(2)}_{j,h-1}Q_{j+1,h-1}q_h,$$

where

$$D^{(2)}_{i,j}=d^{(2)}_i\cdots d^{(2)}_j,\quad q_r:=1,\quad Q_{j+1,j}:=1.$$

Appendix D

Second moment of cumulative ultimate net losses after XL-reinsurance

To compute $\mathrm{E}[(\tilde{X}^{XL}_{t-j,r})^2 | I_{t-j,j}]$ (where we take only losses that are at $I_{t-j,j}$ open, we don't write it explicitly) we split $\tilde{X}^{XL}_{t-j,r}$ again into two parts, the first part containing the losses that occurred in the triangle up to the development year j and the second part with losses that occur in the triangle after the development year j and get

$$\mathrm{E}\left[\left(\tilde{X}^{XL}_{t-j,r}\right)^2 \middle| I_{t-j,j}\right] = \mathrm{E}\left[\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}^{l}_{t-j,r,k} + \sum_{l=j+1}^{r}\sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}^{l}_{t-j,r,k}\right)^2 \middle| I_{t-j,j}\right].$$

This is equal to

$$\mathrm{E}\left[\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}^{l}_{t-j,r,k}\right)^2 \middle| I_{t-j,j}\right] + \mathrm{E}\left[\left(\sum_{l=j+1}^{r}\sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}^{l}_{t-j,r,k}\right)^2 \middle| I_{t-j,j}\right] +$$

$$+2\mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}^{l}_{t-j,r,k} \middle| I_{t-j,j}\right] \mathrm{E}\left[\sum_{l=j+1}^{r}\sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}^{l}_{t-j,r,k} \middle| I_{t-j,j}\right] = A + B + C.$$

We denote the three parts of the expression A, B and C and compute each part separately. For the first part,

$$A = \mathrm{E}\left[\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} \min\left(Y^{l}_{t-j,r,k}, a_{t-j}\right)\right)^2 \middle| I_{t-j,j}\right],$$

we have

$$A = A_1 + A_2,$$

where

$$A_1 = \mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} \left(\min\left(Y^{l}_{t-j,r,k}, a_{t-j}\right)\right)^2 \middle| I_{t-j,j}\right],$$

and

$$A_2 = \mathrm{E}\left[\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\sum_{l'=0}^{j}\sum_{\substack{k'=1\\(l,k)\neq(l',k')}}^{\tilde{N}_{t-j,l'}} \min\left(Y^{l}_{t-j,r,k}, a_{t-j}\right)\min\left(Y^{l'}_{t-j,r,k'}, a_{t-j}\right)\Big| I_{t-j,j}\right].$$

Using previous notation we get for A_1 the expression

$$A_1 = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} K^{a(2)}_{j,r-1}\left(Y^{l}_{t-j,j,k}\right),$$

where

$$K^{a(2)}_{j,r-1}\left(Y^{l}_{t-j,j,k}\right) = \left(Y^{l}_{t-j,j,k}\right)^2 \mathrm{E}\left[\left(\min\left(W^{l}_{t-j,j,r,k}, \frac{a_{t-j}}{Y^{l}_{t-j,j,k}}\right)\right)^2 \Big| I_{t-j,j}, J^{l}_{t-j,j,k}=1\right]. \tag{D.1}$$

With the notion of the reinsurance relief effect of n-th power we can write

$$K^{a(2)}_{j,r-1}\left(Y^{l}_{t-j,j,k}\right) = \left(Y^{l}_{t-j,j,k}\right)^2 \mathrm{E}\left[\left(W^{l}_{t-j,j,r,k}\right)^2 \Big| I_{t-j,j}\right] r^{(2)}_{W^{l}_{t-j,j,r,k}|J^{l}_{t-j,j,k}=1}\left(\frac{a_{t-j}}{Y^{l}_{t-j,j,k}}\right). \tag{D.2}$$

For A_2 we can write in a similar way

$$A_2 = \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\sum_{l'=0}^{j}\sum_{\substack{k'=1\\(l,k)\neq(l',k')}}^{\tilde{N}_{t-j,l'}} \mathrm{E}\left[\min\left(Y^{l}_{t-j,j,k}W^{l}_{t-j,j,r,k}, a_{t-j}\right)\Big| I_{t-j,j}, J^{l}_{t-j,j,k}=1\right]$$

$$\cdot\mathrm{E}\left[\min\left(Y^{l'}_{t-j,j,k'}W^{l'}_{t-j,j,r,k'}, a_{t-j}\right)\Big| I_{t-j,j}, J^{l}_{t-j,j,k}=1\right],$$

which is equal to

$$\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\sum_{l'=0}^{j}\sum_{\substack{k'=1\\(l,k)\neq(l',k')}}^{\tilde{N}_{t-j,l'}} Y^{l}_{t-j,j,k}Y^{l'}_{t-j,j,k'}\mathrm{E}\left[\min\left(W^{l}_{t-j,j,r,k}, \frac{a_{t-j}}{Y^{l}_{t-j,j,k}}\right)\Big| I_{t-j,j}, J^{l}_{t-j,j,k}=1\right]$$

$$\cdot\mathrm{E}\left[\min\left(W^{l'}_{t-j,j,r,k'}, \frac{a_{t-j}}{Y^{l'}_{t-j,j,k'}}\right)\Big| I_{t-j,j}, J^{l}_{t-j,j,k}=1\right].$$

With previous notation this can be rewritten as

$$\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\sum_{l'=0}^{j}\sum_{\substack{k'=1\\(l,k)\neq(l',k')}}^{\tilde{N}_{t-j,l'}} K^{a}_{j,r-1}\left(Y^{l}_{t-j,j,k}\right)K^{a}_{j,r-1}\left(Y^{l'}_{t-j,j,k'}\right).$$

This can be also expressed as

$$\left(\sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}} K_{j,r-1}^{a}\left(Y_{t-j,j,k}^{l}\right)\right)^2 - \sum_{l=0}^{j}\sum_{k=1}^{\tilde{N}_{t-j,l}}\left(K_{j,r-1}^{a}\left(Y_{t-j,j,k}^{l}\right)\right)^2.$$

For the term

$$B = \mathrm{E}\left[\left(\sum_{l=j+1}^{r}\sum_{k=1}^{\tilde{N}_{t-j,l}} \tilde{Y}_{t-j,r,k}^{l}\right)^2 \middle| I_{t-j,j}\right],$$

we have similarly

$$B = B_1 + B_2,$$

where

$$B_1 = \mathrm{E}\left[\sum_{l=j+1}^{r}\sum_{k=1}^{\tilde{N}_{t-j,l}}\left(\min\left(Y_{t-j,r,k}^{l}, a_{t-j}\right)\right)^2 \middle| I_{t-j,j}\right],$$

and

$$B_2 = \mathrm{E}\left[\sum_{l=j+1}^{r}\sum_{k=1}^{\tilde{N}_{t-j,l}}\sum_{l'=j+1}^{r}\sum_{\substack{k=1\\(l,k)\neq(l',k')}}^{\tilde{N}_{t-j,l'}} \min\left(Y_{t-j,r,k}^{l}, a_{t-j}\right)\min\left(Y_{t-j,r,k'}^{l'}, a_{t-j}\right) \middle| I_{t-j,j}\right] = B_1 + B_2.$$

Using similar computation as in previous chapters we get for B_1

$$B_1 = \sum_{l=j+1}^{r}\mathrm{E}\left[\tilde{N}_{t-j,l}\middle|I_{t-j,j}\right]\mathrm{E}\left[\left(\min\left(Y_{t-j,r,k}^{l}, a_{t-j}\right)\right)^2\middle|I_{t-j,j}\right] = N_{t-j,j}\sum_{l=j+1}^{r} v_{t-j,j,l-1,r}^{a(2)},$$

where $v_{t-j,j,l-1,r}^{a(2)}$ has a similar form to $v_{t-j,j,l-1,r}^{a}$,

$$v_{t-j,j,l-1,r}^{a(2)} = A_{t-j,j,l-1}^{(2)}\mathrm{E}\left[W_{t-j,l,r,k}^{l}\right]^2 r_{Y_{t-j,r,k}^{l}}^{(2)}(a_{t-j}), \tag{D.3}$$

where $A_{t-j,j,l-1}^{(2)}$ is defined in (B.3). In a similar way, we have for B_2

$$B_2 = \sum_{l=j+1}^{r}\sum_{\substack{l'=j+1\\l\neq l'}}^{r}\mathrm{E}\left[\tilde{N}_{t-j,l}\middle|I_{t-j,j}\right]\mathrm{E}\left[\tilde{N}_{t-j,l'}\middle|I_{t-j,j}\right]\cdot$$

$$\cdot\mathrm{E}\left[\min\left(Y_{t-j,r,k}^{l}, a_{t-j}\right)\middle|I_{t-j,j}\right]\mathrm{E}\left[\min\left(Y_{t-j,r,k}^{l'}, a_{t-j}\right)\middle|I_{t-j,j}\right],$$

which is with previous notation equal to

$$B_2 = N_{t-j,j}^2\sum_{l=j+1}^{r}\sum_{\substack{l'=j+1\\l\neq l'}}^{r} v_{t-j,j,l-1,r}^{a} v_{t-j,j,l'-1,r}^{a}.$$

This can be alternatively expressed as

$$B_2 = N_{t-j,j}^2 \left(\left(\sum_{l=j+1}^{r} v_{t-j,j,l-1,r}^a \right)^2 - \sum_{l=j+1}^{r} \left(v_{t-j,j,l-1,r}^a \right)^2 \right).$$

For the last term C we use the same computation and notation as in previous sections and get

$$C = 2 \left(\sum_{l=0}^{j} \sum_{k=1}^{\tilde{N}_{t-j,l}} K_{j,r-1}^a \left(Y_{t-j,j,k}^l \right) \right) \left(N_{t-j,j} \sum_{l=j+1}^{r} v_{t-j,j,l-1,r}^a \right).$$

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