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Weaker versions of the continuum hypothesis and partitions of ccc posets

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Abstract: Weaker forms of the continuum hypothesis – CH_n and CH_ω – are certain combinatorial statements which follow from the continuum hypothesis. They were introduced in the paper by Stevo Todorčević "Remarks on Martin's axiom and the continuum hypothesis" in 1991. The statements talk about existence of certain decompositions of ccc posets. Many known consequences of the continuum hypothesis are already implied by some of these weaker statements. This thesis introduces the concepts behind the weaker forms of the continuum hypothesis and studies their consequences. The original content of the thesis consists of analysis of posets introduced in the paper by Todorčević "Two examples of Borel partially ordered sets with the countable chain condition" in the context of these axioms.

Keywords: ccc poset, continuum hypothesis, linked sets, set theory, cardinal characteristics of the continuum

Contents

Introduction

The continuum hypothesis, CH, is a well-known statement concerning two infinite cardinals – $\mathfrak{c}$, the cardinality of the set of all real numbers, and $\aleph_1$, the least uncountable cardinal. The Cantor's well-known diagonal argument implies that there are uncountably many real numbers, hence $\mathfrak{c} \geq \aleph_1$. CH states that in fact $\mathfrak{c} = \aleph_1$. The truth value of CH is undecidable in ZFC (Zermello–Fraenkel set theory with the axiom of choice), meaning that (unless ZFC is inconsistent) it can be neither proved nor disproved from the axioms of ZFC.

If CH is added as a new axiom to ZFC, it resolves various mathematical questions which are otherwise undecidable. CH and various statements related to it have been studied extensively in the set theory. A generalized continuum hypothesis, GCH, is a stronger version of CH. It claims that for each infinite cardinal κ, 2^κ is equal to the least cardinal larger than κ. A diamond principle, denoted $\diamondsuit$, strengthens the continuum hypothesis in a different way. It postulates the existence of a sequence of bounded subsets of ω_1 which in some sense well approximates any subset of ω_1. Among other things, it implies CH, but it is consistent with ZFC that CH holds and $\diamondsuit$ fails.

Some consequences of CH were studied on their own. As an example we mention the proposition $2^\omega < 2^{\omega_1}$, which if CH holds is clearly true. This proposition was shown by Devlin and Shelah ([4]) to be equivalent to a certain combinatorial principle known as the weak diamond principle. The negation of this proposition is sometimes refered to as the Luzin hypothesis. Another example, Martin's axiom, MA, states that a certain combinatorial principle, which provably holds for $\aleph_0$, holds in fact for all cardinals below $\mathfrak{c}$. It is especially interesting to assume $MA + \neg CH$. This combination is known to be consistent and has many interesting consequences.

In the paper of Todorčević [7] some weaker forms of CH were introduced and studied. These are statements, which trivially follow from CH, but presumably do not imply it. Our main interest is which known consequences of CH are already consequences of these weaker forms. Studying these forms is the main topic of this thesis.

In the first chapter we present the basic terminology and theory of ccc posets, which is necessary to formulate these weaker forms. The statements themselves are defined at the end of the chapter.

In the second chapter we introduce the basic theory of cardinal characteristics of the continuum. These are an important tool in studying "in what ways can CH fail" and they are also a natural source of various conjectures, which can be studied in the context of weaker forms of CH.

In the third chapter we present detailed proofs of consequences of these forms, which were described in [7].

In the fourth chapter we seek for new results regarding these cardinals by analysing posets described in [8].

1. Posets and their basic properties

In the whole thesis we use the standard set-theoretic notation (see e.g. [2]).

A *partially ordered set* (commonly abbreviated as *poset*) is an ordered pair $(P, \leq)$, where P is a set and $\leq$ is a binary relation on P which is reflexive, transitive and antisymmetric. In this thesis we will study infinite posets. We now define a number of terms related to posets. For the rest of this section let P be a fixed poset.

If $p \in P$ and $X \subseteq P$, then p is a *lower bound* of X, if $p \leq x$ for all $x \in X$. For $a, b \in P$ we say that a and b are *compatible*, if $\{a, b\}$ has a lower bound (i.e. if there exists $c \in P$ such that $c \leq a$ and $c \leq b$), otherwise we say that they are *incompatible*.

For a set $X \subseteq P$ we say that X is an *antichain*, if every two distinct elements of X are incompatible. We say that P *is ccc* (or *has the countable chain condition*), if every antichain in P is at most countable.

Let $n \geq 2$ be a natural number. A set $X \subseteq P$ is *n-linked*, if every n-element subset of X has a lower bound. In particular, it is 2-linked if all its elements are pairwise compatible. X is *centered*, if every finite subset of X has a lower bound (i.e. if it is n-linked for all n). We say that P is σ-n-linked, if it can be expressed as a countable union of n-linked sets, and σ-centered, if it is a countable union of centered sets. We may write just linked and σ-linked in place of 2-linked and σ-2-linked respectively.

Properties σ-centered, σ-n-linked and ccc form a hierarchy in the following sense.

Lemma 1. *Let $n \geq m \geq 2$ be natural numbers. Then P is σ-centered $\implies$ P is σ-n-linked $\implies$ P is σ-m-linked $\implies$ P is ccc.*

Proof. First implication follows immediately from the definition.

For the second one, let $P = \bigcup_{k\in\omega} P_k$ be a decomposition of P into n-linked sets. Whenever P_k is finite, we split it into $|P_k|$ singleton sets. This way we obtain a decomposition of P into countably many n-linked sets, each of them being either infinite or singleton. It is easy to see that for each such set, n-linked implies m-linked, so this is also a decomposition into m-linked sets.

For the last one, assume P is σ-m-linked, so by the previous paragraph it is also σ-linked. Let $P = \bigcup_{k\in\omega} P_k$ be a decomposition of P into linked sets. Let $X \subseteq P$ be an antichain. For each $k \in \omega$, since every two elements of P_k are compatible and every two distinct elements of X are incompatible, $|P_k \cap X| \leq 1$. Thus, X is countable. □

Example We demonstrate the above defined concepts in a simple example. Let P be a set of all nonempty open subsets of $\mathbb{R}$. For $a, b \in P$ put $a \leq b$ iff $a \subseteq b$. Now let a, b be some nonempty open sets. If $a \cap b \neq \emptyset$, then $a \cap b$ is a nonempty open set (thus element of P), which is below both a and b, hence a and b are compatible. On the other hand, if $a \cap b = \emptyset$, for no $c \in P$ could possibly happen both $c \leq a$ and $c \leq b$, so a and b are incompatible. We see that two elements of

P are compatible precisely if they are not disjoint. Similarly, a finite set has a lower bound if and only if its intersection is nonempty.

Antichains in P are precisely systems of pairwise disjoint nonempty open subsets of $\mathbb{R}$. Since every such set intersects $\mathbb{Q}$ and $\mathbb{Q}$ is countable, there can be only countably many such pairwise disjoint sets. Hence, P is ccc.

Actually, P is even σ-centered. To see this, for each $q \in \mathbb{Q}$ define $P_q = \{a \in P : q \in a\}$. P_q is a centered set for each q and $P = \bigcup_{q \in \mathbb{Q}} P_q$, because $\mathbb{Q}$ is dense in $\mathbb{R}$.

One could ask whether notions of ccc and (for example) σ-linked are not in fact equivalent. The following examples will show that this is not the case. To present them, we will need the following lemma, which is in general useful to show that various posets are ccc.

Definition 2. *Let X be a set and $S \subseteq \mathcal{P}(X)$. We say that S is a* Δ-system, *if there exists $r \subseteq X$ such that for each $x, y \in S$, $x \neq y \implies x \cap y = r$. The set r is called the* kernel *of X.*

Lemma 3. *(Δ-system lemma) Let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be a system of finite sets. Then there exists an uncountable $I \subseteq \omega_1$ such that $\{a_\alpha : \alpha \in I\}$ is a Δ-system.*

Proof. Since there are only countably many possible values of $|a_\alpha|$ for each α, we can always find $n \in \omega$ such that $|a_\alpha| = n$ for uncountably many values of α. Without loss of generality, we can assume that $|a_\alpha| = n$ for all $\alpha < \omega_1$ (otherwise we remove all sets with cardinality not equal to n).

We proceed by induction on n. For $n = 0$ we can simply put $I = \omega_1$. Now let $n \geq 1$ and assume the induction hypothesis is true for $n-1$. For each $x \in \bigcup_{\alpha < \omega_1} a_\alpha$ put $J_x = \{\alpha \in \omega_1 : x \in a_\alpha\}$. We distinguish two cases:

1. J_x is uncountable for some x. Then consider the system $\langle a_\alpha \setminus \{x\} : \alpha \in J_x \rangle$, which is an uncountable system of sets of cardinality $n-1$. By the induction hypthesis, we find an uncountable Δ-system with kernel r included in this system. When we add x back to all sets from this Δ-system, we get a Δ-system included in J_x with kernel $r \cup \{x\}$.

2. J_x is countable for all x. Let $I \subseteq \omega_1$ be a maximal (with respect to inclusion) set such that sets $\langle a_\alpha : \alpha \in I \rangle$ are pairwise disjoint. Such set exists by an easy application of Zorn's lemma. $\{a_\alpha : \alpha \in I\}$ is a Δ-system with kernel $\emptyset$, it remains to show that I is uncountable. Assume for contradiction that I is countable. Then $X = \bigcup_{\alpha \in I} a_\alpha$ is a countable union of finite sets, thus countable, and $J = \bigcup_{x \in X} J_x$ is a countable union of countable sets, thus countable. Pick any $\beta \in \omega_1 \setminus J$. Then a_β is disjoint from a_α for all $\alpha \in I$ (therefore $\beta \notin I$), which contradicts the maximality of I. $\square$

Example We now give an example of a poset which is ccc, but not σ-linked. Let $\mathrm{Fin}(\mathfrak{c}^+, 2)$ be the poset consisting of all functions from finite subsets of $\mathfrak{c}^+$ to $\{0, 1\}$. For $f, g \in \mathrm{Fin}(\mathfrak{c}^+, 2)$ put $f \leq g$ iff $f \supseteq g$.

Theorem 4. $\mathrm{Fin}(\mathfrak{c}^+, 2)$ *is ccc.*

Proof. Let $\{f_\alpha : \alpha < \omega_1\}$ be distinct elements of $\mathrm{Fin}(\mathfrak{c}^+, 2)$. By the Δ-system lemma, there is an uncountable $I \subseteq \omega_1$ such that $\{\mathrm{dom}(f_\alpha) : \alpha \in I\}$ is a Δ-system with kernel r. There are only finitely many functions from r to $\{0,1\}$, so we can always find $\alpha, \beta \in I$ such that $\alpha \neq \beta$ and $f_\alpha \upharpoonright r = f_\beta \upharpoonright r$. This means that $f_\alpha \cup f_\beta$ is a function which is below both f_α and f_β. Hence $\{f_\alpha : \alpha < \omega_1\}$ is not an antichain. □

Theorem 5. $\mathrm{Fin}(\mathfrak{c}^+, 2)$ *is not* σ*-linked.*

Proof. Let $\{M_n : n \in \omega\}$ be a system of linked subsets of $\mathrm{Fin}(\mathfrak{c}^+, 2)$. We construct an element of $\mathrm{Fin}(\mathfrak{c}^+, 2)$ which is not included in any M_n.

Since M_n is linked for each n, every two functions in M_n agree on the intersection of their domains. Thus $\bigcup M_n$ is a function from some subset of $\mathfrak{c}^+$ to $\{0,1\}$. We will denote this function F_n.

We will define a function G from $\mathfrak{c}^+$ to ${}^\omega 2$. For $\alpha \in \mathfrak{c}^+$ and $n \in \omega$ put $G(\alpha)(n) = F_n(\alpha)$ if $\alpha \in \mathrm{dom}(F_n)$ and 0 otherwise. Since $|{}^\omega 2| = \mathfrak{c} < \mathfrak{c}^+$, there are $\alpha, \beta \in \mathfrak{c}^+$, $\alpha \neq \beta$, such that $G(\alpha) = G(\beta)$. This means that $F_n(\alpha) = F_n(\beta)$ for all $n \in \omega$. Define $f : \{\alpha, \beta\} \to \{0,1\}$ as $f(\alpha) = 0$, $f(\beta) = 1$. Then $f \not\subseteq F_n$ for all n, so it can not be included in any M_n. □

Note that the poset described above has cardinality $\mathfrak{c}^+$. For $\kappa \leq \mathfrak{c}$ the poset $\mathrm{Fin}(\kappa, 2)$ is already σ-centered, this can be deduced from Lemma 40. The next example shows another poset which is ccc and not σ-linked, but with cardinality only $\mathfrak{c}$. The example comes from [8].

Example Let $\mathbb{T}(\mathbb{R})$ be the poset consisting of all sets $F \subseteq \mathbb{R}$ which are a finite union of converging sequences including their limit points. For such F, let F' denote the set of all accumulation points of F (i.e. limit points of those sequences). For $F_1, F_2 \in \mathbb{T}(\mathbb{R})$ put $F_1 \leq F_2$ iff $F_1 \supseteq F_2$ and $F_1' \cap F_2 = F_2'$. It is not hard to check that $\leq$ is a transitive relation.

Theorem 6. $\mathbb{T}(\mathbb{R})$ *is ccc.*

Proof. Assume $\{F_\alpha : \alpha < \omega_1\}$ is an uncountable antichain in $\mathbb{T}(\mathbb{R})$. Since F_α' is finite for all α, without loss of generality we can assume that for some fixed $n \in \omega$, $|F_\alpha'| = n$ for all α. We will denote elements of F_α in the following way. Let $F_\alpha' = \{x_\alpha^i : i < n\}$ and $F_\alpha \setminus F_\alpha' = \bigcup_{i<n} s_\alpha^i$, where $s_\alpha^i = (s_\alpha^i(k))_{k \in \omega}$ is a sequence of reals converging to x_α^i for all $i < n$.

By the Δ-system lemma, there is $I \in [\omega_1]^{\omega_1}$ such that $\{F_\alpha' : \alpha \in I\}$ is a Δ-system with kernel r. Without loss of generality assume that I is the whole ω_1. Note that F_α and F_β are compatible if and only if $(F_\alpha \setminus F_\alpha') \cap F_\beta' = \emptyset$ and $(F_\beta \setminus F_\beta') \cap F_\alpha' = \emptyset$. Thus, if we remove r and all sequences s_α^k converging to elements of r from each F_α, we get again an antichain in $\mathbb{T}(\mathbb{R})$. Hence, we can without loss of generality assume that $r = \emptyset$.

We can also assume that $\alpha < \beta$ implies $(F_\alpha \setminus F_\alpha') \cap F_\beta' = \emptyset$. To see this, we will construct by transfinite induction an increasing sequence of countable ordinals $\{\alpha_\beta : \beta < \omega_1\}$ such that $\{F_{\alpha_\beta} : \beta < \omega_1\}$ satisfies this requirement. Let $\beta < \omega_1$ and assume we have defined α_γ for all $\gamma < \beta$. Let $\xi = \sup\{\alpha_\gamma : \gamma < \beta\}$. The supremum of countably many countable ordinals is countable, so $\xi < \omega_1$. If we denote $X = \bigcup_{\gamma<\beta} \left(F_{\alpha_\gamma} \setminus F_{\alpha_\gamma}'\right)$, then X is countable. Since all F_α' are pairwise disjoint, among uncountably many ordinals between ξ and ω_1 we can always find η such that $X \cap F_\eta' = \emptyset$. Then we can put $\alpha_\beta = \eta$.

For $\alpha < \omega_1$ denote $X_\alpha = \langle x^0_\alpha, x^1_\alpha, \ldots, x^{n-1}_\alpha \rangle \in \mathbb{R}^n$. Fix $\gamma < \omega_1$ such that $\{X_\alpha : \alpha < \gamma\}$ is a dense subset of $\{X_\alpha : \alpha < \omega_1\}$. This can be done, because $\{X_\alpha : \alpha < \omega_1\}$, being a subset of $\mathbb{R}^n$, has a countable dense set $\{X_{\gamma_n} : n \in \omega\}$, so we can take $\gamma = \sup_{n\in\omega} \gamma_n + 1$. Pick $\delta > \gamma$. We know that $F'_\delta \cap F'_\gamma = \emptyset$ and also that $F'_\delta \cap (F_\gamma \setminus F'_\gamma) = \emptyset$. Thus, $x^i_\delta \in \mathbb{R} \setminus F_\gamma$ for all $i < n$ and $X_\delta \in (\mathbb{R} \setminus F_\gamma)^n$, which is an open set. By the density condition we find $\beta < \gamma$ such that $X_\beta \in (\mathbb{R} \setminus F_\gamma)^n$, thus $F'_\beta \cap F_\gamma = \emptyset$. This implies $F'_\beta \cap (F_\gamma \setminus F'_\gamma) = \emptyset$ and $(F_\beta \setminus F'_\beta) \cap F'_\gamma = \emptyset$ follows from $\beta < \gamma$. Therefore, F_β and F_γ are compatible, a contradiction. □

Theorem 7. $\mathbb{T}(\mathbb{R})$ *is not σ-linked.*

Proof. Let $\{B_n : n \in \omega\}$ be a system of linked subsets of $\mathbb{T}(\mathbb{R})$. For $n \in \omega$ put $C_n = \bigcup_{F\in B_n} F'$. Every subset of $\mathbb{R}$ has at most countably many isolated points. Hence, we can always find $x \in \mathbb{R}$ which is not an isolated point of C_n for any $n \in \omega$. We define a sequence $(x_n)_{n\in\omega}$ as follows. For each $n \in \omega$, if $x \notin C_n$, set $x_n = x + \frac{1}{n}$. If $x \in C_n$, then pick x_n such that $x_n \in C_n \setminus \{x\}$ and $|x - x_n| < \frac{1}{n}$. Such x_n exists, because x is not an isolated point of C_n. Clearly $\lim_{n\to\infty} x_n = x$, so $F = \{x_0, x_1, \ldots, x\} \in \mathbb{T}(\mathbb{R})$. Assume for contradiction that $F \in B_n$ for some n. Then $x \in C_n$, thus also $x_n \in C_n$, so there exists $G \in B_n$ such that $x_n \in G'$. This implies $x_n \in G' \cap (F \setminus F')$, therefore F and G are incompatible, a contradiction. □

1.1 Continuum hypothesis and its weaker forms

We now turn our attention to weaker forms of CH introduced in [7]. These statements are the central topic of this thesis. The general structure of these axioms is as follows: every ccc poset of cardinality at most $\mathfrak{c}$ can be decomposed into at most $\aleph_1$ parts which are in some sense small. Since CH implies that every ccc poset of cardinality at most $\mathfrak{c}$ can be decomposed into $\aleph_1$ singleton sets, such principles are trivially implied by CH.

The precise formulations are as follows:

Definition 8. *Let $n \geq 2$ be a natural number. By CH_n we denote the following statement:* Every ccc poset of size at most $\mathfrak{c}$ can be decomposed into $\aleph_1$ n-linked sets.

Definition 9. *By CH_ω we denote the following statement:* Every ccc poset of size at most $\mathfrak{c}$ can be decomposed into $\aleph_1$ centered sets.

Obviously, $CH \implies CH_\omega \implies CH_m \implies CH_n$ for all $m \geq n \geq 2$.

In the previous section we have seen that the statement *every ccc poset of size at most $\mathfrak{c}$ can be decomposed into $\aleph_0$ linked sets* is provably false – $\mathbb{T}(\mathbb{R})$ is a counterexample. Axioms introduced above can be thought of as an attempt to fix statements of this kind, by replacing $\aleph_0$ by $\aleph_1$. The standard way to derive consequences of the weaker forms is as follows. First we need to come up with an appropriate poset with cardinality at most $\mathfrak{c}$. We prove that this poset is ccc. Then using CH_n or CH_ω we decompose this poset into $\aleph_1$ parts with certain property. Finally, from the existence of such decomposition we derive what we wanted to prove.

We shall comment on what types of consequences we are interested in. Generally, the following situation is common in mathematics. We have an argument showing that some property holds for a countable system of some objects. This argument can not be generalized in a straightforward way to uncountable case, and it is often evident that the property fails e.g. for systems of cardinality $\mathfrak{c}$. As an example, a countable union of Lebesgue measure zero sets has zero measure as well, but the same definitely does not hold for unions of $\mathfrak{c}$ many such sets. In the absence of CH it is natural to ask how far the argument can be carried out. In this case, how many measure zero sets we have to take so that their union has positive measure. The main interest of this thesis is what the weaker forms of CH can say about such situations.

In the introduction we mentioned Martin's axiom MA. It is interesting to compare MA with the axioms presented above. Similar to these axioms, MA claims that some property holds for all ccc posets. A typical way to use MA in a proof is also the same as what we described above. However, the types of consequences of MA and of the weaker forms of CH are very different, in some sense exactly opposite. To follow up on the previous paragraph, MA generally enables us to take an argument of certain form which works in the countable case, and to extend it to all cardinals below $\mathfrak{c}$, thus making it fail as late as possible. The weaker forms of CH, on the other hand, typically imply that certain arguments fail already for $\aleph_1$ – i.e. as soon as possible. As an example, in Theorem 7 we showed that no countable system of linked sets can cover $\mathbb{T}(\mathbb{R})$. However, CH_2 implies that for $\aleph_1$ linked sets it is already possible.

Both CH_n and CH_ω are undecidable in ZFC. They are consistent, because CH is. Also their negations are consistent, this will follow from results proved later. However, it is an interesting open question whether any of them already implies CH, or whether some of them are not in fact equivalent. To compare with Martin's axiom, it is known that MA does not imply CH.

2. Cardinal characteristics of the continuum

2.1 Introduction

Cardinal characteristics of the continuum provide examples of the phenomenon described at the end of the previous chapter. They correspond to statements which can be proved in countable case, clearly fail in case of $\mathfrak{c}$ and the cases between these two cardinals are unclear. To be more precise, a cardinal characteristic of the continuum is a definition of a cardinal, which typically can be proved in ZFC to be greater then or equal to $\aleph_1$, smaller then or equal to $\mathfrak{c}$, but its exact value can not be determined in ZFC. When studying cardinal characteristics, the main interest is typically what inequalities can be proved between them. In this section we introduce two well-known examples of cardinal characteristics – $\mathfrak{b}$ and $\mathfrak{d}$.

We define an ordering on the set of all functions from ω to ω as follows: for $f, g \in {}^{\omega}\omega$ let $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. We will also take $f <^* g$ to mean that $f(n) < g(n)$ for all but finitely many n. Relation $\leq^*$ is clearly transitive and reflexive (and not antisymmetric). For $M \subseteq {}^{\omega}\omega$ we say that M is:

1. *unbounded*, if $(\forall f \in {}^{\omega}\omega)(\exists g \in M)(g \not\leq^* f)$,
2. *dominating*, if $(\forall f \in {}^{\omega}\omega)(\exists g \in M)(f \leq^* g)$.

A question arises about how small unbounded and dominating sets can be. Since every nonempty class of cardinals has a least element, following definitions are justified.

Definition 10. *The* bounding number*, denoted* $\mathfrak{b}$*, is the least cardinal for which there exists an unbounded* $M \subseteq {}^{\omega}\omega$ *such that* $|M| = \mathfrak{b}$*.*

Definition 11. *The* dominating number*, denoted* $\mathfrak{d}$*, is the least cardinal for which there exists a dominating* $M \subseteq {}^{\omega}\omega$ *such that* $|M| = \mathfrak{d}$*.*

We can prove some simple inequalities about these cardinals.

Lemma 12. $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$

Proof. $\mathfrak{b} \leq \mathfrak{d}$ follows from the fact that every dominating set is unbounded. The inequality $\mathfrak{d} \leq \mathfrak{c}$ holds because ${}^{\omega}\omega$ is itself a dominating set. The only nontrivial inequality is the first one.

We need to show that every countable set of functions from ω to ω is bounded. Let $M \subseteq {}^{\omega}\omega$ be countable. We enumerate all elements of M as $\{f_n : n \in \omega\}$. Now consider a function $f : \omega \to \omega$ defined as $f(n) = \max\{f_0(n), f_1(n), \ldots, f_n(n)\}$ for $n \in \omega$. Then we have $f(n) \geq f_k(n)$ whenever $n \geq k$, thus $f_k \leq^* f$. Therefore M is bounded by f. □

If the continuum hypothesis holds, then no further questions remain about these cardinals – previous lemma implies that $\aleph_1 = \mathfrak{b} = \mathfrak{d} = \mathfrak{c}$. Without the assumption of CH, the situation is less clear. For example, it is consistent with ZFC that all these four cardinals are distinct.

As mentioned above, cardinals $\mathfrak{b}$ and $\mathfrak{d}$ are examples of cardinal characteristics of the continuum. There are many cardinal characteristics of the continuum being investigated in the literature, and various inequalities between them can be proved. Of course, for classical invariants these inequalities are never strict, because a strict inequality would imply $\neg CH$. A study of cardinal characteristics gives insights into what the combinatorial properties of cardinals between $\aleph_1$ and $\mathfrak{c}$ can look like. It gives rise to a rich theory which is trivialized once CH is assumed.

In the rest of this chapter we introduce some more cardinal characteristics of the continuum and recapitulate what is known about them.

2.2 Ideals

To define more cardinal characteristics of the continuum, a notion of an ideal is necessary.

Definition 13. *Let X be a set and $I \subseteq \mathcal{P}(X)$. We say that I is an* ideal *on X, if all following conditions are satisfied:*

1. $\emptyset \in I$,
2. $(\forall a, b \in I)(a \cup b \in I)$,
3. $(\forall a \in I)(\forall b \subseteq a)(b \in I)$.

Intuitively, an ideal gives us a notion about which subsets of X are "small." We consider a set to be small if it is included in I. We require an empty set to be small, a subset of a small set to be small and the union of two small sets to be small.

If $X \in I$, then necessarily $I = \mathcal{P}(X)$. This case is not interesting and often dismissed. We say that an ideal I is *proper*, if $X \notin I$. Another common requirement is that I contains all singletons, equivalently $\bigcup I = X$.

In the following text, we will need cardinal characteristics, which emerge naturally from ideals. For a proper ideal I such that $\bigcup I = X$, the following cardinals are defined:

1. add(I), the *additivity* of I, is the smallest cardinality of a system $A \subseteq I$ such that $\bigcup A \notin I$,
2. cov(I), the *covering number* of I, is the smallest cardinality of a system $A \subseteq I$ such that $\bigcup A = X$,
3. non(I), the *uniformity* of I, is the smallest cardinality of a set $M \subseteq X$ such that $M \notin I$,
4. cof(I), the *cofinality* of I, is the smallest cardinality of a system $A \subseteq I$ such that $(\forall M \in I)(\exists N \in A)(M \subseteq N)$.

It follows easily from definitions that $\aleph_0 \leq \text{add}(I) \leq \text{cov}(I) \leq \text{cof}(I)$ and $\text{add}(I) \leq \text{non}(I) \leq \text{cof}(I)$. If $\text{add}(I) \geq \aleph_1$ (i.e. if I is closed under countable unions), we say that I is a *σ-ideal*.

Example Let X be an infinite set and let I be the set of all finite subsets of X. It is straightforward to check that I is indeed an ideal. For this ideal $\text{add}(I) = \text{non}(I) = \aleph_0$ and $\text{cov}(I) = \text{cof}(I) = |X|$.

We now define two σ-ideals on $\mathbb{R}$ which give rise to 8 cardinal characteristics of the continuum.

Definition 14. *By $\mathcal{L}$ we denote the ideal consisting of all* null *subsets of* $\mathbb{R}$. *Remember that a set is null, if it is a subset of a set with Lebesgue measure zero.*

Definition 15. *By $\mathcal{B}$ we denote the ideal of all* meagre *subsets of* $\mathbb{R}$. *Remember that a set is meagre, if it is a countable union of nowhere dense sets (a set is nowhere dense, if its closure has a dense complement).*

The letter $\mathcal{L}$ stands for "Lebesgue" and $\mathcal{B}$ stands for "Baire". The Baire category theorem states that no nonempty open set is meagre, in particular that $\mathcal{B}$ is a proper ideal.

Both these ideals are in fact σ-ideals. For $\mathcal{B}$ it follows directly from the definition, for $\mathcal{L}$ it follows from the σ-additivity of Lebesgue measure. Additivity, covering, uniformity and cofinality of these ideals are other examples of cardinal characteristics of the continuum – they can be equal to $\aleph_1$, $\mathfrak{c}$, but also something in between.

Some inequalities follow directly from definitions, e.g. $\text{add}(\mathcal{B}) \leq \text{cov}(\mathcal{B})$. One can also prove some relations between the two ideals and the corresponding cardinal inequalities, for example it can be shown that $\text{add}(\mathcal{L}) \leq \text{add}(\mathcal{B})$. One can also prove some inequalities between these ideals and characteristics $\mathfrak{b}$ and $\mathfrak{d}$ defined above. In the rest of this chapter, we prove one such inequality. To do it, we endow the set ${}^{\omega}\omega$ with the so called Baire space topology and show it is in some sense very similar to $\mathbb{R}$.

2.3 The Baire space ${}^{\omega}\omega$

Definition 16. *For $n, k \in \omega$ let $C_n[k] = \{f \in {}^{\omega}\omega : f(n) = k\}$. The system $\{C_n[k] : n, k \in \omega\}$ is a subbasis of the Baire space topology on ${}^{\omega}\omega$. Through this thesis, whenever we talk about topology on ${}^{\omega}\omega$, we mean this topology.*

The set ${}^{\omega}\omega$ can be thought of as a cartesian product of countably many copies of ω. If we endow each of these copies with the discrete topology, the resulting product topology will be exactly the topology defined above. This topological space is interesting for its simple definition and for the following fact, which we state without proof:

Theorem 17. *The Baire space ${}^{\omega}\omega$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.*

We can define meagre subsets of ${}^{\omega}\omega$ in the same way we defined them for $\mathbb{R}$, and since these two spaces differ in just countably many points (in the sense of the previous theorem), corresponding ideals of meagre sets will behave the same way – specifically, their additivity, covering number, uniformity and cofinality will be the same. Characteristics related to $\mathcal{L}$ can be also defined in terms of ${}^{\omega}\omega$ and studied from a combinatorial point of view.

In the next section we show an example of how to use this topological space to prove a relation between seemingly unrelated cardinals – $\mathfrak{d}$ and $\mathrm{cov}(\mathcal{B})$. Topology of ${}^{\omega}\omega$ will be also used in other parts of this thesis (especially the fact that it has a countable basis of open sets). We now prove some simple facts regarding the Baire space, which we will use in later proofs.

For $f, g \in {}^{\omega}\omega$, we say that $f \leq g$, if $f(n) \leq g(n)$ for all n. We say that f and g are *infinitely equal*, if $f(n) = g(n)$ for infinitely many $n \in \omega$.

Lemma 18.

1. *For all $n \in \omega$ and $a \in {}^{n}\omega$, the set $\{f \in {}^{\omega}\omega : f \upharpoonright n = a\}$ is open.*

2. *For all $f \in {}^{\omega}\omega$, the set $\{g \in {}^{\omega}\omega : g \leq f\}$ is closed.*

3. *For all $f \in {}^{\omega}\omega$, the set $\{g \in {}^{\omega}\omega : g$ is not infinitely equal to $f\}$ is meagre.*

Proof.

1. The set can be expressed as $\bigcap_{i<n} C_i\,[a(i)]$, which is a finite intersection of open sets, hence open.

2. The complement of this set can be expressed as $\bigcup_{n\in\omega} \bigcup_{k>f(n)} C_n\,[k]$, which is a union of open sets, hence open. The set itself is therefore closed.

3. For $n \in \omega$ let $A_n = \{g \in {}^{\omega}\omega : (\forall m \geq n)(g(m) \neq f(m))\}$. Our set can be expressed as $\bigcup_{n\in\omega} A_n$. We show that each A_n is nowhere dense. It is closed because its complement is equal to $\bigcup_{m\geq n} C_m\,[f(m)]$. It remains to check that its complement is dense, i.e. that it intersects every nonempty finite intersection of elements of the subbasis. Indeed, given $n_0, k_0, \ldots, n_{l-1}, k_{l-1} \in \omega$ such that $G = \bigcap_{i<l} C_{n_i}\,[k_i]$ is nonempty, we can easily find $g \in G$ such that $g \notin A_n$. Hence, all A_n are nowhere dense and their union is meagre.

□

2.3.1 The relation between $\mathfrak{d}$ and $\mathrm{cov}(\mathcal{B})$

Now we use the above results to show the relationship between $\mathfrak{d}$ and $\mathrm{cov}(\mathcal{B})$ – specifically that $\mathrm{cov}(\mathcal{B}) \leq \mathfrak{d}$.

Lemma 19. *Let $M \subseteq {}^{\omega}\omega$ be such that $|M| < \mathrm{cov}(\mathcal{B})$. Then there exists $g \in {}^{\omega}\omega$ such that for all $f \in M$, f is infinitely equal to g.*

Proof. For $f \in M$ let $A_f = \{g \in {}^{\omega}\omega : g$ is not infinitely equal to $f\}$. By Lemma 18, each A_f is meagre. Since $|M| < \mathrm{cov}(\mathcal{B})$, the set $\bigcup_{f\in M} A_f$ can not cover the whole ${}^{\omega}\omega$, so we can always find $g \in {}^{\omega}\omega$ which is infinitely equal to all members of M. □

Theorem 20. $\mathrm{cov}(\mathcal{B}) \leq \mathfrak{d}$

Proof. Assume for contradiction that $\mathfrak{d} < \mathrm{cov}(\mathcal{B})$. Then there exists a dominating set $M \subseteq {}^{\omega}\omega$ such that $|M| = \mathfrak{d} < \mathrm{cov}(\mathcal{B})$. By the previous lemma, we find $g \in {}^{\omega}\omega$ such that g is infinitely equal to all elements of M. Since M is dominating, there should be $f \in M$ such that $g + 1 \leq^* f$, but this is in contradiction with f and g being infinitely equal. □

Lemma 19 can be shown to be optimal (see [3, Theorem 5.9]) – there exists $M \subseteq {}^{\omega}\omega$ with cardinality $\mathrm{cov}(\mathcal{B})$ such that no element of ${}^{\omega}\omega$ is infinitely equal to all elements of M. This gives an alternative, purely combinatorial definition of $\mathrm{cov}(\mathcal{B})$. Other cardinal characteristics related to $\mathcal{B}$ and $\mathcal{L}$ can be also rephrased in such way.

2.4 Cichoń's diagram

A complete list of all provable inequalities involving the 10 cardinals described in this chapter is known. These inequalities are visualized in the Cichoń's diagram. Arrows between characteristics correspond to inequalities – if there is an arrow from x to y, it means that ZFC proves $x \leq y$. The diagram is complete in the sense that whenever x and y are not transitively connected by arrows, then each of $x > y$ and $x < y$ is consistent with ZFC.

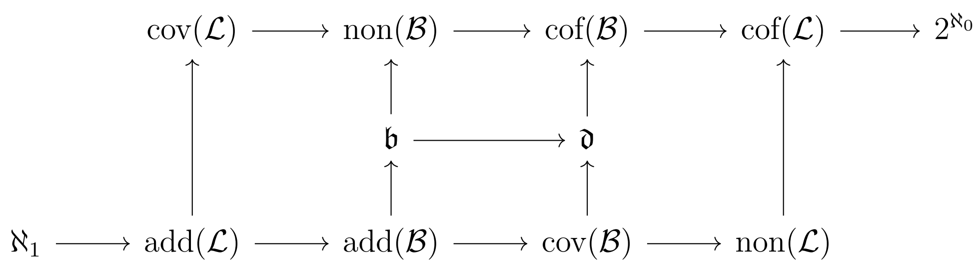

Figure 2.1: Cichoń's diagram

In addition to these inequalities, there are two more relations that can be proved about the characteristics:

$$\mathrm{add}(\mathcal{B}) = \min\{\mathrm{cov}(\mathcal{B}), \mathfrak{b}\},$$
$$\mathrm{cof}(\mathcal{B}) = \max\{\mathrm{non}(\mathcal{B}), \mathfrak{d}\}.$$

3. Weaker forms of CH and cardinal characteristics

In this chapter we look at some consequences of weaker forms of CH on cardinal characteristics of the continuum. Typically, they imply that some cardinal characteristics are equal to $\aleph_1$. In [7], several consequences of this form were proved. This chapter contains a detailed presentation of proofs of these consequences.

3.1 The bounding number

In this section we show that $CH_2 \implies \mathfrak{b} = \aleph_1$. First, a few lemmas.

Lemma 21. *$\mathfrak{b}$ is a regular cardinal.*

Proof. Let $\{f_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded set of functions from ω to ω. Let $\{\xi_\gamma : \gamma < \mathrm{cf}(\mathfrak{b})\}$ be a sequence of ordinals below $\mathfrak{b}$ converging to $\mathfrak{b}$. For each $\gamma < \mathrm{cf}(\mathfrak{b})$ the set $\{f_\alpha : \alpha < \xi_\gamma\}$ has cardinality smaller than $\mathfrak{b}$, so it is bounded by some function g_γ. The set $\{g_\gamma : \xi < \mathrm{cf}(\mathfrak{b})\}$ is then unbounded, because any function bounding it would also bound $\{f_\alpha : \alpha < \mathfrak{b}\}$. Hence $\mathrm{cf}(\mathfrak{b})$ is at least $\mathfrak{b}$, so $\mathfrak{b}$ is regular. □

Lemma 22. *Let $(X, \leq)$ be a well-ordered set and let $\{x_\alpha : \alpha < \omega_1\}$ be a sequence of distinct elements of X. Then there exists an increasing sequence of countable ordinals $\{\gamma_\alpha : \alpha < \omega_1\}$ such that for all $\alpha < \beta < \omega_1$, $x_{\gamma_\alpha} < x_{\gamma_\beta}$.*

Proof. Since every well-order is isomorphic to an ordinal, we can assume that $\{x_\alpha : \alpha < \omega_1\}$ is a permutation of elements of δ for some $\omega_1 \leq \delta < \omega_2$, where $\leq$ corresponds to standard ordering of ordinals. We now define by transfinite induction ordinals $\{\gamma_\alpha : \alpha < \omega_1\}$ such that the sequence $(x_{\gamma_\alpha})_{\alpha<\omega_1}$ is increasing and $x_{\gamma_\alpha} < \omega_1$ for all α.

Let $\alpha < \omega_1$ and assume we have already defined γ_β for all $\beta < \alpha$. Let $\xi = \sup\{\gamma_\beta : \beta < \alpha\}$ and $\mu = \sup\{x_{\gamma_\beta} : \beta < \alpha\}$, both ξ and μ are below ω_1. Since μ is countable, we can always find γ such that $\xi < \gamma < \omega_1$ and $\gamma \neq x_\beta$ for all $\beta \leq \mu$. Put $\gamma_\alpha = \gamma$. □

The following lemma gives a useful way to construct ccc posets. Notice that its proof is similar to that of Theorem 6. The lemma comes from [5, Lemma 13].

Lemma 23. *Let X be a topological space with countable basis of open sets. Let $\leq$ be a well-ordering of X. Suppose F is a function from X such that for all $x \in X$, $F(x)$ is a closed subset of $\{y \in X : y \leq x\}$. Let*

$$P = \{s \in [X]^{<\omega} : x \notin F(y) \text{ for all distinct } x, y \in s\}.$$

For $a, b \in P$ put $a \leq b$ iff $a \supseteq b$. Then P is a ccc poset.

Proof. For contradiction, let $\{s_\alpha : \alpha < \omega_1\}$ be an antichain in P. Because of the Δ-system lemma we can assume that it is a Δ-system with kernel r. If s_α and s_β are incompatible, there must be some $x \in s_\alpha$ and $y \in s_\beta$ such that $x \in F(y)$ or

$y \in F(x)$. Since no such pair of elements is present in any of s_α, s_β, it must be the case that $x \in s_\alpha \setminus r$ and $y \in s_\beta \setminus r$. Thus, $\{s_\alpha \setminus r : \alpha < \omega_1\}$ is an antichain as well, so without loss of generality we can assume that in fact $r = \emptyset$ and all s_α are pairwise disjoint. We can also assume that for some $n \in \omega$, $|s_\alpha| = n$ holds for all α.

For all α, let $\{s_\alpha^0, \dots, s_\alpha^{n-1}\}$ be an increasing enumeration (with respect to ordering of X) of elements of s_α. By a repeated application of Lemma 22, we can assume that for all $i < n$, $(s_\alpha^i)_{\alpha<\omega_1}$ is an incresing sequence.

Let B be a countable basis of open sets of X. Pick $\alpha < \omega_1$. Since $F(s_\alpha^j)$ is a closed set not containing s_α^i for all $i \neq j$, we can find sets $U_{\alpha,0}, \dots, U_{\alpha,n-1} \in B$ such that for all $i \neq j$, $s_\alpha^i \in U_{\alpha,i}$ and $F(s_\alpha^j) \cap U_{\alpha,i} = \emptyset$. Since there are only countably many choices of these sets, some choice of $U_0, \dots, U_{n-1}$ will work for uncountably many α. Assume it works for all α.

Choose $\alpha < \beta < \omega_1$. We know that s_α and s_β are incompatible, but by now we have eliminated most ways how it could happen. For $i \neq j$ we know that $s_\alpha^i \in U_i$ and $F(s_\beta^j) \cap U_i = \emptyset$, hence $s_\alpha^i \notin F(s_\beta^j)$, and similarly $s_\beta^i \notin F(s_\alpha^j)$. For $i < n$ we know that $s_\alpha^i < s_\beta^i$, hence $s_\beta^i \notin F(s_\alpha^i)$. The only remaining option is that $s_\alpha^i \in F(s_\beta^i)$ for some i.

For $\alpha < \omega_1$ let $S_\alpha = \langle s_\alpha^0, \dots, s_\alpha^{n-1} \rangle \in X^n$. X^n has a countable basis of open sets, so $\{S_\alpha : \alpha < \omega_1\}$ has such basis as well. Therefore, there is $\gamma < \omega_1$ such that $\{S_\alpha : \alpha < \gamma\}$ is dense in $\{S_\alpha : \alpha < \omega_1\}$. Choose $\delta > \gamma$. For all $i < n$, $X \setminus F(s_\gamma^i)$ is an open set containing s_δ^i. Hence, $S_\delta \in \Pi_{i<n} \left(X \setminus F(s_\gamma^i) \right)$, so by density we find $\beta < \gamma$ such that $S_\beta \in \Pi_{i<n} \left(X \setminus F(s_\gamma^i) \right)$ as well. Then $s_\beta^i \notin F(s_\gamma^i)$ for all i, which is a contradiction with the previous paragraph. □

To show $CH_2 \implies \mathfrak{b} = \aleph_1$, we need to come up with a poset on which we will apply CH_2. From the definition of $\mathfrak{b}$ we know there exists a sequence of functions $A = \{f_\alpha : \alpha < \mathfrak{b}\}$ which is unbounded. We want A to be "nice" in some way. Specifically, we want the following to hold:

1. Each f_α is an increasing function.
2. $f_\alpha <^* f_\beta$ for all $\alpha < \beta$.

This can be assumed without loss of generality. If it did not hold, then we construct a sequence $\{g_\alpha : \alpha < \mathfrak{b}\}$ by a transfinite induction as follows. For $\alpha < \mathfrak{b}$, if we have g_β defined for all $\beta < \alpha$, pick $g \in {}^\omega\omega$ such that $g \geq^* g_\beta$ for all $\beta < \alpha$. Since $\alpha < \mathfrak{b}$, it follows from the definition of $\mathfrak{b}$ that such g always exists. Now put $g'_\alpha = \max\{g, f_\alpha\} + 1$ and $g_\alpha(n) = \max\{g'_\alpha(0), \dots, g'_\alpha(n)\} + n$. The result of this construction $\{g_\alpha : \alpha < \mathfrak{b}\}$ is unbounded (because $g_\alpha \geq^* g'_\alpha \geq^* f_\alpha$), each g_α is increasing and $g_\alpha <^* g_\beta$ for all $\alpha < \beta$.

Given a system A with the properties described above, we define our poset as

$$P = \{X \in [A]^{<\omega} : f \not\leq g \text{ for all distinct } f, g \in X\}.$$

Remember that $f \leq g$ if $f(n) \leq g(n)$ for all $n \in \omega$. In words, P consists of all pairwise $\leq$-incomparable finite subsets of A. We order P by the reverse inclusion ($X \leq Y$ iff $X \supseteq Y$). Notice that $|P| = \mathfrak{b} \leq \mathfrak{c}$.

Lemma 24. *P is ccc.*

Proof. We apply Lemma 23. Our topological space will be A with the topology inherited from ${}^\omega\omega$. By our assumptions, it is well-ordered by $\leq^*$. For $f \in A$ put $F(f) = \{g \in A : g \leq f\}$. Clearly $F(f) \subseteq \{g \in A : g \leq^* f\}$. Under this definition, our poset is precisely the one from Lemma 23. It remains to check that $F(f)$ is closed in A. Indeed, by Lemma 18, $\{g \in {}^\omega\omega : g \leq f\}$ is closed in ${}^\omega\omega$, so $F(f) = \{g \in {}^\omega\omega : g \leq f\} \cap A$ is closed in A. □

The following is the Lemma 16 of [5].

Lemma 25. *Let $\{f_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded $<^*$-increasing sequence of increasing functions from ω to ω. Then there exists $\alpha < \beta < \mathfrak{b}$ such that $f_\alpha \leq f_\beta$.*

Proof. Since ${}^\omega\omega$ has a countable basis and $\mathrm{cf}(\mathfrak{b}) > \aleph_0$, we can find $\gamma < \mathfrak{b}$ such that $\{f_\alpha : \alpha < \gamma\}$ is dense in $\{f_\alpha : \alpha < \mathfrak{b}\}$. For each $\delta > \gamma$ there is $n_0 \in \omega$ such that $(\forall n \geq n_0)(f_\gamma(n) < f_\delta(n))$. Since $\mathrm{cf}(\mathfrak{b}) > \aleph_0$, some value of n_0 will work for $\mathfrak{b}$ values of δ. We can assume that this n_0 works for all $\delta > \gamma$. We can also assume that for some $t_0 \in {}^{n_0}\omega$, $f_\delta \restriction n_0 = t_0$ holds for all $\delta > \gamma$.

Let $n_1 \in \omega$ be minimal such that $\{f_\delta(n_1) : \delta > \gamma\}$ is unbounded in ω. In other words, n_1 is the largest natural number such that a set $\{f_\delta \restriction n_1 : \delta > \gamma\}$ is finite, in particular $n_1 \geq n_0$. So, there is $t_1 \in {}^{n_1}\omega$ and an increasing sequence $\{\delta_i : i \in \omega\}$ of ordinals above γ such that for all $i \in \omega$, $f_{\delta_i} \restriction n_1 = t_1$ and $f_{\delta_{i+1}}(n_1) > f_{\delta_i}(n_1)$.

By Lemma 18, $\{f \in {}^\omega\omega : f \restriction n_1 = t_1\}$ is an open set and we know it intersects $\{f_\alpha : \alpha < \mathfrak{b}\}$, so by the density condition there is $\alpha < \gamma$ such that $f_\alpha \restriction n_1 = t_1$. Since $f_\alpha <^* f_\gamma$, we find $n_2 \geq n_1$ such that for all $n \geq n_2$, $f_\alpha(n) < f_\gamma(n)$. Take $i \in \omega$ large enough that $f_{\delta_i}(n_1) > f_\alpha(n_2)$. We claim that $f_\alpha \leq f_{\delta_i}$. Indeed:

1. For $n < n_1$ we have $f_\alpha(n) = f_{\delta_i}(n) = t_1(n)$.
2. For $n_1 \leq n < n_2$ we have $f_\alpha(n) < f_\alpha(n_2) < f_{\delta_i}(n_1) < f_{\delta_i}(n)$. Here we use that all our functions are increasing.
3. For $n \geq n_2$ we have $f_\alpha(n) < f_\gamma(n) < f_\delta(n)$.

□

Theorem 26. $CH_2 \implies \mathfrak{b} = \aleph_1$

Proof. Remember that we have chosen an $A = \{f_\alpha : \alpha < \mathfrak{b}\}$ and defined a poset P with cardinality $\mathfrak{b}$, which is ccc by Lemma 24. By CH_2, we can decompose P into $\aleph_1$ linked sets – let $\{P_\gamma : \gamma < \omega_1\}$ be such decomposition. Assume for contradiction that $\mathfrak{b} \geq \aleph_2$. For each $\alpha < \mathfrak{b}$ let $\gamma_\alpha < \omega_1$ be such that $\{f_\alpha\} \in P_{\gamma_\alpha}$. Since $\mathfrak{b}$ is regular, there is γ such that $\gamma = \gamma_\alpha$ for $\mathfrak{b}$ values of α. If we define $B = \{f_\alpha : \alpha < \mathfrak{b},\ \{f_\alpha\} \in P_\gamma\}$, then B is a subsequence of $\{f_\alpha : \alpha < \mathfrak{b}\}$ with cardinality $\mathfrak{b}$, so it is unbounded and with order type $\mathfrak{b}$. We can therefore use Lemma 25 on B and find $\alpha < \beta < \mathfrak{b}$ such that $f_\alpha \leq f_\beta$ and $\{f_\alpha\}, \{f_\beta\} \in P_\gamma$. But the first condition means that $\{f_\alpha\}$ and $\{f_\beta\}$ are incompatible, which is in contradiction with P_γ being linked. Therefore, $\mathfrak{b} = \aleph_1$. □

Remark. It is known to be consistent with ZFC that $\mathfrak{b} > \aleph_1$ (for example, $MA + \neg CH$ implies it). Hence, $\neg CH_2$ is consistent with ZFC.

3.2 The dominating number

Showing that $\mathfrak{d} = \aleph_1$ would be a stronger result than $\mathfrak{b} = \aleph_1$. It is not known whether it follows from CH_2 alone, but the stronger CH_3 already implies $\mathfrak{d} = \aleph_1$. In this section we show the proof of this implication.

Definition 27. *For $f, g \in {}^\omega\omega$ let $\Delta(f,g)$ be the least $n \in \omega$ such that $f(n) \neq g(n)$. If $f = g$, we put $\Delta(f,g) = \infty$.*

As before, we need to come up with a good poset. Let

$$P = \{X \in [{}^\omega\omega]^{<\omega} : \text{ for no distinct } f,g,h \in X \text{ is } \Delta(f,g) = \Delta(f,h) = \Delta(g,h)\}.$$

In other words, P consists of such finite sets X, that no three functions from X split for the first time at the same point. We order P by the reverse inclusion.

Lemma 28. *P is ccc.*

Proof. Let $\{X_\alpha : \alpha < \omega_1\}$ be a system of elements of P. We show that two of its elements are compatible. We can assume that for some $n \in \omega$, $|X_\alpha| = n$ for all α. For each α let $\{f_{\alpha,0}, \dots, f_{\alpha,n-1}\}$ be an enumeration of elements of X_α.

For each α we can find $N \in \omega$ such that $f_{\alpha,0} \upharpoonright N, \dots, f_{\alpha,n-1} \upharpoonright N$ are all distinct (in other words, $\Delta(f_{\alpha,i}, f_{\alpha,j}) < N$ for all $i < j < n$). We can assume that some choice of N works for all α. We can also assume that for all $\alpha < \beta < \omega_1$ and $i < n$, $f_{\alpha,i} \upharpoonright N = f_{\beta,i} \upharpoonright N$.

Now pick some distinct $\alpha, \beta < \omega_1$. We show that X_α and X_β are compatible. Assume not, then there are some $f,g,h \in X_\alpha \cup X_\beta$ distinct such that $\Delta(f,g) = \Delta(f,h) = \Delta(g,h) = c$. Neither of X_α and X_β contains such three functions, so (without loss of generality) $f = f_{\alpha,i}$, $g = f_{\alpha,j}$ and $h = f_{\beta,k}$ for some $i,j,k < n$, $i \neq j$. Then $\Delta(f,g) < N$, so $c < N$. Denote $f' = f_{\beta,i}$ and $g' = f_{\beta,j}$. We know that $f \upharpoonright N = f' \upharpoonright N$ and $g \upharpoonright N = g' \upharpoonright N$. Hence, $\Delta(f',g') = \Delta(f,g) = c$. Notice that if $i = k$ (i.e. $h = f'$), then f and h are equal below N, so $\Delta(f,h) \geq N$. On the other hand, if $i \neq k$, then $\Delta(f',h) = \Delta(f,h) = c < N$. Same relations hold for j. Now we distinguish 3 cases:

1. $i = k \neq j$. Then $\Delta(f,h) \geq N$ and $\Delta(g,h) < N$, a contradiction.

2. $i \neq k = i$. Similarly, $\Delta(g,h) \geq N$ and $\Delta(f,h) < N$, a contradiction.

3. $i \neq k \neq j$. Then $\Delta(f',h) = c$ and also $\Delta(g',h) = c$. Thus, $\Delta(f',h) = \Delta(g',h) = \Delta(f',g') = c$, which is a contradiction with $f',g',h \in X_\beta$.

As we see, all choices lead to contradiction. □

Remark. The proof can be easily modified to show that P is in fact σ-linked. First we split P into countably many parts based on the cardinality of elements (n from the proof). Then we split each part into countably many smaller parts based on the value of N from the proof. Finally, we split each of these smaller parts based on the value of $\langle f_{\alpha,0} \upharpoonright N, \dots, f_{\alpha,n-1} \upharpoonright N\rangle$. The last paragraph of the proof shows that all of these final parts are linked.

Lemma 29. *Let $X \subseteq {}^\omega\omega$ be such that for no distinct $f,g,h \in X$ is $\Delta(f,g) = \Delta(f,h) = \Delta(g,h)$. Then X is bounded.*

Proof. We show that for all $n \in \omega$, $\{f(n) : f \in X\}$ is finite. We prove it by induction on n. Let $n \in \omega$, assume the lemma holds for all $k < n$ and we show it for n. Since $\{f(k) : f \in X\}$ is finite for all $k < n$, $M = \{f \restriction n : f \in X\}$ is finite as well. For all $t \in M$, the set $\{f(n) : f \in X, f \restriction n = t\}$ has cardinality at most 2. Otherwise, we would get three functions splitting for the first time at the same point. Hence, $|\{f(n) : f \in X\}| \leq 2|M| < \omega$.

Now, for $n \in \omega$ put $g(n) = \sup\{f(n) : n \in X\}$, this is a correct definition because of the previous paragraph. Then clearly $f \leq^* g$ (even $f \leq g$) for all $f \in X$. □

Theorem 30. $CH_3 \implies \mathfrak{d} = \aleph_1$

Proof. By Lemma 28 P is ccc, clearly $|P| = \mathfrak{c}$, so by CH_3 there is a decomposition $\{P_\alpha : \alpha < \omega_1\}$ of P into 3-linked sets. For $\alpha < \omega_1$ let $X_\alpha = \{f \in {}^\omega\omega : \{f\} \in P_\alpha\}$. Observe that $\bigcup_{\alpha<\omega_1} X_\alpha = {}^\omega\omega$. Since P_α is 3-linked, X_α satisfies the condition of Lemma 29, so we find g_α which dominates all elements of X_α. Then $\{g_\alpha : \alpha < \omega_1\}$ is a dominating set of cardinality $\aleph_1$. □

3.3 The cofinality of $\mathcal{L}$

In this section we look at CH_ω, the strongest of the considered axioms. It turns out that $CH_\omega \implies \text{cof}(\mathcal{L}) = \aleph_1$. Note that this already implies that all cardinal characteristics from Cichoń's diagram are equal to $\aleph_1$.

Let $\mathcal{E}$ be the set of all finite unions of open intervals in $\mathbb{R}$ with rational endpoints (thus $|\mathcal{E}| = \aleph_0$). For the proof of the implication we will need the following well-known property of Lebesgue measure which we state without proof.

Theorem 31 (regularity of Lebesgue measure)**.** *Let $X \subseteq \mathbb{R}$ be Lebesgue measurable with $\lambda(X) < \infty$ and let $\varepsilon > 0$.*

1. *There exists a compact set $F \subseteq X$ such that $\lambda(F) > \lambda(X) - \varepsilon$.*
2. *There exists an open set $G \supseteq X$ such that $\lambda(G) < \lambda(X) + \varepsilon$.*
3. *There exists $E \in \mathcal{E}$ such that $\lambda(X \triangle E) < \varepsilon$. Here, $A \triangle B$ stands for $(A \setminus B) \cup (B \setminus A)$.*

We consider the poset

$$P = \{F \subseteq \mathbb{R} : F \text{ is compact and } \lambda(F) > 1\}.$$

For $a, b \in P$ put $a \leq b$ if $a \subseteq b$. Notice that $|P| = \mathfrak{c}$.

Lemma 32. *P is ccc.*

Proof. We show it is σ-linked. For each $n \in \omega$ let $P_n = \left\{F \in P : \lambda(F) > 1 + \frac{1}{n}\right\}$, so $P = \bigcup_{n\in\omega} P_n$. For $n \in \omega$ and $E \in \mathcal{E}$ put $P_{n,E} = \left\{F \in P_n : \lambda(F \triangle E) < \frac{1}{2n}\right\}$. By the regularity of measure, $\bigcup_{E\in\mathcal{E}} P_{n,E} = P_n$. Now we show that each $P_{n,E}$ is linked.

Fix $n \in \omega$ and $E \in \mathcal{E}$. For any $F_1, F_2 \in P_{n,E}$ we see that

$$\begin{aligned} \lambda(F_1 \cap F_2) &= \lambda(F_1 \cup F_2) - \lambda(F_1 \triangle F_2) \\ &\geq \lambda(F_1) - \lambda(F_1 \triangle F_2) \\ &> 1 + \frac{1}{n} - \lambda(F_1 \triangle F_2) \\ &\geq 1 + \frac{1}{n} - (\lambda(F_1 \triangle E) + \lambda(F_2 \triangle E)) \\ &> 1 + \frac{1}{n} - \left(\frac{1}{2n} + \frac{1}{2n}\right) = 1, \end{aligned}$$

so $F_1 \cap F_2 \in P$. Hence, F_1 and F_2 are compatible and $P_{n,E}$ is linked. □

Lemma 33. *Let $X \subseteq P$ be centered. Then $\lambda\left(\bigcap X\right) \geq 1$.*

Proof. $\bigcap X$ is a compact set, hence measurable. Assume $\lambda\left(\bigcap X\right) < 1$. By regularity of measure, we can find an open $G \supseteq \bigcap X$ such that $\lambda(G) \leq 1$. Now consider the system $Y = \{F \setminus G : F \in X\}$. Since X is centered, every finite intersection of elements of X has measure more than 1. Hence, every finite intersection of elements of Y has positive measure. Since Y consists of compact sets, it follows that $\bigcap Y \neq \emptyset$. However, $\bigcap Y = \bigcap X \setminus G = \emptyset$, a contradiction. □

For $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ define $A + x = \{a + x : a \in A\}$. For $A, B \subseteq \mathbb{R}$ put $A + B = \{a + b : a \in A, b \in B\}$.

Lemma 34. *Let $X \subseteq \mathbb{R}$ have finite positive measure. Then the complement of $X + \mathbb{Q}$ has measure zero.*

Proof. Denote $Y = X + \mathbb{Q}$ and $Z = \mathbb{R} \setminus Y$. Fix $\varepsilon > 0$. We can find $E \in \mathcal{E}$ such that $\lambda(E) > 0$ and $\lambda(X \triangle E) < \frac{1}{2}\lambda(X)\varepsilon$. Then

$$\begin{aligned} \frac{\lambda(Y \cap E)}{\lambda(E)} &\geq \frac{\lambda(X \cap E)}{\lambda(E)} \\ &\geq \frac{\lambda(X) - \lambda(X \triangle E)}{\lambda(X) + \lambda(X \triangle E)} \\ &> \frac{\left(1 - \frac{\varepsilon}{2}\right)\lambda(X)}{\left(1 + \frac{\varepsilon}{2}\right)\lambda(X)} \\ &= \frac{1 - \frac{\varepsilon}{2}}{1 + \frac{\varepsilon}{2}} \\ &> 1 - \varepsilon. \end{aligned}$$

Call a set $M \subseteq \mathbb{R}$ with finite positive measure *good*, if $\frac{\lambda(Y \cap M)}{\lambda(M)} > 1 - \varepsilon$. We know that E is good. Since $E \in \mathcal{E}$, we can find $n \in \mathbb{N}$ such that E is equal to a union of finitely many intervals of the form $\left(\frac{i}{n}, \frac{i+1}{n}\right)$, $i \in \mathbb{Z}$, and a finite set. It then follows that for some $i \in \mathbb{Z}$, $\left(\frac{i}{n}, \frac{i+1}{n}\right)$ is also good. Since $Y = Y + q$ for all $q \in \mathbb{Q}$, this already implies that $\left(\frac{i}{n}, \frac{i+1}{n}\right)$ is in fact good for all $i \in \mathbb{Z}$. But then interval $(0, 1)$, being a finite union of good intervals plus finitely many points, must be good as well.

We proved that $\lambda(Y \cap (0,1)) > 1 - \varepsilon$. Since this holds for all $\varepsilon > 0$, it must be the case that $\lambda(Y \cap (0,1)) = 1$ and hence $\lambda(Z \cap (0,1)) = 0$. Similarly, $\lambda(Z \cap (n, n+1)) = 0$ for all $n \in \mathbb{Z}$. By σ-additivity of measure, we see that $\lambda(Z) = \sum_{n\in\mathbb{Z}} \lambda(Z \cap (n, n+1)) = 0$. □

Theorem 35. $CH_\omega \implies \mathrm{cof}(\mathcal{L}) = \aleph_1$

Proof. By Lemma 32 P is ccc, also $|P| = \mathfrak{c}$, so by CH_ω we get a decomposition $P = \bigcup_{\alpha<\omega_1} P_\alpha$ with P_α centered. By Lemma 33, every $X_\alpha = \bigcap P_\alpha$ has measure at least 1. By Lemma 34, every $Y_\alpha = \mathbb{R} \setminus (X_\alpha + \mathbb{Q})$ has measure zero. We claim that every null set $N \subseteq \mathbb{R}$ is a subset of Y_α for some $\alpha < \omega_1$. It is enough to show it for N measurable with measure zero.

Choose $N \subseteq \mathbb{R}$ such that $\lambda(N) = 0$. The set $N + \mathbb{Q} = \bigcup_{q\in\mathbb{Q}}(N + q)$ is a countable union of zero measure sets, hence it has also measure zero. Let $M = [0,2] \setminus (N + \mathbb{Q})$, then $\lambda(M) = 2$. By regularity of measure, we find a compact $F \subseteq M$ such that $\lambda(F) > 1$. Since F does not intersect $N + \mathbb{Q}$, N does not intersect $F + \mathbb{Q}$. We have $F \in P$, so let $\alpha < \omega_1$ be such that $F \in P_\alpha$. Then $X_\alpha \subseteq F$, so $X_\alpha + \mathbb{Q} \subseteq F + \mathbb{Q}$ and $Y_\alpha \supseteq \mathbb{R} \setminus (F + \mathbb{Q}) \supseteq N$. We see that each null set is included in some Y_α, so indeed $\mathrm{cof}(\mathcal{L}) = \aleph_1$. □

3.4 The cofinality of $\mathfrak{c}$

Note that by König's inequality, $\mathfrak{c}^{\mathrm{cf}(\mathfrak{c})} > \mathfrak{c}$. On the other hand $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$, hence $\mathrm{cf}(\mathfrak{c}) \geq \aleph_1$. Of course, $\mathrm{cf}(\mathfrak{c}) \leq \mathfrak{c}$. From this perspective, $\mathrm{cf}(\mathfrak{c})$ can be also thought of as a cardinal characteristic of the continuum. In this section we show that $CH_3 \implies \mathrm{cf}(\mathfrak{c}) = \aleph_1$.

Fix some $f : {}^\omega\omega \to {}^\omega\omega$. We will study the poset

$$P_f = \Big\{X \in [{}^\omega\omega]^{<\omega} : (\forall x, y, z \in X)\, (\Delta(x,y) < \Delta(x,z) \implies \min\{\Delta(x,y), \Delta(f(x), f(y))\} < \min\{\Delta(x,z), \Delta(f(x), f(z))\})\Big\}.$$

This is a modified version of the poset from [7, p. 844, (16)], which seems to contain a mistake in its defintion.

As usual, we need to show that the poset is ccc. The next proof is very similar to the proof of Lemma 28. As in Lemma 28, it can be easily modified to show that P_f is even σ-linked.

Lemma 36. *P_f is ccc.*

Proof. Let $\{X_\alpha : \alpha < \omega_1\}$ be a system of elements of P_f. Assume that for some n, all X_α have cardinality n. For all α, let $\{x_{\alpha,0}, \dots, x_{\alpha,n-1}\}$ be an enumeration of elements of X_α.

We can assume that there is $N \in \omega$ such that for all α, $x_{\alpha,0} \restriction N, \dots, x_{\alpha,n-1} \restriction N$ are all distinct. Similarly, we can assume that whenever $f(x_{\alpha,i}) \neq f(x_{\alpha,j})$ for some $i < j < n$, then also $f(x_{\alpha,i}) \restriction N \neq f(x_{\alpha,j}) \restriction N$. We can also assume that $\langle x_{\alpha,0} \restriction N, \dots, x_{\alpha,n-1} \restriction N, f(x_{\alpha,0}) \restriction N, \dots, f(x_{\alpha,n-1}) \restriction N\rangle$ does not depend on α.

Pick some distinct $\alpha, \beta < \omega_1$. We show that X_α, X_β are compatible. If they were not, we could find $x, y, z \in X_\alpha \cup X_\beta$ such that $\Delta(x,y) < \Delta(x,z)$ and $\min\{\Delta(x,y), \Delta(f(x), f(y))\} \geq \min\{\Delta(x,z), \Delta(f(x), f(z))\}$. This implies that

$\Delta(x,y) \geq \Delta(f(x), f(z))$. It can be easily checked that x, y, z must be all distinct. Without loss of generality let $x \in X_\alpha$.

Note that if $t \in X_\alpha$, then $t = x_{\alpha,i}$ for some $i < n$. In such case, by t' we denote $x_{\beta,i}$. If $t \in X_\beta$, by t' we mean t. It is easy to see that for $t \in X_\alpha \cap X_\beta$ these definitions are compatible.

The idea is that x', y', z' behave the same way as x, y, z below N, but since $x', y', z' \in X_\beta$, they can not satisfy the condition which x, y, z are supposed to satisfy. The precise argument goes as follows.

1. Assume $\Delta(x', y') \neq \Delta(x, y)$. This can happen only if $y = x'$. Then $\Delta(x, y) \geq N$, so also $\Delta(x, z) \geq N$. Therefore, z is equal to either x or y, a contradiction.

2. If $\Delta(x', z') \neq \Delta(x, z)$, then $z = x'$. Then $\Delta(f(x), f(z)) \geq N$, so also $\Delta(x, y) \geq N$, so y is either x or z, a contradiction.

3. $\Delta(f(x), f(y)) \neq \Delta(f(x'), f(y'))$ can happen only if $N \leq \Delta(f(x), f(y)) < \infty$ and $\Delta(f(x'), f(y')) = \infty$.

4. If $\Delta(f(x), f(z)) \neq \Delta(f(x'), f(z'))$, then $\Delta(f(x), f(z)) \geq N$, hence also $\Delta(x, y) \geq N$, thus $y = x'$, but this we already refuted.

We see that all relevant expressions are preserved when we replace x, y, z with x', y', z', except possibly for $\Delta(f(x), f(y))$ which can become larger. Because of $x', y', z' \in X_\beta$ we have

$$\Delta(x', y') < \Delta(x', z') \implies \min\{\Delta(x', y'), \Delta(f(x'), f(y'))\} < \min\{\Delta(x', z'), \Delta(f(x'), f(z'))\},$$

so the same must hold for x, y, z, a contradiction. □

Lemma 37. *Let $X \subseteq {}^\omega\omega$ be such that for all $x, y, z \in X$, $\Delta(x, y) < \Delta(x, z)$ implies $\min\{\Delta(x, y), \Delta(f(x), f(y))\} < \min\{\Delta(x, z), \Delta(f(x), f(z))\}$. Then f is continuous on X.*

Proof. It is enough to show that for each $[a] \in {}^{<\omega}\omega$, $M_a = f^{-1}[a] \cap X$ is an open set in X. Pick some $x \in M_a$ and we find an open neighbourhood of x which is also included in M_a.

We claim there is $b \in {}^{<\omega}\omega$ such that $x \in [b]$ and $[b] \cap X \subseteq M_a$. Assume not, then for each $n \in \omega$ we can find $y_n \in X \setminus M_a$ such that $\Delta(x, y_n) \geq n$ (otherwise we could set $b = x \restriction n$). We can assume that $(\Delta(x, y_n))_{n\in\omega}$ is an increasing sequence. Then, $(\min\{\Delta(x, y_n), \Delta(f(x), f(y_n))\})_{n\in\omega}$ is an increasing sequence as well, so we can find $n \in \omega$ such that $\Delta(f(x), f(y_n)) \geq \mathrm{dom}(a)$, hence $f(y_n) \in [a]$. But this means that $y \in M_a$, a contradiction. □

We use the next theorem to prove $CH_3 \implies \mathrm{cf}(\mathfrak{c}) = \aleph_1$, but the theorem is an interesting consequence of CH_3 on its own.

Theorem 38. *CH_3 implies that for each $f : {}^\omega\omega \to {}^\omega\omega$, f can be expressed as a union of $\aleph_1$ many continuous functions.*

Proof. Consider the poset P_f. By Lemma 36, P_f is ccc. $|P_f| = \mathfrak{c}$, so by CH_3 we have a decomposition $P = \bigcup_{\alpha<\omega_1} P_\alpha$ into 3-linked sets. For $\alpha < \omega_1$ put $X_\alpha = \bigcup P_\alpha$. It follows from P_α being 3-linked that X_α satisfies the condition of Lemma 37, hence $f \restriction X_\alpha$ is a continuous function. However, for all $x \in {}^\omega\omega$ we have $\{x\} \in P$, so $\{x\}$ must be included in some P_α. This imples that $\bigcup_{\alpha<\omega_1} X_\alpha = {}^\omega\omega$, so $\{f \restriction X_\alpha : \alpha < \omega_1\}$ gives us a decomposition of X into $\aleph_1$ many continuous functions. □

Theorem 39. $CH_3 \implies \mathrm{cf}(\mathfrak{c}) = \aleph_1$

Proof. We just need to find $f : {}^\omega\omega \to {}^\omega\omega$ such that f is not continuous on any set of cardinality $\mathfrak{c}$. Then, by Theorem 38, we will get a decomposition of ${}^\omega\omega$ into $\aleph_1$ sets of cardinality smaller than $\mathfrak{c}$.

Every continuous function from a subset of ${}^\omega\omega$ to ${}^\omega\omega$ can be extended to a continuous function on a G_δ set (i.e. a countable intersection of open sets) – see [6, chapter 4]. Since the space ${}^\omega\omega$ has a countable basis, there are only $\mathfrak{c}$ many G_δ subsets of ${}^\omega\omega$. Let A be some subset of ${}^\omega\omega$. Since ${}^\omega\omega$ has a countable basis, there is a countable $D \subseteq A$, which is dense in A. Every continuous function from A to ${}^\omega\omega$ is fully determined by its values on D. Hence, there are at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ continuous functions from A to ${}^\omega\omega$. It follows that there are at most $\mathfrak{c}$ continuous functions from G_δ subsets of ${}^\omega\omega$ to ${}^\omega\omega$.

Let $\{x_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of elements of ${}^\omega\omega$ and $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions from G_δ subsets of ${}^\omega\omega$ to ${}^\omega\omega$. For each $\alpha < \mathfrak{c}$ the set $M_\alpha = \{f_\beta(x_\alpha) : \beta < \alpha, x_\alpha \in \mathrm{dom}(f_\beta)\}$ has cardinality $|\alpha| < \mathfrak{c}$. Hence, we can always find $y \in {}^\omega\omega \setminus M_\alpha$. Let $f(x_\alpha)$ be any such y. This way we obtained a function $f : {}^\omega\omega \to {}^\omega\omega$ such that for all $\beta < \mathfrak{c}$, $\{x \in {}^\omega\omega : f(x) = f_\beta(x)\}$ is a subset of $\{x_\alpha : \alpha \leq \beta\}$. In particular, f agrees with f_β in less than $\mathfrak{c}$ points.

Let $X \subseteq {}^\omega\omega$ be such that $f \restriction X$ is continuous. Then, there is $\beta < \mathfrak{c}$ such that $f \restriction X \subseteq f_\beta$. Since f agrees with f_β in less than $\mathfrak{c}$ points, it follows that $|X| < \mathfrak{c}$. This means that sets $\{X_\alpha : \alpha < \omega_1\}$ from the proof of Theorem 38 cover the whole space ${}^\omega\omega$, but each of them has cardinalty smaller than $\mathfrak{c}$. Hence, $\mathrm{cf}(\mathfrak{c}) \leq \aleph_1$, so in fact $\mathrm{cf}(\mathfrak{c}) = \aleph_1$. □

4. Decompositions of two ccc posets

In this chapter we will look at posets described in [8] and attempt to obtain new consequences of the weaker forms of CH by applying them to these posets. We show that the first of these posets can be decomposed into $\aleph_1$ centered sets in ZFC (hence the weaker forms does not give us any new results there) and the existence of the decomposition of the second one is related to a certain combinatorial statement.

4.1 Finite antichains in pseudotrees

Poset $(T, \leq)$ is a *pseudotree*, if for each $x \in T$ the set $\{y \in T : y \leq x\}$ is linearly ordered by $\leq$. For a given pseudotree T, we will be interested in the poset

$$\mathbb{P}(T) = \{X \in [T]^{<\omega} : x \not\leq y \text{ for all distinct } x, y \in X\}.$$

$\mathbb{P}(T)$ is ordered by reverse inclusion.

The first poset from [8] is of this form. In this section we show that for the posets constructed this way the weaker forms of CH give no new information. That is, we prove that if $\mathbb{P}(T)$ is ccc and has cardinaly at most $\mathfrak{c}$, then it can be always decomposed into $\aleph_1$ centered sets.

Notice that $P \subseteq \mathbb{P}(T)$ is centered iff $\bigcup P$ contains no two distinct comparable elements.

Lemma 40. *Let X be a set of cardinality at most $\mathfrak{c}$. Then there exists a countable system $\mathcal{F} \subseteq {}^{X}2$ such that for all $Y \in [X]^{<\omega}$ and $f : Y \to 2$ there is $F \in \mathcal{F}$ satisfying $F \restriction Y = f$.*

Proof. Without loss of generality assume $X \subseteq {}^{\omega}\omega$. For each $n \in \omega$, $f : {}^{n}2 \to 2$ and $g \in {}^{\omega}\omega$ let $F_{n,f}(g) = f(g \restriction n)$. Let $\mathcal{F}$ consist of all such functions $F_{n,f}$. We claim that $\mathcal{F}$ satisfies the required property.

Choose some $Y \in [X]^{<\omega}$ and $g : Y \to 2$. Let $\{f_0, \ldots, f_{k-1}\}$ be an enumeration of elements of Y. Pick $n \in \omega$ such that $f_0 \restriction n, \ldots, f_{k-1} \restriction n$ are all distinct. Then we can find $h : {}^{n}2 \to 2$ such that for all $i < k$, $h(f_i \restriction n) = g(f_i)$. It follows that $F_{n,h} \restriction Y = g$. □

Lemma 41. *Let T be a pseudotree such that $|T| \leq \mathfrak{c}$ and sizes of all chains in T are bounded by a fixed natural number. Then $\mathbb{P}(T)$ is σ-centered.*

Proof. We proceed by induction on the length of the longest chain in T. For the length 0 (i.e. $T = \emptyset$) the lemma is obvious. Now let A be the set of all minimal elements of T and $B = T \setminus A$. By the induction hypothesis, $\mathbb{P}(B)$ can be decomposed into $\aleph_0$ centered sets. Let $\{Q_n : n \in \omega\}$ be such decomposition. Now we need to decompose $\mathbb{P}(T)$ into $\aleph_0$ centered sets.

Apply Lemma 40 on the set A to get a countable system $\mathcal{F} \subseteq {}^{A}2$. For $a \in A$ let $B_a = \{b \in B : a \leq b\}$. For each $n < \omega$ and $F \in \mathcal{F}$ put

$$T_{n,F} = \{a \in A : F(a) = 0\} \cup \left(\bigcup Q_n \cap \bigcup \{B_a : F(a) = 1\}\right).$$

We show that sets $P_{n,F} = [T_{n,F}]^{<\omega}$ give us a decomposition of $\mathbb{P}(T)$ into centered sets.

1. $P_{n,F}$ is a centered subset of $\mathbb{P}(T)$. For that we need to show that no two distinct elements of $T_{n,F}$ are comparable. Let $a, b \in T_{n,F}$, $a \neq b$. Clearly if $a, b \in A$, then they are incomparable. If $a, b \in T_{n,F} \cap B \subseteq \bigcup Q_n$, then (since Q_n is centered) they are also incomparable. Finally let $a \in A$ and $b \in B$. Since T is a pseudotree, b has a unique predecessor a' in A and to show that a and b are incomparable, we just have to show that $a' \neq a$. However, it follows from the definition of $T_{n,F}$ that $F(a) = 0$ and $F(a') = 1$, so $a' \neq a$.

2. $\bigcup P_{n,F} = \mathbb{P}(T)$. Let $p = \{a_0, \ldots, a_{k-1}, b_0 \ldots, b_{l-1}\} \in \mathbb{P}(T)$ with $a_i \in A$ for $i < k$ and $b_j \in B$ for $j < l$. Since $\{b_0, \ldots, b_{l-1}\} \in \mathbb{P}(B)$, we can find n such that $\{b_0, \ldots, b_{l-1}\} \in Q_n$. For each $j < l$ let a'_j be the unique predecessor of b_j in A. Since $p \in \mathbb{P}(T)$, $a_i \neq a'_j$ for all $i < k$, $j < l$. Now we use the property of $\mathcal{F}$ to find $F \in \mathcal{F}$ such that $F(a_i) = 0$ for all i and $F(a'_j) = 1$ for all j. Then $p \subseteq T_{n,F}$, so $p \in P_{n,F}$. □

Theorem 42. *Let T be a pseudotree of size at most $\mathfrak{c}$. If $\mathbb{P}(T)$ is ccc, then $\mathbb{P}(T)$ can be decomposed into $\aleph_1$ centered sets.*

Proof. By Zorn's lemma let f be a maximal (with respect to inclusion) partial function from T to ω_1 such that:

1. the domain of f is downwards closed (i.e. if $a \in \mathrm{dom}(f)$ and $b \leq a$, then also $b \in \mathrm{dom}(f)$),

2. for each $\alpha \in \omega_1$, no two distinct elements of $f^{-1}(\alpha)$ are comparable.

Assume $x \in T$ and x is not in the domain of f. Since T is a pseudotree, the set $\{\{y\} : y \leq x\}$ is an antichain in $\mathbb{P}(T)$, thus countable, so x has only countably many predecessors. We extend f by defining it at all predecessors of x (including x). The only requirement is that they all need to map into different values, which can be done, because there are only countably many of them. The resulting function will satisfy requirements above, so it contradicts the maximality of f. Hence, f is defined on all of T.

For each $M \in [\omega_1]^{<\omega}$ consider $T_M = f^{-1}[M]$. T_M is a pseudotree, in which sizes of all chains are bounded by a natural number. By Lemma 41, $\mathbb{P}(T_M)$ is σ-centered. Notice that the system of posets $\{\mathbb{P}(T_M) : M \in [\omega_1]^{<\omega}\}$ covers the whole $\mathbb{P}(T)$. This means we decomposed $\mathbb{P}(T)$ into $\aleph_1$ σ-centered posets, which also gives a decomposition into $\aleph_1$ centered sets. □

Remark. If we consider arbitrary posets instead of psuedotrees, the statement is no longer provable. The poset considered in Section 3.1 has the correct form, it is ccc, but existence of its decomposition into $\aleph_1$ linked sets implies $\mathfrak{b} = \aleph_1$, which is not provable in ZFC.

4.2 σ-ideal-independent systems

In this section we are interested in systems with the following property.

Definition 43. *Let X be a set. The system $\mathcal{A} \subseteq \mathcal{P}(X)$ will be called* σ-ideal-independent*, if for each $A \in [\mathcal{A}]^\omega$ and $B \in \mathcal{A} \setminus A$ we have $B \not\subseteq \bigcup A$.*

The problem of decomposition of the second poset from [8] turns out to be related to the following statement:

($*$) There exists a σ-ideal-independent system $\mathcal{A} \subseteq \mathcal{P}(\omega_1)$ such that $|\mathcal{A}| = \mathfrak{c}$.

If the continuum hypothesis holds, then ($*$) is clearly true – simply take $\mathcal{A}$ to contain all singletons of ω_1. ($*$) also holds if $\mathfrak{c} = \aleph_2$, as we show in a moment.

Interesting examples of σ-ideal-independent systems are given by AD systems.

Definition 44. *Let κ be an infinite cardinal. A system $\mathcal{A} \subseteq [\kappa]^\kappa$ is* AD *(shorthand for* almost disjoint*), of for all $A, B \in \mathcal{A}$, $A \neq B \implies |A \cap B| < \kappa$. A system is* MAD *(maximal almost disjoint), if it is AD and also maximal with respect to inclusion.*

By a straghtforward application of Zorn's lemma, every AD system can be expanded into a MAD system.

Lemma 45. *Every AD system on ω_1 is σ-ideal-independent.*

Proof. Let $\mathcal{A} \subseteq \mathcal{P}(\omega_1)$ be AD. Let $A \in [\mathcal{A}]^\omega$ and $B \in \mathcal{A} \setminus A$. Then $B \cap \bigcup A = \bigcup_{C \in A} B \cap C$, which is a countable union of countable sets, thus countable. But $|B| = \omega_1$, so $B \neq B \cap \bigcup A$, therefore $B \not\subseteq \bigcup A$. □

If we talked about system of pairwise disjoint uncountable subsets of ω_1, then clearly any such system has cardinality at most $\aleph_1$. Interestingly, if we relax the disjointness condition by allowing countable intersections (as in the definition of the AD system), the situation changes.

Lemma 46. *There exists an AD system on ω_1 of cardinality $\aleph_2$.*

Proof. Clearly there exists an AD system with cardinality $\aleph_1$ – we can decompose ω_1 into $\aleph_1$ disjoint uncountable sets. Extend this AD system to a MAD system. We show that resulting MAD system has cardinality at least $\aleph_2$.

Assume not, so the MAD system has cardinality $\aleph_1$. Enumerate its elements as $\{A_\alpha : \alpha < \omega_1\}$. By Lemma 45, for each $\alpha < \omega_1$ we can pick $x_\alpha \in A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$. Then $X = \{x_\alpha : \alpha < \omega_1\}$ is an uncountable set such that $X \cap A_\alpha = \{x_\alpha\}$ for all α. Hence X could be added to the MAD system, contradicting its maximality.

Therefore our MAD system has cardinality at least $\aleph_2$. We can take a subsystem having cardinality exactly $\aleph_2$. □

Corollary. $\mathfrak{c} \leq \aleph_2 \implies (*)$

It is consistent with ZFC that there exists an AD system on ω_1 of cardinality $2^{\aleph_1}$, but it is also consistent that $\mathfrak{c} > \aleph_2$ and there is no AD system on ω_1 of cardinality larger than $\aleph_2$. We do not know whether $\neg(*)$ is consistent with ZFC. Clearly, ($*$) is consistent, because it follows from CH.

4.2.1 Todorčević orderings

The following construction is known as Todorčević ordering. It was introduced in [8] and further studied in [1]. The poset $\mathbb{T}(\mathbb{R})$ mentioned in the first chapter is a special case of this construction.

Definition 47. *Let X be a Hausdorff topological space. The* Todorčević ordering $\mathbb{T}(X)$ *consists of all sets $F \subseteq X$, which are a finite union of converging sequences including their limit points. For $F_1, F_2 \in \mathbb{T}(X)$ put $F_1 \leq F_2$ iff $F_1 \supseteq F_2$ and $F_1' \cap F_2 = F_2'$.*

Clearly a subset of $\mathbb{T}(X)$ is linked iff it is centered. Thus CH_2 and CH_ω give the same consequences when applied to Todorčević orderings.

The following two theorems show the relation between $(*)$ and decompositions of posets constructed this way.

Theorem 48. *Let X be a Hausdorff space of size $\mathfrak{c}$. If $(*)$ holds, then $\mathbb{T}(X)$ can be decomposed into $\aleph_1$ centered subsets.*

Proof. Let $\mathcal{A} \subseteq \mathcal{P}(\omega_1)$ be a σ-ideal-independent system of cardinality $\mathfrak{c}$. Let $\{A_x : x \in X\}$ be an enumeration of all elements of $\mathcal{A}$. For each $m \in [\omega_1]^{<\omega}$ let

$$B_m = \{F \in \mathbb{T}(X) : (\forall x \in F)(x \in F' \iff m \cap A_x = \emptyset)\}.$$

We show that $\{B_m : m \in [\omega_1]^{<\omega}\}$ is a decomposition of $\mathbb{T}(X)$ into centered sets.

1. B_m is linked (and thus centered). Let $F, G \in B_m$. It follows from the definition of B_m that $F' \cap (G \setminus G') = (F \setminus F') \cap G' = \emptyset$. Thus, $F \cup G$ is below both F and G.

2. $\bigcup B_m = \mathbb{T}(X)$. Let $F \in \mathbb{T}(X)$. $F \setminus F'$ is countable, so for each $x \in F'$ we can pick $\alpha_x \in A_x \setminus (\bigcup_{y \in F \setminus F'} A_y)$. Put $m = \{\alpha_x : x \in F'\}$, then $F \in B_m$.

□

Theorem 49. *CH_2 implies that either $(*)$ or $\operatorname{cov}(\mathcal{B}) = \aleph_1$ holds.*

Proof. Assume $\operatorname{cov}(\mathcal{B}) > \aleph_1$. We consider the poset $\mathbb{T}(\mathbb{R})$. This poset is ccc by Theorem 6, so by CH_2 it can be decomposed into $\aleph_1$ linked subsets. Let $\{B_\alpha : \alpha < \omega_1\}$ be such decomposition. For $\alpha < \omega_1$ let $B_\alpha' = \bigcup_{F \in B_\alpha} F'$. Let $\pi : \omega_1 \times \mathbb{Q}^2 \to \omega_1$ be an arbitrary injection (remember that $\mathbb{Q}$ denotes the set of all rational numbers). For $x \in \mathbb{R}$ put

$$A_x = \{\pi(\alpha, p, q) : \alpha < \omega_1;\ p, q \in \mathbb{Q};\ p < x < q;\ x \in B_\alpha'\}.$$

We show that $\{A_x : x \in \mathbb{R}\}$ is a σ-ideal-independent system.

Assume by contradiction that there is a sequence $\{x_n\}_{n<\omega}$ of real numbers and $x \in \mathbb{R}$ such that $x \neq x_n$ for each n and $A_x \subseteq \bigcup_{n \in \omega} A_{x_n}$. Fix an increasing sequence $\{p_n\}_{n<\omega}$ and a decreasing sequence $\{q_n\}_{n<\omega}$ of rational numbers such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = x$. Let $S = \{\alpha < \omega_1 : x \in B_\alpha'\}$. S is nonempty, because we can find $F \in \mathbb{T}(\mathbb{R})$ such that $x \in F'$ and this F must be included in some B_α. For each $\alpha \in S$ and $n \in \omega$ we have $\pi(\alpha, p_n, q_n) \in A_x$, so there is m such that $\pi(\alpha, p_n, q_n) \in A_{x_m}$. Let $f_\alpha(n)$ be an arbitrary such m. Now $\{f_\alpha : \alpha \in S\}$ is a system of functions from ω to ω of size at most $\aleph_1$. Since $\mathrm{cov}(\mathcal{B}) > \aleph_1$, we can use Lemma 19 to find $f \in {}^\omega\omega$ such that for each $\alpha \in S$ we have $f(n) = f_\alpha(n)$ for some $n \in \omega$. We will also require the f to be such that for each $n \in \omega$ there is $\alpha \in S$ such that $f(n) = f_\alpha(n)$ (this can be done, because S is nonempty).

Let $n \in \omega$. Since $f(n) = f_\alpha(n)$ for some α, we have $\pi(\alpha, p_n, q_n) \in A_{x_{f(n)}}$, so $p_n < x_{f(n)} < q_n$. Hence, $\lim_{n\to\infty} x_{f(n)} = x$ and $F = \{x_{f(n)}\}_{n\in\omega} \cup \{x\}$ is an element of $\mathbb{T}(\mathbb{R})$. Let α be such that $F \in B_\alpha$ and let n be such that $f(n) = f_\alpha(n)$. From the definition of f_α, $\pi(\alpha, p_n, q_n) \in A_{x_{f(n)}}$. Therefore there is $G \in B_\alpha$ such that $x_{f(n)} \in G'$. However, $F \in B_\alpha$ and $x_{f(n)} \in F \setminus F'$, so F is incompatible with G and B_α is not linked, a contradiction. $\square$

Conclusion

In this thesis we introduced the reader to the weaker forms of the continuum hypothesis. In the first two chapters we described the basic theory of ccc posets and cardinal characteristics of the continuum, which was required to present the results related to the weaker forms of CH. In the third chapter we gave detailed exposition of proofs of consequences of CH_n and CH_ω regarding cardinal characteristics, which were described in [7]. In the fourth chapter we attempted to obtain new results by considering two posets from [8]. While the first one was shown to not lead to any new results, the second one leads to the principle $(*)$ which might be of independent interest.

The result of Section 4.1 gives us a class of posets for which the statements of CH_n and CH_ω are provably true. One could search for other classes of posets for which ccc already implies existence of the decomposition. In Section 4.2 we introduced the statement $(*)$ and showed $CH_2 \implies (*) \vee \mathrm{cov}(\mathcal{B}) = \aleph_1$. It would be interesting to know whether $\neg(*)$ is consistent with ZFC and whether the result of Theorem 49 can not be strengthened to either $CH_2 \implies (*)$ or $CH_2 \implies \mathrm{cov}(\mathcal{B}) = \aleph_1$. All results mentioned in this thesis are related to either CH_2, CH_3 or CH_ω – interesting consequences of CH_n for $n \geq 4$ are still waiting to be discovered.

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