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Envelopes of implicit surfaces

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Title: Envelopes of implicit surfaces

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Abstract: The aim of the thesis is to study envelopes and characteristic curves of one parameter systems of quadratic surfaces in real three dimensional space. We define one parameter systems and their envelopes generally and present algebraic geometry approach for envelope computation using Gröbner bases and elimination theory. We convey a proof of rationality of envelopes of rational one parameter systems of spheres, cones and cylinders of revolution using dual space and different models of Laguerre geometry. Then we present a new approach to one parameter systems and their envelopes. We introduce the systems as curves in homogeneous spaces which allows us to study all characteristic curves at a single surface. This approach allows us to prove rationality and even provide an explicit parameterization of characteristic curves and the envelope of a one parameter system of isometric cones of revolution. We provide several other examples illustrating the concepts and results.

Keywords: envelope surfaces, characteristic curves, Lie groups, rational parameterization

Contents

Introduction

In this thesis we study envelopes of one parameter systems of surfaces. Primarily we are interested in systems of quadratic surfaces in real three dimensional space that depend rationally on a parameter t. Intuitively, an envelope of such system is a *silhouette* of the system, a surface that is tangent to every element of the system along a *characteristic curve*.

One class of one parameter systems of quadrics are systems of spheres with either constant or non-constant radii. Envelopes of these systems are called pipe and canal surfaces, respectively. It was shown in [1] that if the centres of the spheres lie on a rational curve and the radius is constant, the pipe surface defined as the envelope of the system is rational. This result was later generalised to canal surfaces, see e. g. [2]. Both the proofs are constructive and introduce a general way of computing a parametrization of the envelope surface.

Another interesting class of one parameter systems of surfaces are the motions of a surface moving through space. Such one parameter systems of planes, spheres and cylinders of revolution and their characteristic curves are described quite cumbersomely in [3]. Some notes about systems of cones are also given. Envelopes of systems of moving cylinders and cones of revolution find motivation in CNC machining. A truncated cone or cylinder may represent a machining tool and the set of all its positions while it moves along a trajectory defines a one parameter system. There are many publications addressing the problem, e. g. [4], [5] or [6].

The thesis aims to properly define and describe envelopes and characteristic curves of one parameter systems of quadratic surfaces, illustrate known methods by examples. Furthermore, we present a new efficient way to compute a parameterization of envelopes which exploits homogenity of a one parameter system and linearity of derivative. The method allows us to prove rationality of envelope surfaces of one parameter systems of isometric cones of revolution.

The thesis is organised as follows. In the first chapter, we introduce one parameter systems of surfaces, specially one parameter systems of quadrics given by an implicit equation in the real three dimensional space. We provide several examples, mostly in three, but also in two dimensions, where the visualisation of the systems is more illustrative. We define envelope surfaces of such systems, prove a well known characterisation and demonstrate the resulting method of computation. Then we take an algebraic geometry approach and use Gröbner bases, elimination theory and resultants to compute envelopes. However, these methods have several drawbacks; not only is it computationally difficult, but also we can only obtain the implicit equation of the envelope. If we are interested in a segment of the envelope, we need to find its parameterization.

The second chapter presents two proofs of rationality of envelopes of one parameter systems of isometric cones of revolution. First of them, published in [6], computes a parametrization of an envelope in dual space and transfers it back to $\mathbb{R}^3$. We apply the construction from the proof to a specific one parameter system. The second proof, published in [7], uses Laguerre geometry and some of its models. The proof shows that a rational parameterization of the envelope surface exists, and provides a symbolic parameterization in one of the models of

Laguerre geometry. We present Laguerre geometry formally in details and convey the proof. We apply the method to an example to illustrate the process.

In the last chapter we offer a kinematic approach to one parameter systems of surfaces. We describe one parameter systems of implicit surfaces as images of a fixed surface under a mapping g, where g is an element of a certain Lie group. The Lie group acts transitively on a suitable set of surfaces, among which the one parameter system is included. This concept lets us express the one parameter system as a curve in the Lie group. Similar concepts are used e. g. in [8]. We present the necessary theory and prove a theorem depicting characteristic curves of a one parameter system as images of curves lying on a fixed surface. As a result, we prove that a one parameter system of isometric cones of revolution possesses a rational envelope and we provide explicit parameterization of its characteristic curves. We provide examples of envelopes of other one parameter systems defined by various Lie groups.

The main contributions of the thesis are a clearer definition of an envelope surface and a new and efficient method of parameterization of envelopes, which is also used to find a new proof of the fact that the envelope of a one parameter system of isometric cones of revolution is rational. Furthermore we build the theory necessary to understand the dual proofs from the second chapter and provide clarifying examples to make the proofs more accessible. We provide original and illustrative examples of the used concepts throughout the whole thesis.

All figures in the thesis were created in Wolfram Mathematica, [9].

1. Envelopes of one parameter systems of surfaces

1.1 One parameter systems

In this thesis, we are interested in envelopes of one parameter systems of quadratic surfaces in real three dimensional space. From time to time we illustrate concepts used in the thesis by planar examples. Therefore let us define one parameter systems of surfaces given by implicit equations in a real space of general dimension n.

Definition 1. *Let $f(x_1, x_2, \dots, x_n, t) : \mathbb{R}^n \times I \to \mathbb{R}$ be function polynomial in $x_1, x_2, \dots, x_n$ and rational in t, where $I \subseteq \mathbb{R}$ is an interval.. We define one parameter system of surfaces as the system of sets*

$$\mathcal{F}_t = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n,\ f(x_1, x_2, \dots, x_n, t) \ = \ 0,\ t \in I\},$$

Primarily we are interested in one parameter systems of quadratic surfaces in $\mathbb{R}^3$ and we denote the variables x_1, x_2, x_3 simply by x, y, z. The function f from the definition above can be then written as

$$f(x,y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz + a_{14}x + a_{24}y + a_{34}z + a_{44} = 0,$$

or in the form

$$f(x,y,z) = (x\ y\ z\ 1)\mathbf{A}(x\ y\ z\ 1)^T = 0,$$

where the $\mathbf{A} \in \mathbb{R}^{4\times 4}$ is a symmetric matrix given as

$$\mathbf{A} = \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} & \frac{1}{2}a_{14} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} & \frac{1}{2}a_{24} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} & \frac{1}{2}a_{34} \\ \frac{1}{2}a_{14} & \frac{1}{2}a_{24} & \frac{1}{2}a_{34} & a_{44} \end{pmatrix}.$$

The entries of the matrix $\mathbf{A}$ depend rationally on the parameter t, in other words, $a_{11}, \dots, a_{44} \in \mathbb{R}(t)$ are rational functions in one variable t.

Remark. We refer to an element of a one parameter system $\mathcal{F}_t$ corresponding to $t_0 \in I$ as to $F_{t_0} \in \mathcal{F}_t$, $F_{t_0} = \{(x,y,z) \in \mathbb{R}^3, f(x,y,z,t_0) = 0\}$. Hence F_{t_0} is a set of all points in $\mathbb{R}^3$ satisfying a polynomial equation.

We refer to an element of $\mathcal{F}$ just as to $F \in \mathcal{F}_t$ if we are not interested in any particular value of the parameter t.

Let us first present a one parameter system of lines in plane.

Example 2. Let $\mathcal{F}_t$ be a one parameter system of lines in plane defined as follows:

$$\mathcal{F}_t = \{(x,y) \in \mathbb{R}^2, f(x,y,t) = 0,\ t \in \mathbb{R}\},$$

where

$$f(x,y,t) = \frac{2(t+1)}{t^2+2t+2}x - \frac{t(t+2)}{t^2+2t+2}y - 1 = 0.$$

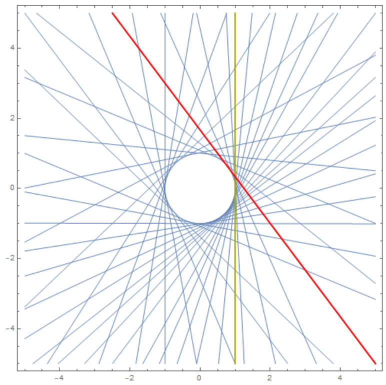

Figure 1.1: Illustration of a one parameter system of lines $\mathcal{F}_t$. Elements F_0 and $F_{-\frac{1}{2}}$ are highlighted by yellow and red respectively.

See Figure 1.1 for an illustration of the system. We illustrate the system (and any other one parameter system later in the text) by displaying some elements of the system for discrete values of the parameter t.

For $t = 0$ the corresponding $F_0 \in \mathcal{F}_t$ is a line with implicit equation

$$f(x, y, z, 0) = x - 1.$$

Any other element of the system can be seen as the line F_0 rotated by an angle $\alpha \in (-\pi, \pi]$ around the origin. If we parameterize the rotation rationally for any angle and plug it into the equation $f(x, y, z, 0)$, we obtain the general implicit equation $f(x, y, z, t)$ of the system. For instance, for $t = -\frac{1}{2}$ we have

$$f(x, y, z, -\frac{1}{2}) = \frac{1}{5}(4x + 3y - 5).$$

We did not define an envelope of a one parameter system, but in this case, intuitively, it is the unit circle centred at the origin.

We use this kinematic concept to describe one parameter systems and their envelopes later in the last chapter. For the time being we think of the system only in terms of their algebraic properties given by the implicit equations.

Let us next introduce an example that we use throughout the whole thesis to illustrate different methods and approaches to parameterization of envelopes.

Example 3. Let $\mathcal{F}_t$ be a one parameter system of quadrics defined as follows:

$$\mathcal{F}_t = \{(x, y, z) \in \mathbb{R}^3, f(x, y, z, t) = 0,\ t \in \mathbb{R}\},$$

where

$$
\begin{aligned}
f(x,y,z,t) = & \frac{1}{100\left(1108t^2+788t+517\right)^2} \\
& (x^2(107430976t^4+233406592t^3+157385824t^2+10446368t+4338116) \\
& + y^2(21322564+68579872t+186419296t^2+195440768t^3+108682304t^4) \\
& + z^2(26727964+80672992t+2451616t^2-86590592t^3+24508864t^4) \\
& + xy(22004736+61155328t-12087296t^2-78061568t^3+29392896t^4) \\
& - xz(289536+125152768t+291359744t^2+45606912t^3-77635584t^4) \\
& + yz(142272+61441536t+118962688t^2-43902976t^3-74400768t^4) \\
& - x(23582412-96000024t-264336032t^2+21383744t^3+705373632t^4 \\
& + 477161600t^5) \\
& - \ y(54543076+317551848t+729372064t^2+730133312t^3+222281536t^4 \\
& + 23555200t^5) \\
& - \ z(79820724+245231112t+50123296t^2-507820992t^3-456273856t^4 \\
& + 150288000t^5) \\
& + 91188121+350105108t+500306044t^2+125408352t^3-143577744t^4 \\
& + 255892800t^5+475790400t^6) = 0.
\end{aligned}
$$

For each $t \in \mathbb{R}$ the equation $f(x,y,z,t)$ defines an infinite cone. The cones are isometric and the angle between their axes and any line on the cone (*a generator*) is $\arctan(\frac{1}{5})$. The system is illustrated in Figure 1.2 for several cones F_t, $t \in [-2, \frac{2}{5}]$.

For $t = -\frac{4}{5}$ the cone $F_{-\frac{4}{5}}$ is defined by an equation

$$
\begin{aligned}
f(x,y,z,-\frac{4}{5}) = & \frac{1}{22180144900}\big(13254688516x^2+10844660736xy \\
& -19679552256xz-33014762732x+18886008964y^2 \\
& +11955582912yz+14500367484y+11332386524z^2 \\
& +37760657196z+30722517817)
\end{aligned}
\tag{1.1}
$$

The cone $F_{-\frac{4}{5}}$ is highlighted in red in Figure 1.2.

1.2 Envelopes of one parameter systems

In this section, we define the envelope surface of a one parameter system of surfaces. We prove a characterisation of envelopes which can be used for computing them for simple one parameter systems.

In literature the definition of an envelope surface is usually not rigorously stated and only its well known characterisation is used (see e.g. [2], [3]). Although even for us the characterisation is sufficient, let us state the precise definition of an envelope of a one parameter system of surfaces in $\mathbb{R}^3$. This definition is a generalisation of the definition of a *canal surface*, i.e. the envelope of a one parameter systems of spheres, which can be found in [10].

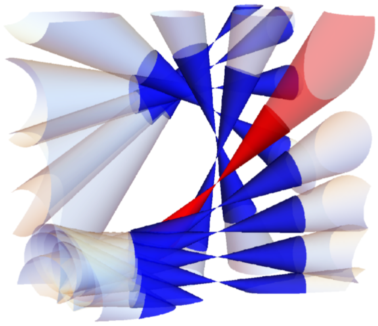

Figure 1.2: Illustration of the one parameter system of cones $\mathcal{F}_t$. Trimmed cones are highlighted for clarity. The cone $F_{-\frac{4}{5}}$ is highlighted in red.

Definition 4. *Let* $\mathcal{F}_t = \{(x, y, z) \in \mathbb{R}^3, f(x, y, z, t) = 0, t \in I\}$ *be a one parameter system. Consider the set*

$$M = \{(x, y, z, t) \in \mathbb{R}^3 \times I, f(x, y, z, t) = 0\} \subseteq \mathbb{R}^4$$

with the natural projection $\pi : M \to \mathbb{R}^3 : (x, y, z, t) \mapsto (x, y, z)$.

We define the envelope χ *of the system* $\mathcal{F}_t$ *as the set of all critical values of* π.

Then we can define *characteristic curves* of a one parameter system.

Definition 5. *Let* $\mathcal{F}_t$ *be a one parameter system of surfaces. Then the* characteristic curves *are the sets*

$$\chi_t = \{(x, y, z) \in \mathbb{R}^3; f(x, y, z, t) = 0 \wedge \frac{\partial f}{\partial t}(x, y, z, t) = 0\}$$

From Definition 4 we can prove a theorem that characterises envelope surfaces.

Theorem 6. *Let* $\mathcal{F}_t$ *be a one parameter system of surfaces. Its envelope surfaces is the union of all characteristic curves,* $\chi = \cup_t \chi_t$.

Proof. First, a point $(x_0, y_0, z_0) \in \mathbb{R}^3$ is a critical value of π if and only if there exist $t_0 \in I$ such that $f(x_0, y_0, z_0, t_0) = 0$ and $\frac{\partial f}{\partial t}(x_0, y_0, z_0, t_0) = 0$.

Let $(x_0, y_0, z_0) \in \cup_t \chi_t$. Then there is a $t_0 \in I$ such that $f(x_0, y_0, z_0, t_0) = 0$ and $\frac{\partial f}{\partial t}(x_0, y_0, z_0, t_0) = 0$.

Hence (x_0, y_0, z_0) is a critical value of π because $\frac{\partial f}{\partial t}(x_0, y_0, z_0, t_0) = 0$.

On the other hand, let $(x_0, y_0, z_0) \in \chi$. Then there exists $t_0 \in I$ such that $f(x_0, y_0, z_0, t_0) = 0$ and $\frac{\partial f}{\partial t}(x, y, z, t_0) = 0$, which concludes the proof. □

Remark. An envelope can be similarly defined for one parameter systems in real space with lower or higher dimension then three.

An envelope is thus characterised by two equations, generally rational in the parameter t. The following lemma allows us to simplify the situation.

Lemma 7. *Let* $\lambda(t) \neq 0$ *be a function in* t, *then sets* χ_t *and* χ_t^λ *of one-parameter systems of surfaces given by implicit equations* $f(x, y, z, t) = 0$ *and* $f^\lambda(x, y, z, t) = \lambda(t) f(x, y, z, t) = 0$, *are the same.*

Proof. From Theorem 6 we see that characteristic curves of the system

$$\begin{aligned}\mathcal{F}_t^\lambda &= \{(x,y,z) \in \mathbb{R}^3;\ f^\lambda(x,y,z,t) = 0\} \\ &= \{(x,y,z) \in \mathbb{R}^3;\ \lambda(t) f(x,y,z,t) = 0\}\end{aligned}$$

are the sets

$$\chi_t^\lambda = \{(x,y,z) \in \mathbb{R}^3; f^\lambda(x,y,z,t) = 0 \wedge \frac{\partial f^\lambda}{\partial t}(x,y,z,t) = 0\}$$

where

$$\frac{\partial f^\lambda}{\partial t}(x,y,z,t) = 0 \iff \lambda'(t) f(x,y,z,t) + \lambda(t) \frac{\partial f}{\partial t}(x,y,z,t) = 0,$$

and since $\lambda(t) \neq 0$, we obtain $f^\lambda(x,y,z,t) = 0 \wedge \frac{\partial f^\lambda}{\partial t}(x,y,z,t) = 0$ if and only if $f(x,y,z,t) = 0$ and $\frac{\partial f}{\partial t}(x,y,z,t) = 0$. Points satisfying these equations are exactly the points of χ_t.

Thus the sets of characteristic curves are equal and hence the envelopes are the same. □

The Lemma 7 clearly holds for one parameter systems in spaces with different number of variables.

Hereafter, we denote $f^\lambda = \lambda(t) f(x,y,z,t)$ and $f_t^\lambda = \frac{\partial f^\lambda}{\partial t}(x,y,z,t)$.

Let us illustrate the computation of an envelope using Theorem 6 by several simple examples.

Example 8. Let us begin with a one parameter system $\mathcal{F}_t$ of curves in the real plane $\mathcal{F}_t = \{(x,y) \in \mathbb{R}^2,\ f(x,y,t) = 0,\ t \in \mathbb{R}\}$, where

$$f(x,y,t) = 2(x+1)\left(\frac{ty}{t^2+1} + 1\right) + (x+1)^2 - 1.$$

See Figure 1.3 for an illustration of the system.

From the characterisation of an envelope (6), we can compute each characteristic curve χ_t of the system as the solution of the system of equations

$$\begin{aligned} f(x,y,t) &= 0, \\ \frac{\partial f}{\partial t}(x,y,t) &= -\frac{2\left(t^2-1\right)(x+1)y}{\left(t^2+1\right)^2} = 0.\end{aligned}$$

The envelope of the system can be then obtained from the two equations by eliminating the variable t.

Using Lemma 7, we multiply the first equation by a non-zero polynomial $\lambda(t) = (t^2+1)$ to clear out its denominator. The envelope is then characterised by the following equations,

$$\begin{aligned} f^\lambda &= t^2x^2 + 4t^2x + 2t^2 + 2txy + 2ty + x^2 + 4x + 2 = 0, \\ f_t^\lambda &= 2tx^2 + 8tx + 4t + 2xy + 2y = 0.\end{aligned}$$

The second equation is of degree 1 in t, hence it can be easily solved and we obtain $t = \frac{-xy-y}{x^2+4x+2}$. Plugging the value into the equation $f^\lambda = 0$, we get the implicit equation of the envelope,

$$\chi = -\frac{(x+1)^2y^2}{x(x+4)+2} + x(x+4) + 2.$$

The envelope is highlighted in Figure 1.3.

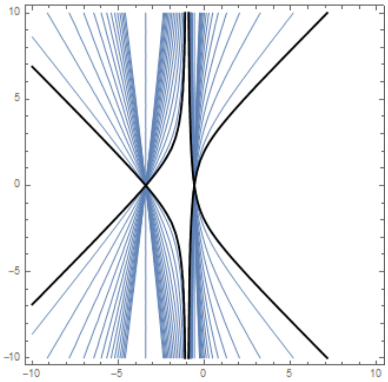

Figure 1.3: Blue curves represents some elements of the one parameter system $\mathcal{F}_t$. The black curve is the envelope χ of the system.

Example 9. Consider a one parameter system $\mathcal{F}_t$ of spheres with non-constant radii,

$$\mathcal{F}_t = \{(x,y,z) \in \mathbb{R}^3,\ x^2+y^2+(z-t)^2 - \frac{t^2}{26} = 0,\ t \in \mathbb{R}\}.$$

The radii of the spheres go to zero as t approaches zero. For $t = 0$, the element $F_0 \in \mathcal{F}$ is one point $[0,0,0]$. We can think of this point as of a sphere with zero radius. See Figure 1.4.

The envelope of this system is then determined by the following equations,

$$\begin{aligned} f(x,y,z,t) &= x^2+y^2+(z-t)^2 - \frac{t^2}{26} = 0, \\ \frac{\partial f}{\partial t}(x,y,z,t) &= -2(z-t) - \frac{t}{13} = 0. \end{aligned}$$

From the second one, we get $t = \frac{26}{25}z$. If we plug it into the first equation, we obtain an implicit equation of the envelope surface, $\chi : x^2+y^2-\frac{1}{25}z^2 = 0$. It is a cone of revolution with the angle between its axis and any generator equal to $\arctan(\frac{1}{5})$.

For example for $t = \frac{7}{4} \in I$, the corresponding characteristic curve is intersection of two surfaces defined by the equations

$$f(x,y,z,\frac{7}{4}) = x^2+y^2+\left(z+\frac{7}{4}\right)^2 - \frac{49}{416} = 0$$

and

$$\frac{\partial f}{\partial t}(x,y,z,\frac{7}{4}) = 2z + \frac{175}{52} = 0.$$

We can conclude that the characteristic curve $\chi_{\frac{7}{4}}$ is a circle centred at the point $(0,0,\bar{z})$ in the plane $z = \bar{z} = -\frac{175}{104}$. See Figure 1.4 for an illustration of the system, its envelope, the sphere $F_{\frac{7}{4}}$ and the plane given by the implicit equation $\frac{\partial f}{\partial t}(x,y,z,\frac{7}{4}) = 0$.

In fact, for any $t \in I$ the characteristic curve is a circle with centre lying on the $z-$axis in the plane given by the equation $\frac{\partial f}{\partial t}(x,y,z,t) = 0$.

Unfortunately, in most cases, the equations that characterise the envelopes are too complicated and we are not able to derive the equation of the envelope from them.

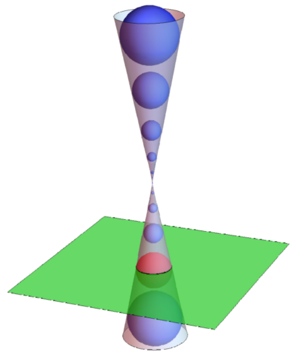

Figure 1.4: A cone as the envelope of a one parameter system of spheres. A characteristic curve (black) as the intersection of two surfaces (red and green).

Example 10. Consider the one parameter system of cones from Example 3. As f^λ we take the numerator of the equation f. Then, the partial derivative of f^λ with respect to t, written as a polynomial in t with coefficients in $\mathbb{R}[x, y, z]$ is as follows,

$$
\begin{aligned}
&\frac{\partial f}{\partial t}(x, y, z, t) = \\
&+ t^5\Big(2854742400\Big) \\
&+ t^4\Big(-2385808000x - 117776000y - 751440000z + 1279464000\Big) \\
&+ t^3\Big(429723904x^2 + 117571584xy + 310542336xz - 2821494528x \\
&\qquad + 434729216y^2 - 297603072yz - 889126144y + 98035456z^2 \\
&\qquad + 1825095424z - 574310976\Big) \\
&+ t^2\Big(700219776x^2 - 234184704xy - 136820736xz - 64151232x \\
&\qquad + 586322304y^2 - 131708928yz - 2190399936y - 259771776z^2 \\
&\qquad + 1523462976z + 376225056\Big) \\
&+ t\Big(314771648x^2 - 24174592xy - 582719488xz + 528672064x \\
&\qquad + 372838592y^2 + 237925376yz - 1458744128y + 4903232z^2 \\
&\qquad - 100246592z + 1000612088\Big) \\
&+ x^2 + 61155328xy - 125152768xz + 96000024x + 68579872y^2 \\
&+ 61441536yz - 317551848y + 80672992z^2 - 245231112z + 350105108
\end{aligned}
$$

We have two polynomials of degrees 6 and 5, respectively, and in this case, we cannot compute the envelope just by eliminating the variable t from the equations.

Let us at least briefly discuss the characteristic curve for some choice of $t \in I$, for instance for $t = -\frac{4}{5}$. The cone F_t is defined by the equation 1.1 and its partial derivative with respect to t is the following:

$$\frac{\partial f}{\partial t}(x,y,z,-\frac{4}{5}) = \frac{1}{625}\big(13254688516x^2 + 10844660736xy - 19679552256xz$$
$$-33014762732x + 18886008964y^2 + 11955582912yz$$
$$+14500367484y + 11332386524z^2 + 37760657196z$$
$$+30722517817\big)\,. \tag{1.2}$$

Both equations are quadratic, thus their intersection is hence an algebraic curve of degree four. Later we show that in this case, the intersection is a rational algebraic curve, but, in general situation of two intersecting quadratic surfaces, it does not have to be the case.

In Figure 1.5 the cone $F_{-\frac{4}{5}}$ and the surface defined by the equation 1.2 are shown.

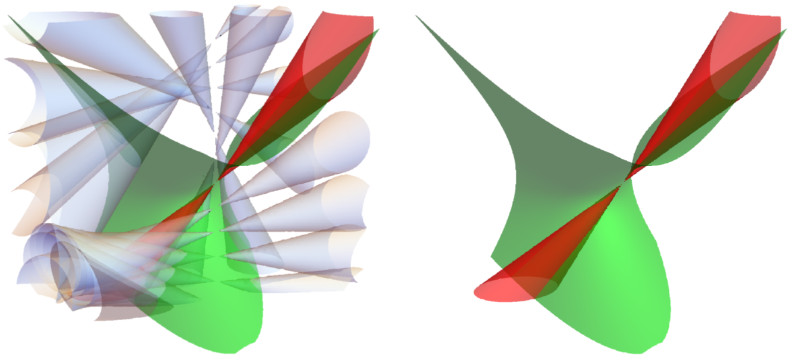

Figure 1.5: One parameter system $\mathcal{F}_t$, a cone $F_{-\frac{4}{5}}$ (red) and a surface defined by the equation $\frac{\partial f}{\partial t}(x,y,z,-\frac{4}{5}) = 0$ (green). Their intersection is the characteristic curve of $\mathcal{F}_t$ at $t = -\frac{4}{5}$.

1.3 Algebraic geometry approach

The characterisation 6 of an envelope surfaces describes characteristic curves as all the points satisfying a pair of rational equations in a parameter t. From Lemma 7 we know that we can multiply the implicit equation of the system by a suitable non-zero function in t and then differentiate it to gen two polynomial equations. Let us briefly recall some basic notions of algebraic geometry. For more details, we refer to any textbook on the topic, for instance [11] and [12]. Since we are interested in envelopes of one parameter systems in the real space, let us define the notions only for the field of real numbers, however, the definitions and theorems hold for any field. Later, when we need an algebraically closed field, we work over $\mathbb{C}$.

In our case, a one parameter system in three dimensional space is defined by an equation, that is polynomial in the variables x, y, z and rational in the parameter t. From the characterisation 6 and the Lemma 7 it is clear that we can compute the envelope by solving a system of two polynomial equations in variables x, y, z, t, since the denominators, which are functions in t, can be cleared out.

If we fix a $t_0 \in I$, the characteristic curve χ_{t_0} at $t_0 \in I$ is the intersection of two affine varieties defined by polynomials $f^\lambda = \lambda(t) f(x, y, z, t_0) = 0$ and $f_t^\lambda = \frac{\partial f^\lambda}{\partial t}(x, y, z, t_0) = 0$, where $\lambda(t)$ is a suitable non-zero polynomial in t,

$$\begin{aligned}\chi_{t_0} &= \mathrm{V}\left(f^\lambda(x, y, z, t_0)\right) \cap \mathrm{V}\left(f_t^\lambda(x, y, z, t_0)\right) \\ &= \mathrm{V}\left(f^\lambda(x, y, z, t_0), f_t^\lambda(x, y, z, t_0)\right).\end{aligned}$$

The whole envelope surface is then the affine variety defined by the two polynomials in x, y, z and t, where t is eliminated.

$$\chi = \mathrm{V}\left(f^\lambda(x, y, z, t_0), f_t^\lambda(x, y, z, t_0)\right) \cap \mathbb{R}^3.$$

Example 11. Consider a one parameter system

$$\mathcal{F}_t = \{(x, y) \in \mathbb{R}^2,\ F(x, y, t) = 0,\ t \in \mathbb{R}\},$$

where

$$f(x, y, t) = \frac{x^2}{(t^2+1)^2} + (y - 2t)^2 - 1.$$

Some elements of the system (together with the envelope) are pictured in Figure 1.6. From the characterisation 6 and the Lemma 7 we know we can compute the envelope of this system from a suitable multiple of the equations

$$\begin{aligned} f(x, y, t) &= 0 \\ \frac{\partial f}{\partial t}(x, y, t) &= t\left(8 - \frac{4x^2}{(t^2+1)^3}\right) - 4y = 0. \end{aligned}$$

By multiplying the first equation by $(1 + t^2)^2$ and differentiating it, we obtain two polynomial equations

$$\begin{aligned} f^\lambda &= 4t^6 - 4t^5 y + t^4 y^2 + 7t^4 - 8t^3 y + 2t^2 y^2 + 2t^2 - 4ty + x^2 + y^2 - 1 = 0, \\ f_t^\lambda &= 24t^5 - 20t^4 y + 4t^3 y^2 + 28t^3 - 24t^2 y + 4ty^2 + 4t - 4y = 0, \end{aligned}$$

and we can then describe the envelope as

$$\chi = \mathrm{V}(f^\lambda, f_t^\lambda) \cap \mathbb{R}^2.$$

1.3.1 Gröbner bases and elimination theory

Even for a simple example as 11, both the polynomial equations characterising the envelope are of a high degree in t. Hence it is difficult to eliminate the parameter from them without an appropriate tool. A standard apparatus for this task are *Gröbner bases.*

To introduce this notion, let us first define an ideal and the ideal of an affine variety.

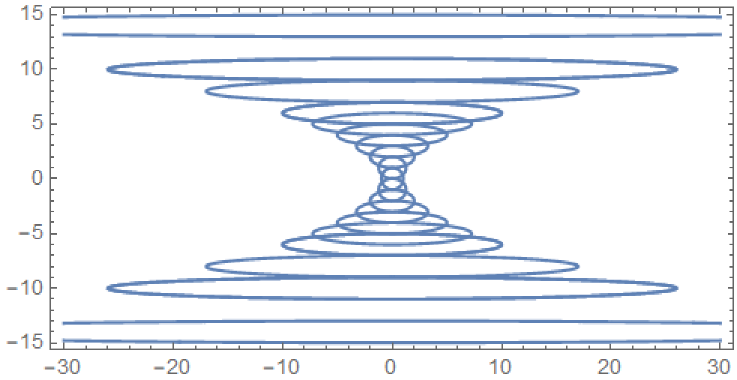

Figure 1.6: One parameter system of ellipses in plane.

Remark. Let $f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n]$ by polynomials, by $\langle f_1, \ldots, f_r \rangle$ we denote the ideal defined by $f_1, \ldots, f_r$,

$$\langle f_1, \ldots, f_r \rangle = \{\sum_{i=1}^{m} p_i f_{j_i},\ p_i \in \mathbb{R}[x_1, \ldots, x_n], j_i \in \{1, \ldots, r\},\ m \in \mathbb{N}\}.$$

Definition 12. *Let $V \subset \mathbb{R}^n$ be an affine variety. We define the* ideal of the variety V *as*

$$I(V) = \{f \in \mathbb{R}[x_1, \ldots, x_n],\ f(a_1, \ldots, a_n) = 0\ \forall (a_1, \ldots, a_n) \in V\}.$$

The fact that an ideal of an affine variety is an ideal is a simple observation.

Theorem 13. *The ideal of an affine variety $V \subset \mathbb{R}^n$ is an ideal in $\mathbb{R}[x_1, \ldots, x_n]$.*

A generating set of an ideal is called a *basis* of the ideal. The following theorem is one of the most important facts about ideals.

Theorem 14 ([12, Ch. 2, Thm. 4], Hilbert Basis Theorem)**.** *Every ideal in $\mathbb{R}[x_1, \ldots, x_n]$ is finitely generated, i.e., for an ideal $I \subset \mathbb{R}[x_1, \ldots, x_n]$ there exist a finite collection of polynomials $\{f_1, \ldots, f_l\} \subset \mathbb{R}[x_1, \ldots, x_n]$ such that any polynomial $g \in I$ can by generated by this set.*

There are many different bases of an ideal, among them the *Gröbner bases* are one of those with a great importance. A Gröbner basis depends on a *monomial ordering*.

Definition 15. *A monomial order of $\mathbb{R}[x_1, \ldots, x_n]$ is any relation $<$ on the set of monomials x^α in $\mathbb{R}[x_1, \ldots, x_n]$ satisfying*

1. *$>$ is a total ordering,*
2. *$>$ is compatible with multiplication in $\mathbb{R}[x_1, \ldots, x_n]$, that is, if $x^\alpha > x^\beta$ and x^γ is a monomial, then $x^\alpha x^\gamma > x^\beta x^\gamma$,*
3. *$>$ is a well ordering.*

Example 16. An example of a monomial order on $\mathbb{R}[x_1, \dots, x_n]$ is the *lexicographic order*. It is defined as follows. Let x^α, x^β be monomials in $\mathbb{R}[x_1, \dots, x_n]$. Let $x^\alpha = x_1^{u_1} \dots x_n^{u_n}, x^\beta = x_1^{v_1} \dots x_n^{v_n}$. We say $x^\alpha >_{lex} x^\beta$, if there exist some $i \in \{1, \dots, n\}$ such that $u_1 = v_1, \dots, u_{i-1} = v_{i-1}$ and $u_i > v_i$.

The ordering compares the monomials by their degree in x_1 and then breaks ties by the degree in x_2, x_3, etc.

An example in $\mathbb{R}[x, y, z]$ is then $x >_{lex} y^3z^2 >_{lex} y^2z^3 >_{lex} z^{10}$.

Another example is the *graded reverse lexicographic order*, or shortly *grevlex order*, which compares the monomials by the total degree and breaks ties by the smallest degree in x_n, x_{n-1} etc.

Following the example above, it holds $z^{10} >_{grevlex} y^3z^2 >_{grevlex} y^2z^3 >_{grevlex} x$.

Given a polynomial and a monomial ordering, we define a *leading monomial* and a *leading coefficient* of the polynomial.

Definition 17. *Let f be a polynomial in $\mathbb{R}[x_1, \dots, x_n]$. The* leading term *of f with respect to a given monomial ordering $<$ is the term $c_\alpha x^\alpha$, where x^α is the largest monomial in f with respect to $<$. We denote if by $lt(f)$. The coefficient c_α is called the* leading coefficient *and x^α the* leading monomial *of f.*

Now we can finally define a Gröbner basis of an ideal.

Definition 18. *Let $>$ be a monomial order on $\mathbb{R}[x_1, \dots, x_n]$, let $I \subset \mathbb{R}[x_1, \dots, x_n]$ be an ideal. A* Gröbner basis *for I (with respect to $<$) is a finite set of polynomials $\{g_1, \dots, g_l\} \subset I$ satisfying the following:*

$$\forall f \in I \ \exists i \in \{1, \dots, l\}, \text{such that } lt(g_i) | lt(f).$$

Even for a given monomial order, a Gröbner basis for an ideal is not defined uniquely. However, assuming more conditions on the Gröbner basis yields uniqueness.

Definition 19. *A* reduced Gröbner basis *for an ideal $I \subset \mathbb{R}[x_1, \dots, x_n]$ is a Gröbner basis G for I such that for all $f, g \in I$, $f \neq g$, no monomial appearing in g is a multiple of $lt(f)$.*

A monic Gröbner basis *is a reduced Gröbner basis in which the leading coefficient of every polynomial equals to one, or it is $\emptyset$, if $I = 0$.*

Then the following statement holds true.

Theorem 20. *[12, Ch. 2, Thm. 5] Let $>$ be a monomial order on $\mathbb{R}[x_1, \dots, x_n]$, then any ideal I in $\mathbb{R}[x_1, \dots, x_n]$ has a unique monic Gröbner basis with respect to $>$.*

Example 21. Different monomial orders usually lead to very different Gröbner bases. Let I be generated by the polynomials $f_1 : x^2 + y + 1, f_2 : x^3 - y^2 - 3yz - 1$. Let us compute the monic Gröbner bases for the lexicographic and the *grevlex* order on $\mathbb{R}[x, y, z]$.

The monic Gröbner basis for I with respect to the lexicographic monomial order is the set

$$\begin{aligned}G = \{&y^4 + 6y^3z + y^3 + 9y^2z^2 + 5y^2 + 6yz + 3y + 2,\\ &xz - \frac{2}{3}x + \frac{1}{3}y^3 + 2y^2z + 3yz^2 - yz + y + z,\\ &xy + x + y^2 + 3yz + 1,\\ &x^2 + y + 1\}.\end{aligned}$$

To verify that G is the Gröbner basis for I, consider an element $f \in \mathrm{I}$. I is generated by f_1, f_2, thus $f = a_1f_1 + a_2f_2$ for some polynomials $a_1, a_2 \in \mathbb{R}[x, y, z]$. The leading term of f is then either $\mathrm{lt}(a_1f_1)$, $\mathrm{lt}(a_2f_2)$ or their sum. In any case, the last polynomial in G divides its leading term. Furthermore, its is easy to see that G is monic.

Next, consider the *grevlex* monomial order. The monic Gröbner basis for I is then the set

$$\begin{aligned}G' = \{&y^3 + 6y^2z + 9yz^2 + 3xz - 3yz - 2x + 3y + 3z,\\ &xy + y^2 + 3yz + x + 1,\\ &x^2 + y + 1\}\end{aligned}$$

One can notice that the Gröbner bases do not even have to have the same number of elements. Furthermore, the Gröbner basis with respect to the *grevlex* monomial order is not only shorter, but also the polynomials have smaller degree and nicer coefficients. Although it is not proven, it is usually the case. Moreover, computation of a Gröbner basis with respect to the *grevlex* order is usually much faster. On the other hand, Gröbner bases with respect to the lexicographic monomial ordering can be used for solving systems of polynomial equations.

As mentioned before, from the characterisation 6 and Lemma 7 it follows that to compute the envelope of an one parameter system, we need to solve a system of two polynomial equations. This can be done by eliminating the parameter t from these equations. The Gröbner bases are often used for this task.

Definition 22. *Let I be an ideal in $\mathbb{R}[x_1, \dots, x_n]$. The l-th* elimination ideal *is*

$$I_l = I \cap \mathbb{R}[x_{l+1}, \dots, x_n].$$

Recall that in our usual settings we wish to eliminate the parameter t from the system of two equations $f^\lambda(x, y, z, t) = 0$ and $f_t^\lambda(x, y, z, t) = 0$, hence for $I = (f^\lambda, f_t^\lambda)$ we have the following elimination ideal

$$\mathrm{I}_t = \mathrm{I} \cap \mathbb{R}[x, y, z].$$

The following theorem depict a Gröbner basis of an elimination ideal.

Theorem 23 ([12, Ch.3, Thm. 2], Elimination Theorem)**.** *Let G be a Gröbner basis with respect to the lexicographic monomial order for an ideal $I \subset \mathbb{R}[x_1, \dots, x_n]$. Then for any $l \in \{1, \dots, n\}$*

$$G_l = G \cap \mathbb{R}[x_l, \dots, x_n]$$

is a Gröbner basis of the l-th elimination ideal I_l.

Definition 24. *Let* $I_l \subset \mathbb{R}[x_l, \dots, x_n]$ *be an* $l-th$ *elimination ideal of an ideal* $I \subset \mathbb{R}[x_1 \dots, x_n]$. *Then a point* $p \in V(I_l) \subset \mathbb{R}^{n-l}$ *is called the* partial solution.

All points in $V(I)$ truncate to a partial solution. The following theorem tells us, which partial solutions extend to a point in $V(I)$. For this theorem, we need to work over an algebraically closed field, in our case, the field of complex numbers.

Theorem 25 ([12, Ch.3, Thm. 3], Extension Theorem)**.** *Let* $I = \langle f_1 \dots, f_r \rangle$ *be an ideal in* $\mathbb{C}[x_1, \dots, x_n]$ *and* I_1 *be its first elimination ideal. For every* $i \in \{1, \dots, r\}$ *write* f_i *as*

$$f_i = c_i(x_2, \dots, x_n)x_1^{N_i} + \textit{terms in which } x_i \textit{ has degree} < N_i,$$

where $N_i \in \mathbb{N}$ *and* $c_i \in \mathbb{C}[x_2, \dots, x_n]$ *is non-zero.*

Suppose we have a partial solution $(a_2, \dots, a_n) \in V(I_1)$. *If*

$$(a_2, \dots, a_n) \notin V(c_1, \dots, c_r),$$

then there exists $a_1 \in \mathbb{C}$ *such that* $(a_1, a_2, \dots, a_n) \in V(I)$.

The Extension Theorem can be inductively extended to other elimination ideals.

Example 26. Recall Example 11. Let us first compute a reduced Gröbner basis $G = \{g_1, g_2, g_3, g_4, g_5\}$ of the ideal $I = \langle f^\lambda, f_t^\lambda \rangle$ with respect to the lexicographic ordering with $t > x > y$.

$$\begin{aligned} g_1 = {} & 11664x^8 + 16x^6y^6 - 585x^6y^4 + 5076x^6y^2 - 38664x^6 - x^4y^{10} - 5x^4y^8 \\ & - 884x^4y^6 - 6916x^4y^4 + 5885x^4y^2 + 42625x^4 - x^2y^{12} - 18x^2y^{10} - 183x^2y^8 \\ & - 1116x^2y^6 - 4575x^2y^4 - 11250x^2y^2 - 15625x^2, \\ g_2 = {} & 16tx^2y^{13} + 1321tx^2y^{11} + 29688tx^2y^9 - 92522tx^2y^7 - 7839424tx^2y^5 \\ & - 31597167tx^2y^3 - 1560600tx^2y - 93312x^6y^4 - 2834352x^6y^2 + 3569184x^6 \\ & - 128x^4y^{10} - 1152x^4y^8 + 110331x^4y^6 + 1346274x^4y^4 + 23526288x^4y^2 \\ & - 7700184x^4 - 144x^2y^{12} - 2153x^2y^{10} + 172287x^2y^8 + 3228925x^2y^6 \\ & + 5234421x^2y^4 - 18569520x^2y^2 + 4131000x^2, \\ g_3 = {} & 3319374168tx^4 + 4112tx^2y^{12} + 419465tx^2y^{10} + 11960430tx^2y^8 \\ & + 9104606tx^2y^6 - 3087637940tx^2y^4 - 14225835735tx^2y^2 - 3841868250tx^2 \\ & - 23981184x^6y^3 - 1194801840x^6y - 32896x^4y^9 - 935808x^4y^7 \\ & + 40771323x^4y^5 + 484488324x^4y^3 + 9021165372x^4y - 37008x^2y^{11} \\ & - 1273033x^2y^9 + 53962761x^2y^7 + 1348048703x^2y^5 + 2769404571x^2y^3 \\ & - 7303869450x^2y, \end{aligned} \tag{1.3}$$

$$
\begin{aligned}
g_4 =& 171502706177t^2y^8 + 2058032474124t^2y^6 + 14749232731222t^2y^4 \\
&+ 51450811853100t^2y^2 + 107189191360625t^2 + 8570784208tx^2y^{11} \\
&+ 447143859077tx^2y^9 + 2637043681390tx^2y^7 - 114540691274161tx^2y^5 \\
&- 650595377992068tx^2y^3 - 329349595559310tx^2y - 49984813501056x^6y^2 \\
&+ 839468449296x^6 - 68566273664x^4y^8 + 1466755633792x^4y^6 \\
&+ 11936450992239x^4y^4 + 413705122761564x^4y^2 + 188859580240104x^4 \\
&- 77137057872x^2y^{10} + 1191027458875x^2y^8 + 53352589022478x^2y^6 \\
&+ 162741630161810x^2y^4 - 276360042692994x^2y^2 - 296888240050025x^2 \\
&+ 171502706177y^8 + 2058032474124y^6 + 14749232731222y^4 \\
&+ 51450811853100y^2 + 107189191360625, \\
g_5 =& 1029016237062t^2x^2 + 3835312tx^2y^{11} + 224556467tx^2y^9 + 2456837050tx^2y^7 \\
&- 42939029714tx^2y^5 - 622020302148tx^2y^3 - 2792792777023tx^2y \\
&- 22367539584x^6y^2 - 142307343504x^6 - 30682496x^4y^8 + 460629376x^4y^6 \\
&+ 9525009465x^4y^4 + 219270724812x^4y^2 + 1396561520952x^4 \\
&- 34517808x^2y^{10} + 312779149x^2y^8 + 27193951123x^2y^6 \\
&+ 222693218689x^2y^4 + 454926520789x^2y^2 - 1254254177448x^2, \\
g_6 =& 8232129896496000t^3 - 15949751674461t^2y^7 - 161384046512557t^2y^5 \\
&- 1204463505481071t^2y^3 - 6169809854717575t^2y + 15430671472tx^2y^{12} \\
&+ 426513396863tx^2y^{10} - 15006102325790tx^2y^8 - 319921877859794tx^2y^6 \\
&+ 3932980083404115tx^2y^4 + 26096343414243210tx^2y^2 \\
&- 5927133525477120tx^2 + 8232129896496000t - 89991676024704x^6y^3 \\
&+ 2209014462034224x^6y - 123445371776x^4y^9 + 5668842986368x^4y^7 \\
&- 43236019313379x^4y^5 + 216584367885756x^4y^3 - 17414290688695056x^4y \\
&- 138876043248x^2y^{11} + 5550943018945x^2y^9 + 43203483205279x^2y^7 \\
&- 2080524984593504x^2y^5 - 7412218093958823x^2y^3 \\
&+ 19070089710359527x^2y - 15949751674461y^7 - 161384046512557y^5 \\
&- 1204463505481071y^3 - 6169809854717575y.
\end{aligned}
$$

From Elimination Theorem 23, the first elimination ideal I_1 is generated by the polynomial g_1, which is a polynomial where the variable t was eliminated.

One can then use Extension Theorem 25 to verify that for every $(x, y) \in V(I_1)$ there exist t such that $(t, x, y) \in V(f^\lambda, f_t^\lambda)$, if we work over $\mathbb{C}$. Write

$$
\begin{aligned}
f^\lambda &= c_1(x,y)t^6 - 4t^7y + t^6y^2 + 11t^6 - 12t^5y + 3t^4y^2 + 9t^4 - 12t^3y \\
&\quad + t^2x^2 + 3t^2y^2 + t^2 - 4ty + x^2 + y^2 - 1 \\
f_t^\lambda &= c_2(x,y)t^5 - 4t^6y + 24t^5 - 12t^4y + 24t^3 - 12t^2y - 4tx^2 + 8t - 4y,
\end{aligned}
$$

where $c_1(x, y) = 4$ and $c_2(x, y) = 24$. Then since for any $(x, y) \in \mathrm{I}_1$, $(x, y) \notin V(4, 8) = \emptyset$, there exists a $t \in \mathbb{C}$ such that $(t, x, y) \in V(f^\lambda, f_t^\lambda)$.

Furthermore, the polynomials g_2 and g_3 are linear in t, thus we cam obtain its value from the equations $g_2 = 0$ and $g_3 = 0$ and we see that for real points (x, y) the corresponding t will be also real. Hence $\mathrm{V}(g_1) = \mathrm{V}(I_1)$ is really the envelope of the one parameter system $\mathcal{F}_t$,

$$
\chi = \mathrm{V}(f^\lambda, f_t^\lambda) = \mathrm{V}(g_1) = \mathrm{V}(I_1).
$$

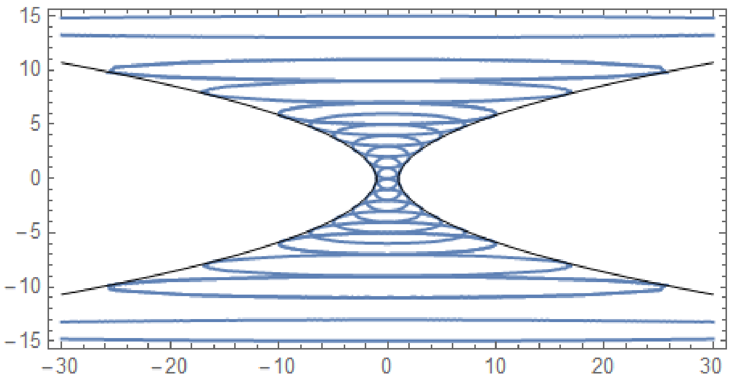

Figure 1.7: One parameter system of ellipses in plane with highlighted envelope (black).

See Figure 1.7 for the one parameter system $\mathcal{F}_t$ with highlighted envelope.

Recall Example 3. In the previous section, we saw that we are not able to compute the envelope just by solving the two defining equations for t. We may try to compute the Gröbner basis with respect to the lexicographic monomial ordering with $t > x > y > z$ as in the example above, however, the computation of the Gröbner basis takes too much time.

A Gröbner bases with respect to a different monomial ordering might be computed much faster, on the other hand, unfortunately, it can not be generally used to eliminate the variable t from the equations.

There are also methods for solving polynomial equations that are not dependent of a monomial ordering, but the most attention is given to the case when the ideal generated by the polynomials $f = 0$ and $\frac{\partial f}{\partial t} = 0$ is zero dimensional, that is, when the affine variety $V(f, \frac{\partial f}{\partial t})$ has only finitely many points, see [11, Chapter 2]. In our settings, in general, the solution to the system of two equations for a specific value of the parameter t is an infinite curve.

1.3.2 Resultants

Another way to find a polynomial which lies in the first elimination ideal, that is independent of Gröbner bases and monomial orderings uses the theory of *resultants.*

Definition 27. *Let $f, g \in \mathbb{R}[x]$ be non-zero polynomials of degree m, l, respectively. Write f, g as*

$$f = c_0 x^l + c_1 x^{l-1} + \cdots + c_l, \;\; c_0 \neq 0$$
$$g = d_0 x^m + d_1 x^{m-1} + \cdots + d_m, \;\; d_0 \neq 0.$$

Then if $l, m > 0$ then the Resultant *of f, g, denoted by $Res(f, g)$ is the determinant of a $(l+m) \times (l+m)$ matrix,*

$$
Res(f,g) = det \begin{pmatrix}
c_0 & & & & d_0 & & & \\
c_1 & c_0 & & & d_1 & d_0 & & \\
c_2 & c_1 & \ddots & & d_2 & d_1 & \ddots & \\
\vdots & c_2 & \ddots & c_0 & \vdots & d_2 & \ddots & d_0 \\
c_l & \vdots & \ddots & c_1 & d_m & \vdots & \ddots & d_1 \\
 & c_l & \ddots & c_2 & & d_m & \ddots & d_2 \\
 & & \ddots & \vdots & & & \ddots & \vdots \\
 & & & c_l & & & & d_m
\end{pmatrix} .
$$

$$
\underbrace{\qquad\qquad\qquad\qquad}_{m \text{ columns}} \underbrace{\qquad\qquad\qquad\qquad}_{l \text{ columns}}
$$

Remark. If we want to emphasise the dependence on x, we write $\text{Res}(f, g, x)$ instead of just $\text{Res}(f, g)$.

Example 28. Consider two polynomials $f, g \in \mathbb{R}[x, y]$,

$$
f = 4x^2y^2 + 3xy - 2x + 2y + 1,
$$

$$
g = -x^2 + 3x^2y + y^2 + x + 2y - .
$$

We can look at these polynomials as at polynomials in x with coefficients in $\mathbb{R}[y]$ and write

$$
f = (4y^2)x^2 + (3y - 2)x + (2y + 1)
$$

$$
g = (-1)x^3 + (3y)x^2 + x + (y^2 + 2y - 2).
$$

Then the resultant $\text{Res}(f, g, x)$ is the determinant of the following 5×5 matrix

$$
S = \begin{pmatrix}
4y^2 & 0 & 0 & -1 & 0 \\
3y - 2 & 4y^2 & 0 & 3y & -1 \\
2y + 1 & 3y - 2 & 4y^2 & 1 & 3y \\
0 & 2y + 1 & 3y - 2 & y^2 + 2y - 2 & 1 \\
0 & 0 & 2y - 2 & 0 & y^2 + 2y - 2
\end{pmatrix},
$$

thus

$$
\begin{aligned}
\text{Res}(f, g, x) = \det(S) = 64y^{10} + 256y^9 - 192y^8 - 788y^7 + 816y^6 - 573y^5 + 644y^4 \\
- 434y^3 + 192y^2 - 114y + 22.
\end{aligned}
$$

There are several interesting properties of the resultants. We need to work over an algebraically closed field (in our case, the field of complex numbers) and we state the properties for resultants $\text{Res}(f, g, x_1)$ for two polynomials $f, g \in \mathbb{C}[x_1, x_2, \ldots, x_n]$. Then, as before, we work with $x_1 = t$, $x_2 = x$, $x_3 = y$ and $x_4 = z$.

Remark. When computing $\text{Res}(f, g, x_1)$ for polynomials $f, g \in \mathbb{C}[x_1, x_2, \ldots, x_n]$, we often understand f and g as polynomials in the variable x_1 and coefficients in $\mathbb{C}[x_2, \ldots, x_n]$.

Theorem 29. *[12, Ch. 3, Prop. 3] Let f and g be two non-zero polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ with positive degrees in x_1. Then $Res(f, g, x_1) \in \mathbb{C}[x_2, \ldots, x_n]$ is an integer polynomial in the coefficients of f and g. Furthermore, the resultant $Res(f, g, x_1) = 0$ if and only if f and g have a common factor in $\mathbb{C}[x_2, \ldots, x_n]$.*

Theorem 30. *[12, Ch. 3, Prop. 5] Let f and g be two non-zero polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ with positive degrees in x_1. Then there are non-zero polynomials $p, q \in \mathbb{C}[x_1, \ldots, x_n]$ such that*

$$Res(f, g, x_1) = p \cdot f + q \cdot g,$$

and the coefficients of p, q are integer polynomials in the coefficients of f and g.

Let us illustrate those properties by a simple example.

Example 31. Let $f = x^3y^2 - 2x^3y + x^2y^2 - 2x^2y$, $g = x^2y^2 - 2x^2y - xy + 2x$ be polynomials in $\mathbb{C}[x, y]$. The resultant $\mathrm{Res}(f, g, x)$ is as follows

$$\mathrm{Res}(f, g, x) = \det \begin{pmatrix} y^2-2y & 0 & y^2-2y & 0 & 0 \\ y^2-2y & y^2-2y & -y+2 & y^2-2 & 0 \\ 0 & y^2-2y & 0 & -y+2 & y^2-2 \\ 0 & 0 & 0 & 0 & -y+2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 0,$$

since the matrix is clearly singular. Thus, from Theorem 29 the polynomials f, g have a common factor in $\mathbb{C}[y]$. Indeed, one can easily verify that

$$\begin{aligned} f &= x^2y(x+1)(y-2), \\ g &= x(xy-1)(y-2), \end{aligned}$$

thus f and g have $(y-2) \in \mathbb{C}[y]$ as the common factor. Furthermore, one can write

$$\mathrm{Res}(f, g, x) = 0 = (xy-1)f - xy(x+1)g,$$

which corresponds to Theorem 30.

Theorem 30 says that for two polynomials $f, g \in \mathbb{C}[x_1, \ldots, x_n]$ the resultant $\mathrm{Res}(f, g, x_1)$ lies in the ideal $\langle f, g \rangle \cap \mathbb{C}[x_2, \ldots, x_n]$, which is the first elimination ideal of the ideal $\langle f, g \rangle$.

Example 32. To apply this theory on the problem of computation of envelopes, recall once again Example 11. We compute the resultant $\mathrm{Res}(f^\lambda, f_t^\lambda, t)$ and we obtain

$$\begin{aligned} \mathrm{Res}(f^\lambda, f_t^\lambda, t) = 16384x^4 \big(&11664x^8 + 16x^6y^6 - 585x^6y^4 + 5076x^6y^2 - 38664x^6 \\ &-x^4y^{10} - 5x^4y^8 - 884x^4y^6 - 6916x^4y^4 + 5885x^4y^2 + 42625x^4 - x^2y^{12} \\ &-18x^2y^{10} - 183x^2y^8 - 1116x^2y^6 - 4575x^2y^4 - 11250x^2y^2 - 15625x^2\big), \end{aligned}$$

which is a polynomial in $\mathbb{R}[x, y]$ and a multiple of g_1 from 1.3 by the factor $16384x^4$, which follows from the fact that the resultant lies in the first elimination ideal, $\mathrm{Res}(f^\lambda, f_t^\lambda, t) \in \mathrm{I}_1 = \langle F^\lambda, f_t^\lambda \rangle \cap \mathbb{R}[x, y]$. One can thus obtain the implicit equation of the envelope by dividing the resultant by the factor $16384x^4$.

Computing determinants of large matrices is computationally difficult and time consuming problem, but there is a method how to compute it more efficiently, similar to the Euclidean algorithm (see again [12, Ch. 3, §6]).

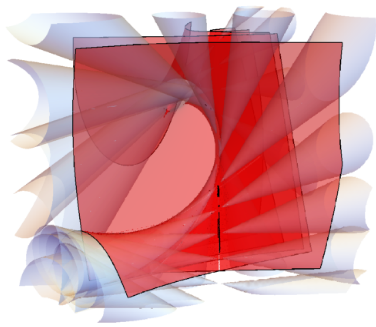

Figure 1.8: Envelope of one parameter system of cone derived from resultant.

Example 33. In our running example 3 we are not able to find the Gröbner basis with respect to the lexicographic monomial order (in a reasonable amount of time), however, the resultant is computed within few seconds. We do not present the resulting polynomial here, since it is rather long, but we at least show the the envelope surface defined by the formula, see Figure 1.8. The imperfections in the figure are due to the fact that the image is created using the implicit equation obtained from the resultant. A nicer and clearer figure can be found at the very end of this thesis, where we prove that the envelope is actually a rational surface and we are able to plot it using the parametric formula.

We have seen that using resultants, one can eliminate the variable t from the equations $f^\lambda = 0$ and $f_t^\lambda = 0$ even for quite complicated examples, where we are not able to compute a Gröbner basis. However, this methods does still have several drawbacks. The resultant lies in the first elimination ideal $I = \langle g_1 \rangle$, but generally it is a multiple of its Gröbner basis (g_1). Thus $\mathrm{V}(\mathrm{Res}(f^\lambda, f_t^\lambda, t))$ can contain more points than $\mathrm{V}(g_1)$ and to get rid of these redundant points, one needs to know the correct factor. Moreover, this method (just like the method based on the elimination theory using Gröbner bases) can help us compute the envelopes only if the parameter t of the one parameter system lies in the whole real line. If we restrict the domain of the parameter to an interval in $\mathbb{R}$, the envelope usually can not be given by an implicit equation (usually, it is not an affine variety) and thus one needs to find some parameterization of the envelope surface.

1.4 Characteristic curves and parameterization

As said many times before, for a one parameter system

$$\mathcal{F}_t = \{(x, y, z) \in \mathbb{R}^3, f(x, y, z, t) = 0, t \in \mathrm{I}\}$$

of quadratic surfaces in $\mathbb{R}^3$, each characteristic curve is defined as an intersection of two surfaces. For $t_0 \in \mathrm{I}$, the corresponding characteristic curve is the intersection of $F_{t_0} \in \mathcal{F}_t$ and a surface defined by implicit equation $\frac{\partial f}{\partial t}(x, y, z, t_0) = 0$. Thanks to the Lemma 7, we can multiply the equations by a common factor to obtain formulas polynomial in x, y, z and t. From the Bezout's theorem (see e.

g. [12, Ch. 8, Thm. 10]), the intersection χ_{t_0} is an algebraic curve of degree 4, which is generally not a *rational curve*.

Given an affine variety $V = \mathrm{V}(f_1, \dots, f_m) \subseteq \mathbb{R}^n$, we say V is rational if there exist a n-tuple $(r_1, \dots, r_n)$ of rational functions $r_1, \dots, r_n \in \mathbb{R}(t_1, \dots, t_k)$ is called *rational parametric representation* of V

- the points
$$\begin{pmatrix} r_1(t_1, \dots, t_k) \\ r_2(t_1, \dots, t_k) \\ \dots \\ r_n(t_1, \dots, t_k) \end{pmatrix}$$
lie in V for almost all $t_1, \dots, t_k \in \mathbb{R}$, and
- for almost all points $P \in V$ there exist $(t_1, \dots, t_n)$ such that
$$P = \begin{pmatrix} r_1(t_1, \dots, t_k) \\ r_2(t_1, \dots, t_k) \\ \dots \\ r_n(t_1, \dots, t_k) \end{pmatrix}.$$

We say an algebraic curve in $\mathbb{R}^3$ is rational if there exist rational functions $r_1, r_2, r_3 \in \mathbb{R}(s)$ satisfying the two conditions above. For more details on rational algebraic curves we refer to [13].

In the next chapters we prove that characteristic curves of some special one parameter systems of surfaces are rational, that is, for each $t \in I$ we describe χ_t as
$$\chi_t(s) = \begin{pmatrix} r_1(s) \\ r_2(s) \\ r_3(r) \end{pmatrix}, s \in \mathbb{R},$$
where $r_1, r_2, r_3 \in \mathbb{R}(s)$ are rational functions. Since the one parameter system $\mathcal{F}_t$ is defined to depend rationally on the parameter t, also the envelope of the system is a rational surface. The parameterization then allows us to express not only the whole envelope surface but also its sections for $t \in I \subset \mathbb{R}$ and $s \in J \subset \mathbb{R}$, which might be useful in practical applications.

2. Dual representation of envelope surfaces

In this chapter we embed real space into projective space and present duality in the projective space. We present one parameter systems and their envelopes in the dual space. Necessary theory can be found e. g. in [14].

The dual approach helps us to prove that envelopes of special one parameter systems are rational. First we prove rationality of an envelope surface of a (truncated) cone using duality between a point and a plane in line geometry and combining line geometry with kinematics. The proof can be found in [6].

Next we present Lagurre geometry and its different models and convey the first proof of the fact that one parameter systems of one parameter systems of spheres, cones and cylinders of revolution posses rational envelopes. The proof was first published in [7] and builds on the paper [15].

2.1 Duality in the projective space

The points of dual projective space $\mathbb{P}^3$ can be identified with hyperplanes in $\mathbb{R}^3$. A surface in the dual projective space can be hence interpreted as the set of all its tangent hyperplanes.

Example 34. Consider a cone F in $\mathbb{P}^3$ given by the implicit equation

$$f(x, y, z) = x^2 + y^2 - \frac{1}{25}z^2.$$

A tangent plane t_P to F at a point $P = \{p_1, p_2, p_3\} \in F$ is dual to the the point P,

$$\begin{aligned} t_P(x, y, z) : 2p_1(x - p_1) + 2p_2(y - p_2) - \frac{2}{25}p_3(z - p_3) \\ = p_1x + p_2y - \frac{1}{25}p_3z = 0, \end{aligned}$$

since the point P is a point lying on F. The dual of the cone F is then the set of all tangent planes,

$$F^* = \{t_P, P \in \Phi(x, y, z)\},$$

in other words, it is a variety defined by the polynomials $p_1x + p_2y - \frac{1}{25}p_3z$ and $p_1^2 + p_2^2 - \frac{1}{25}p_3^2$,

$$F^* = V(p_1x + p_2y - \frac{1}{25}p_3z, p_1^2 + p_2^2 - \frac{1}{25}p_3^2).$$

Eliminating the variables x, y, z from the two polynomials (using for instance the Elimination theory from the previous section), we obtain

$$F^* : V(p_1^2 + p_2^2 - \frac{1}{25}p_3^2),$$

which is a cone in the dual space to $\mathbb{P}^3$ with coordinates p_0, p_1, p_2, p_3. Generally, the dual to a quadric in $\mathbb{P}^3$ is a quadric in the dual projective space, [14, Thm. 1.1.35].

For a one parameter system of cones (and other surfaces) we can use the algebraic methods derived in the previous chapter to compute envelopes in the dual projective space.

Example 35. Consider the following simple example of a one parameter system of cones in $\mathbb{R}^3$,

$$\mathcal{F}_t = \{(x, y, z) \in \mathbb{R}^3, (t + x)^2 + y^2 - \frac{z^2}{25} = 0, t \in \mathbb{R}\}.$$

It is a system of isometric cones of revolution whose vertices lie on the $x-$axis and whose axes are parallel to the $z-$axis. Each cone of the system has a one parameter system of tangent planes, hence for each $t \in I$, the set of all tangent planes to F_t is the following set,

$$T_t = \{p_t(s), s \in \mathbb{R}\},$$

where

$$p_t(s) = -\frac{1}{2(s^2+1)}\left(5\sqrt{2}s^2t + 5\sqrt{2}s^2x + 5\sqrt{2}s^2y + 2s^2z - 10\sqrt{2}st - 10\sqrt{2}sx \right.$$
$$\left. +10\sqrt{2}sy - 5\sqrt{2}t - 5\sqrt{2}x - 5\sqrt{2}y + 2z\right).$$

In the dual coordinates the tangent planes are thus as follows,

$$p_t(s)^* = -\frac{1}{2(s^2+1)}\left(5\sqrt{2}s^2 - 10\sqrt{2}s - 5\sqrt{2}, 5\sqrt{2}s^2 + 10\sqrt{2}s - 5\sqrt{2}, \right.$$
$$\left. 2s^2 + 2, 5s^2t - 10st - 5t\right).$$

For $t = 0$, the tangent planes of F_0 have equations

$$p_0(s) = -\frac{5\sqrt{2}}{2(s^2+1)}\left(s^2 - 2s - 1\right)x + 5\sqrt{2}\left(s^2 + 2s - 1\right)y + 2\left(s^2 + 1\right)z,$$

see Figure 2.1 (left).

Hence we have two parameter system of tangent planes $T(t, s)$ defined by two parameter implicit equation $p(t, s)$. We can compute the envelope of the system of planes by eliminating the variables t, s from the equations

$$p(t, s) = 0$$

$$\frac{\partial p}{\partial s} = -\frac{5\sqrt{2}}{(1+s^2)^2}\left(s^2t + s^2x - s^2y + 2st + 2sx + 2sy - t - x + y\right)$$

$$\frac{\partial p}{\partial t} = -\frac{1}{\sqrt{2}(s^2+1)}\left(5\left(s^2 - 2s - 1\right)\right).$$

We obtain two implicit equations

$$\chi_1 = (20 - 10\sqrt{2})y - (4 - 2\sqrt{2})z,$$
$$\chi_2 = -(20 + 10\sqrt{2})y - (4 + 2\sqrt{2})z,$$

each of them defining one component of the envelope, see Figure 2.1 (right).

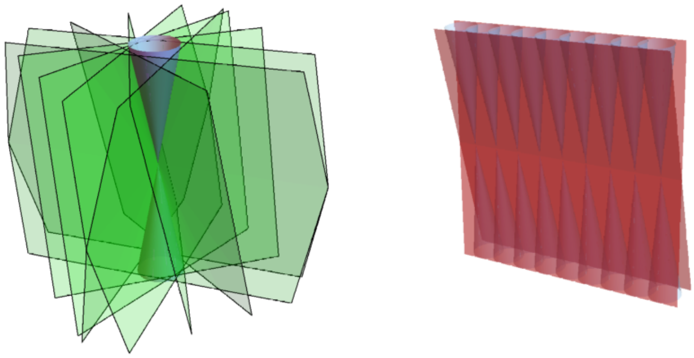

Figure 2.1: Tangent planes of a cone (left), one parameter system of cones and its envelope (right).

Furthermore, we can find parameterization of all the tangent planes of a one parameter system. The following lemma holds.

Lemma 36. *Let $\mathcal{F}_t = \{(x, y, z) \in \mathbb{R}^3,\ f(x, y, z, t) = 0,\ t \in I\}$ be a one parameter system of cones in $\mathbb{R}^3$ and let χ be its envelope. Then every tangent plane of χ is also a tangent plane of a cone $F_{t_0} \in \mathcal{F}_t$ for some $t_0 \in I$.*

The statement is obvious, since envelope is tangent to each element of a system along a characteristic curve, hence at a point Q of the envelope corresponding to a characteristic curve χ_{t_0} the tangent plane coincides with a tangent plane of a cone F_{t_0} at the point Q.

For *non-degenerate* one parameter system of cones of revolution it can be actually shown, that each tangent plane of the system is a tangent plane of the envelope. This fact is used in [6] and in [7], where the authors represent each element of a one parameter system as the set of all its tangent hyperplanes and then parameterize the envelope of the tangent planes.

A degenerate one parameter system is e. g. the one in the example above. Characteristic curves of the system are the generators of the cone. In a general case, characteristic curves are curves of degree four. In the last chapter, we prove that each characteristic curve χ_t passes through the vertex of a cone F_t, which implies the fact above.

Let us comment on the proof from [6]. A cone of revolution is represented by a one parameter system of its tangent planes in *Bezier form*. Each plane is represented by its *plane coordinates*. Then, they describe a rational motion of the cone using *dual quaternions* and *Bezier motion*, which results in a one parameter system of isometric cones. Applying the same motion to the one parameter system of tangent planes, they obtain rational parameterization of all the tangent planes of the system and therefore the parameterization of the dual envelope in the plane coordinates. Point coordinates of the envelope are then computed.

Example 37. Let us illustrate the proof by an example, using the same method as in [6], but different representations. Consider the one parameter system from Example 3. We can represent the system as the set of all positions of a cone F

with implicit equation $x^2 + y^2 - \frac{1}{25}z^2 = 0$ under the following rational motion,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{468t^2+788t-123}{1108t^2+788t+517} & \frac{16(2t+1)(29t+12)}{1108t^2+788t+517} & \frac{16(2t+1)(12t-29)}{1108t^2+788t+517} \\ \frac{16(2t+1)(27t+16)}{1108t^2+788t+517} & \frac{588t^2+268t+387}{1108t^2+788t+517} & -\frac{4(2t+1)(46t-57)}{1108t^2+788t+517} \\ -\frac{16(2t+1)(16t-27)}{1108t^2+788t+517} & \frac{4(2t+1)(18t-71)}{1108t^2+788t+517} & -\frac{972t^2+1292t+3}{1108t^2+788t+517} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t + \frac{3}{2} \\ t + \frac{1}{2} \\ 3t + \frac{3}{2} \end{pmatrix}.$$

We will see this approach to one parameter systems of surfaces again in the next chapter.

We find a one parameter system of tangent planes of the cone F and map them via the rational mapping above. We obtain two parameter system of all tangent planes of the system

$$\begin{aligned}
p(t,s) = {} & \frac{1}{2\,(s^2+1)\,(1108t^2+788t+517)} \\
& \Big((-5800+4040\sqrt{2})s^2t^3 - (2300\sqrt{2}+768)s^2t^2x - (7260\sqrt{2}-736)s^2t^2y \\
& + (1944+1840\sqrt{2})s^2t^2z - (10812-12360\sqrt{2})s^2t^2 - (300\sqrt{2}-1472)s^2tx \\
& - (6060\sqrt{2}+544)s^2ty - (560\sqrt{2}-2584s^2)tz - (6302-11330\sqrt{2})s^2t \\
& - (1575\sqrt{2}-928)s^2x - (3215\sqrt{2}+456)s^2y - (740\sqrt{2}-6)s^2z \\
& - (1173-5080\sqrt{2})s^2 + 30880\sqrt{2}st^3 - 13960\sqrt{2}st^2x + 2760\sqrt{2}st^2y \\
& - 6560\sqrt{2}st^2z + 5880\sqrt{2}st^2 - 16360\sqrt{2}stx + 6760\sqrt{2}sty + 11040\sqrt{2}stz \\
& - 14880\sqrt{2}st - 690\sqrt{2}sx - 1310\sqrt{2}sy + 7160\sqrt{2}sz - 9050\sqrt{2}s \\
& - (4040\sqrt{2}+5800)t^3 - (768-2300\sqrt{2})t^2x + (736+7260\sqrt{2})t^2y \\
& - (1840\sqrt{2}-1944)t^2z - (12360\sqrt{2}+10812)t^2 + (1472+300\sqrt{2})tx \\
& - (544-6060\sqrt{2})ty + (2584+560\sqrt{2})tz - (11330\sqrt{2}+6302)t \\
& + (928+1575\sqrt{2})x - (456-3215\sqrt{2})y + (6+740\sqrt{2})z - 5080\sqrt{2} - 1173\Big).
\end{aligned}$$

Multiplying the equation by its denominator and differentiating it with respect to s and t, we obtain a system of three equations

$$\begin{aligned}
p(t,s) &= 0, \\
\frac{\partial p}{\partial s}(t,s) &= 0, \\
\frac{\partial p}{\partial t}(t,s) &= 0,
\end{aligned}$$

that are linear in x, y and z. Solving the system for x, y, z, we obtain a parameterization of the envelope $\chi(t,s)$. We do not provide the explicit parameterization of the envelope, because the formulas are too long. See at least Figure 2.2 for a segment of the envelope for $t \in (-\frac{3}{2}, 0)$ and $s \in (0, \frac{1}{2})$.

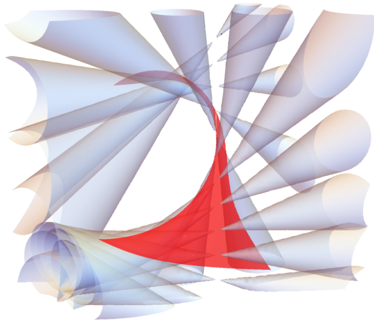

Figure 2.2: Segment of an envelope of a one parameter system of cones.

2.2 Laguerre geometry

Laguerre geometry in Euclidean space $\mathbb{R}^n$ studies points, oriented spheres and oriented hyperplanes. We will describe the objects by their implicit equations,

$$\begin{aligned} &s : (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 + r^2 = 0,\ c_i, r \in \mathbb{R}, \\ &h : a_0 + a_1 x_1 + \cdots + a_n x_n = 0,\ a_i \in \mathbb{R}, \\ &P = (p_1, \ldots, p_n),\ p_i \in \mathbb{R}. \end{aligned}$$

Remark. Oriented spheres and hyperplanes are oriented by their unit normals. A point $P = (p_1, \ldots, p_n) \in \mathbb{R}^n$ in Laguerre geometry is considered to be a sphere with centre $(p_1, \ldots, p_n)$ and zero radius. Points have no orientation.

Remark. Spheres can be oriented by their oriented radii. Hyperplanes can be oriented by the absolute term a_0.

Let us denote the set of all oriented spheres and points in $\mathbb{R}^n$ by $\mathcal{S}^n$. The set $\mathcal{P}^n \subset \mathcal{S}^n$ is the set of all points in $\mathbb{R}^n$. By $\mathcal{H}^n$ we denote the set of all oriented hyperplanes in $\mathbb{R}^n$ with unit normal vector, that is, $||(a_1, \ldots, a_n)||_2 = 1$. By $\mathcal{O}^n = \mathcal{S}^n \cup \mathcal{H}^n$ let us denote the set of all the geometric objects in $\mathbb{R}^n$ we are interested in.

Definition 38. *We say oriented spheres or oriented hyperplanes are in* oriented contact*, if they are tangent and their unit normal vectors at the point of tangency coincide. An oriented sphere $s \in \mathcal{S}^n$ or an oriented hyperplane $h \in \mathcal{H}$ are in* oriented contact *with a point $P \in \mathcal{P}^n$ if $P \in s$ or $P \in h$. Two points $P, Q \in \mathcal{P}^n$ are in* oriented contact*, if $P = Q$.*

Example 39. Consider two oriented spheres

$$\begin{aligned} &s_1 : (x + 1)^2 + (y - 1)^2 + (z - 2)^2 - 2^2 = 0, \\ &s_2 : (x - 2)^2 + (y - 1)^2 + (z + 2)^2 - (-3)^2 = 0. \end{aligned}$$

One can easily check that the spheres are tangent with the point of tangency $P = [\frac{1}{5}, 1, \frac{2}{5}]$.

We compute the unit normal vectors n_1, n_2 at the point P

$$n_1 = n_2 = (1, 0, 0).$$

The unit normals n_1, n_2 coincide, thus the spheres are in oriented contact. In case of the sphere s_1, the normal n_1 faces inside the sphere, it is therefore positively oriented. The sphere s_2 is negatively oriented, since its unit normal n_2 faces outwards.

Consider a hyperplane

$$h : \frac{1}{4}(\sqrt{3}x + 3y - 2z + 4\sqrt{3} + 13) = 0,$$

which is a tangent plane of the sphere s_2 at the point $Q = [-\frac{3\sqrt{3}}{4} - 4, -\frac{5}{4}, \frac{7}{2}]$. If we again compute the unit normals of h and s_2 at Q, we will obtain the same vector $n = (-\frac{\sqrt{3}}{4}, -\frac{3}{4}, \frac{1}{2})$, thus also h and s_2 are in oriented contact. See Figure 2.3.

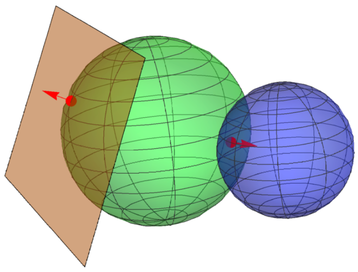

Figure 2.3: Two oriented spheres s_1(blue), s_2(green) from Example 39 in an oriented contact. The unit normals at the point of tangency coincide. The yellow hyperplane h is in oriented contact with s_2.

Definition 40. *The maps $\phi : \mathcal{S}^n \to \mathcal{S}^n$ and $\phi^* : \mathcal{H}^n \to \mathcal{H}^n$ are called* Laguerre transformations*, if they are bijective and preserve the oriented contact.*

Remark. An example of a Laguerre transformations are similarities of $\mathbb{R}^n$ or Möbius transformations.

Example 41. A simple example of a Laguerre transform is *dilatation d*:

$$d : \mathcal{S}^n \to \mathcal{S}^n$$
$$(x - c_1)^2 + \cdots + (x_n - c_n)^2 - r^2 = 0 \mapsto (x - c_1)^2 + \cdots + (x_n - c_n)^2 - (r + k)^2 = 0,$$

$$d^* : \mathcal{H}^n \to \mathcal{H}^n$$
$$a_0 + a_1x_1 + \cdots + a_nx_n = 0 \mapsto (a_0 + k) + a_1x_1 + \cdots + a_nx_n = 0,$$

which adds a real number $k \in \mathbb{R}$ to the oriented radius of an oriented sphere or to an absolute term of an oriented hyperplane.

Consider the transforms d, d^* with $k = -3$. Images of the spheres s_1, s_2 are

$$d(s_1) : (x + 1)^2 + (y - 1)^2 + (z - 2)^2 - (-1)^2 = 0,$$
$$d(s_2) : (x - 2)^2 + (y - 1)^2 + (z + 2)^2 - (-6)^2 = 0.$$

The images are tangent and their unit normal vectors at the point of tangency coincide in $(1, 0, 0)$, thus they are still in oriented contact.

We observe that the Laguerre transformation does not preserve points. The image of the point P under d is the sphere

$$\left(x - \frac{1}{5}\right)^2 + (y-1)^2 + \left(z - \frac{2}{5}\right)^2 - (-3)^2 = 0,$$

the red sphere in Figure 2.4.

By adding the same constant number k to the absolute term a_0 of the oriented hyperplane h, we obtain the oriented hyperplane

$$d^*(h) : \frac{1}{4}\left(-\sqrt{3}x - 3y + 2z - 4\sqrt{3} - 16\right).$$

This hyperplane is in oriented contact with $d(s_2)$ with the point of tangency $[-\frac{3\sqrt{3}}{2} - 4, -\frac{7}{2}, 5]$.

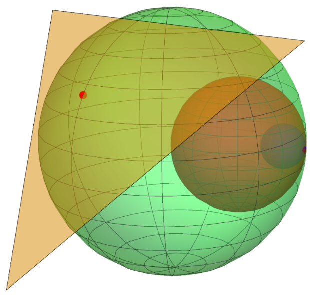

Figure 2.4: Laguerre transform preserves the oriented contact and does not preserve points.

2.2.1 Models of Laguerre geometry

We describe three different models of Laguerre geometry.

Definition 42. *The* cyclographic model *of Laguerre geometry is the pair of bijective maps* ζ, ζ^**,*

$$\zeta : \mathcal{S}^n \to \mathbb{R}^{n+1}$$
$$(x - c_1)^2 + \cdots + (x - c_n)^2 - r^2 = 0 \mapsto [c_1, \ldots, c_n, r]$$

$$\zeta^* : \mathcal{H}^n \to \mathbb{R}^{n+1}$$
$$a_0 + a_1x_1 + \cdots + a_nx_n = 0 \mapsto (x_1, \ldots, x_n, x_{n+1}) : \quad a_0 + a_1x_1 + \cdots + a_nx_n + x_{n+1} = 0.$$

Example 43. The spheres s_1, s_2 from Example 39 in the cyclographic model are mapped to the points $\zeta(s_1) = [-1, 1, 2, 3]$, $\zeta(s_2) = [2, 1, -2, -3] \in \mathbb{R}^4$. The image of the point P is $\zeta(P) = [\frac{1}{5}, 1, \frac{2}{5}, 0]$. The hyperplane h is mapped to the oriented hyperplane $\zeta^*(h) = \frac{1}{4}(\sqrt{3}x + 3y - 2z + 4w + 4\sqrt{3} + 13) = 0$, where w stands for the variable x_4.

Example 44. We can understand a hyperplane $\zeta^*(h)$ in $\mathbb{R}^{n+1}$ to be the set of all points in $\mathbb{R}^n$ such that their preimages in the cyclographic model are spheres tangent to the hyperplane $h \in \mathbb{R}^n$.

The cyclographic image of a line $h : \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + 1 = 0 \subset \mathbb{R}^2$ is the plane $\zeta^*(h) : \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + z + 1 = 0$.

All spheres tangent to the line h are the spheres

$$\left(\frac{r}{\sqrt{2}} - t + x\right)^2 + \left(\frac{r}{\sqrt{2}} + t + y + \sqrt{2}\right)^2 - r^2 = 0,$$

for some parameter $t \in \mathbb{R}$ passing through the line h and a radius $r \in \mathbb{R}$. The cyclographic image of any of the spheres is the point

$$[t - \frac{r}{\sqrt{2}}, -\sqrt{2} - t - \frac{r}{\sqrt{2}}, r],$$

which satisfies the equation $\zeta^*(h) = 0$. See Figure 2.5 for illustration.

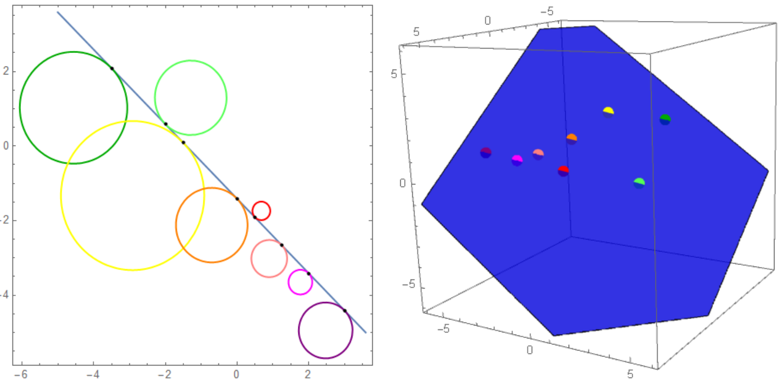

Figure 2.5: The cyclographic image of a hyperplane can be seen as the set of all points corresponding to the spheres tangent to the hyperplane.

Definition 45. *An Euclidean space $\mathbb{R}^{n+1}$ together with an indefinite inner product, defined for vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n+1}$ by*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T D \boldsymbol{y}$$

where D is a diagonal $n+1$ by $n+1$ matrix, $D = diag(1, 1, \dots, 1, -1)$, is called Minkowski space *and is denoted by $\mathbb{R}^{n,1}$.*

For a vector $\mathbf{x} \in \mathbb{R}^{n+1}$ the value $\langle \mathbf{x}, \mathbf{x} \rangle \in \mathbb{R}$ does not have to be positive for $\mathbf{x} \neq \mathbf{0}$ and even can be negative. We distinguish the three possibilities.

Definition 46. *A vector $\boldsymbol{x} \in \mathbb{R}^{n+1}$ called*

- space-like, *if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$,*
- time-like, *if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$,*

- light-like*, if* $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$*.*

A line in $\mathbb{R}^{n+1}$ *with directional vector* $\boldsymbol{x}$ *is called*

- hyperbolic*, if* $\boldsymbol{x}$ *is space-like,*
- elliptic*, if* $\boldsymbol{x}$ *is time-like,*
- parabolic*, if* $\boldsymbol{x}$ *is light-like.*

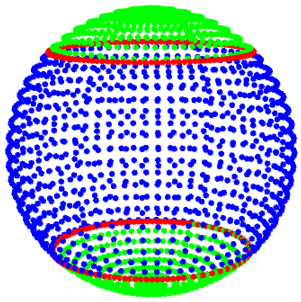

Figure 2.6: Points on the unit sphere in $\mathbb{R}^3$ corresponding to light-like (red), time-like (green) and space-like (blue) vectors.

Example 47. In $\mathbb{R}^3$, a unit vector $\mathbf{v} = (x, y, z)$ is light-like, if $x^2 + y^2 = z^2$. A point with such coordinates lies on a unit sphere and its $z-$coordinate is $z = \pm\frac{1}{\sqrt{2}}$. For time-like vectors, $|z| > \frac{1}{\sqrt{2}}$ and, $z \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ for space-like vectors. See Figure 2.6.

If we are more interested in oriented hyperplanes than in spheres, we might us a model that maps the oriented hyperplanes in $\mathbb{R}^n$ to points in $\mathbb{R}^{n+1}$.

Definition 48. *The* Blaschke model *of Laguerre geometry is the bijective map*

$$\begin{aligned} delta : \mathcal{S}^n &\to \mathbb{R}^{n+1} \\ \delta : \mathcal{H}^n &\to \mathbb{R}^{n+1} \\ a_0 + a_1 x_1 + \cdots + a_n x_n = 0 &\mapsto [a_1, \ldots, a_n, a_0]. \end{aligned}$$

The oriented hyperplanes are mapped to points satisfying $a_1^2 + \cdots + a_n^2 = 1$, which form so called *Blaschke hypercylinder* $\Delta^{n+1} : x_1^2 + \cdots + x_n^2 - 1 = 0 \subset \mathbb{R}^{n+1}$. Oriented spheres are mapped to hyperplanar cuts of the Blaschke hypercylinder, each point of the cut corresponding to a tangent hyperplane of the sphere.

Example 49. Consider a sphere

$$s : (x-1)^2 + (y-1)^2 - 1 = 0 \subset \mathbb{R}^2,$$

we can write all its tangent lines as

$$\begin{aligned} t_a :& (a-1)x - \sqrt{2a - a^2}y + \sqrt{2a - a^2} - a = 0, \qquad (2.1) \\ t'_a :& (a-1)x + \sqrt{2a - a^2}y - \sqrt{2a - a^2} - a = 0, \ \ a \in [0, 2]. \end{aligned}$$

In the Blaschke model, these lines are mapped to the points

$$P_a = [a-1, -\sqrt{2a-a^2}, \sqrt{2a-a^2}-a],$$
$$P'_a = [a-1, \sqrt{2a-a^2}, -\sqrt{2a-a^2}-a],$$

lying on the Blaschke cylinder $\Delta^3 : x^2 + y^2 - 1 = 0 \subset \mathbb{R}^3$.

In Figure 2.7, the tangent lines of the circle s at the points $[\frac{4}{3}, 1 - \frac{2\sqrt{2}}{3}] \in s$ and $[1, 2] \in s$ are

$$t_{\frac{4}{3}} : \frac{1}{3}\left(x - 2\sqrt{2}y + 2\sqrt{2} - 4\right) = 0 \text{ (red)},$$
$$t'_1 : y - 2 = 0 \text{ (green)}$$

These lines $t_{\frac{4}{3}}, t'_1$ are mapped to the points $[\frac{1}{3}, -\frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}-4}{3}]$, $[0, 1, -2] \in \Delta^3$.

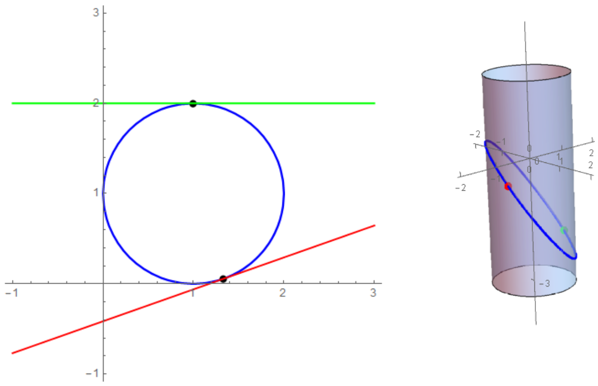

Figure 2.7: Lines in $\mathbb{R}^2$ in the Blaschke model are mapped to points on Blaschke cylinder, circles to planar cuts of the Blaschke cylinder.

Finally, let us introduce the *isotropic model*.

Definition 50. *The* isotropic model *of Laguerre geometry is the map*

$$\Lambda : \mathcal{H}^n \to \mathbb{R}^n \cup \mathbb{R}$$
$$a_0 + a_1 x + \cdots + a_n x_n = 0 :$$
$$(a_1, \ldots, a_n) \neq (0, \ldots, 0, 1) \mapsto \frac{1}{1-a_n}[a_1, \ldots, a_{n-1}, a_0],$$
$$a_0 + x_n \mapsto a_0.$$

Let us justify Definition 50. We embed the euclidean space $\mathbb{R}^n$ into the projective space $\mathbb{P}^n$ by the map $P = [p_1, \ldots, p_n] \in \mathbb{R}^n \mapsto [p_1, \ldots, p_n, 1] \in \mathbb{P}^n$. Let $w \in \mathbb{P}^n$ be the generator line of Δ^{n+1} which contains the point $W = [0, \ldots, 0, 1, 0] \in \mathbb{P}^n$. Consider the hyperplane $\overline{\mathbb{R}}^n : x_n = 0 \in \mathbb{R}^{n+1}$, which is parallel to w and set the coordinate functions in $\overline{\mathbb{R}}^n$ to be $y_1 = x_1, \ldots, y_{n-1} = x_{n-1}, y_n = x_{n+1}$.

Next, consider the stereographic projection $\sigma : \Delta^{n+1} \setminus w \to \overline{\mathbb{R}}^n$. A point $[a_1, \ldots, a_n, a_0] \neq [0, \ldots, 0, 1, a_0]$ of the Blaschke cylinder is projected to the point $\frac{1}{1-a_n}[a_1, \ldots, a_{n-1}, a_0] \in \overline{\mathbb{R}}^n$. We extend the map σ to the points $[0, \ldots, 0, 1, a_0] \in$

Δ^{n+1} and we obtain a map $\overline{\sigma} : [0, \ldots, 0, 1, a_0] \mapsto a_0 \in \mathbb{R}$. Composing $\overline{\sigma}$ with the Blaschke model δ, we obtain the map Λ from the definition. We can write

$$\begin{aligned}
\Lambda : \mathcal{H}^n &\to \overline{\mathbb{R}}^n \cup \mathbb{R} \\
a_0 + a_1 x + \cdots + a_n x_n = 0 : & \\
(a_1, \ldots, a_n) \neq (0, \ldots, 0, 1) &\mapsto \frac{1}{1 - a_n}[a_1, \ldots, a_{n-1}, a_0], \\
a_0 + x_n &\mapsto a_0.
\end{aligned}$$

Definition 51. *The space* $\overline{\mathbb{R}}^n \cup \mathbb{R}$ *is called the* isotropic conformal closure *of* $\mathbb{R}^n$ *and is denoted by* I^n*.*

Example 52. Images of all lines in $\mathbb{R}^2$ in the Blaschke model lie on the Blaschke cylinder $\Delta^3 : x^2 + y^2 - 1 = 0$. Isotropic images of the lines $a_0 + a_1 x + a_2 y = 0$ with $(a_0, a_1, a_2) \neq (a_0, 0, 1)$ lie in the hyperplane $\overline{\mathbb{R}}^2 : y = 0$ (yellow in Figure 2.8) with the coordinate functions x, z. The lines $a_0 + y = 0$ are mapped to $a_0 \in \mathbb{R}$.

Example 53. The line $t_{\frac{4}{3}} : \frac{1}{3}\left(x - 2\sqrt{2}y + 2\sqrt{2} - 4\right) = 0$ from Example 49 is mapped to the point $\frac{1}{1+\frac{2\sqrt{2}}{3}}[\frac{1}{3}, \frac{2\sqrt{2}-4}{3}] \in \overline{\mathbb{R}}^2$ (the red point in Figure 2.8).

The line $t_1' : y - 2 = 0$ has no image in $\overline{\mathbb{R}}^2$, its image in the isotropic model is $\Lambda(t_1') = -2 \in \mathbb{R}$.

As before, we can write the set of all tangent lines of the sphere $s : (x-1)^2 + (y-1)^2 - 1 = 0$ in the form 2.1.The isotropic images of tangent lines are the points

$$\begin{aligned}
P_a &= \frac{1}{1 + \sqrt{2a - a^2}}[a - 1, -a + \sqrt{2a - a^2}] \in \overline{\mathbb{R}}^2, \ a \in [0, 2], \\
P_a' &= \frac{1}{1 - \sqrt{2a - a^2}}[a - 1, -a - \sqrt{2a - a^2}] \in \overline{\mathbb{R}}^2, \ a \in [0, 2] \setminus \{1\}.
\end{aligned}$$

We can observe that these point lie on a hyperbola in $\overline{\mathbb{R}}^2$.

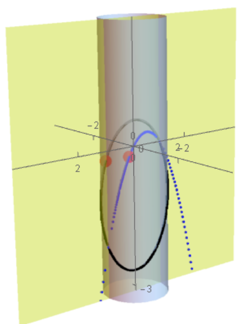

Figure 2.8: A circle as the set of its tangent lines in the Blaschke and the isotropic model. In the isotropic model, the lines form a hyperbola in the plane $y = 0$.

Theorem 54. *The set of all tangent hyperplanes of an oriented sphere*

$$s : (x_1 - c_1)^2 + \cdots + (x_n - c_n)^2 - r^2 \subset \mathbb{R}^n$$

is mapped via the isotropic model Λ *to the set of points of a paraboloid of revolution or a hyperplane* Ψ *satisfying*

$$\Psi : 2y_n + (y_1^2 + \cdots + y_{n-1}^2)(r + c_n) + 2y_1c_1 + \cdots + 2y_{n-1}c_{n-1} + r - c_n = 0$$

extended by the real number $r + c_n$.

Proof. The tangent hyperplane of s at a point $P = [p_1, \ldots, p_n] \in s$ with unit normal vector is given by the equation

$$t_P : \frac{1}{r}\left((p_1 - c_1)(x_1 - p_1) + \cdots + (p_n - c_n)(x_n - p_n)\right) = 0.$$

If $\frac{p_n - c_n}{r} \neq 1$, its isotropic image is the point

$$\Lambda(t_P) = \frac{1}{1 - \frac{p_n + c_n}{r}} \frac{1}{r} \left[p_1 - c_1, \ldots, p_{n-1} - c_{n-1}, c_1p_1 - p_1^2 + \cdots + c_np_n - p_n^2\right],$$

which vanishes on Ψ.

If $\frac{p_n - c_n}{r} = 1$, then

$$t_P : x_n - p_n = 0,$$

and its image is the real number $p_n = r + c_n$.

On the other hand, consider a point $Q \in \Psi$,

$$Q = [q_1, \ldots, q_n] = \frac{1}{q}[qq_1, \ldots, qq_n],$$

where q is the common denominator of $q_1, \ldots, q_n$. Then Q is the isotropic image of the hyperplane

$$\begin{aligned} &qq_n + qq_1x + \cdots + qq_{n-1}x_{n-1} + (1 + q)x_n \\ &= \frac{1}{r}\left(qq_nr + qq_1rx + \cdots + qq_{n-1}rx_{n-1} + (1 + q)rx_n\right), \end{aligned}$$

which is the tangent hyperplane of s at the point

$$P = [c_1 + qq_1r, \ldots, c_{n-1} + qq_{n-1}r, r + qr - c_n] \in s.$$

□

Example 55. Consider a sphere $(x - 1)^2 + (y + 1)^2 - 1 = 0$. The set of all its tangent lines is

$$\begin{aligned} t_a &: (a - 1)x - \sqrt{2a - a^2}y - a - \sqrt{2a - a^2} = 0, \\ t'_a &: (a - 1)x + \sqrt{2a - a^2}y - a + \sqrt{2a - a^2} = 0, \ a \in [0, 2]. \end{aligned}$$

The isotropic image of the lines $t_a,\ a \in [0,2]$ and of the lines $t'_a,\ a \in [0,2]\setminus\{1\}$ are the points

$$P_a = \frac{1}{1+\sqrt{2a-a^2}}[a-1, -a-\sqrt{2a-a^2}],$$
$$P'_a = \frac{1}{1-\sqrt{2a-a^2}}[a-1, -a+\sqrt{2a-a^2}],$$

satisfying

$$\Psi : 2z + x^2(1+(-1)) + 2x + 1 - (-1) = 2z + 2x + 2 = 0 \subset \overline{\mathbb{R}}^2.$$

The tangent line $t'_1 : y = 0$ is mapped to the real number 0 The line Ψ (green), and the line t'_1 together with its Blaschke images (red) are shown in Figure 2.9.

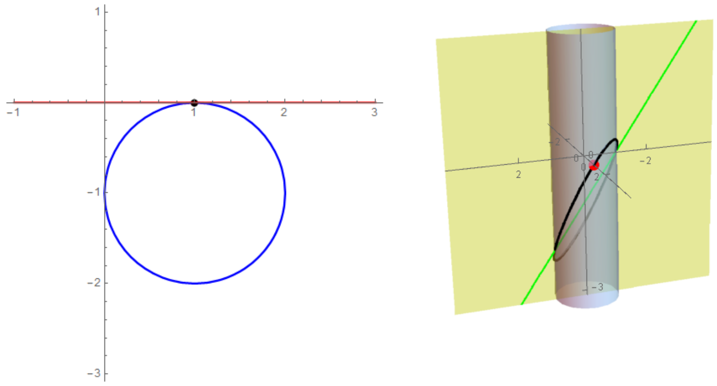

Figure 2.9: A circle in $\mathbb{R}^2$ and its tangent line (left). Blaschke (black and red) and isotropic (green) images of the circle and the line (right).

It can be shown that rational surfaces in the isotropic model are images of rational surfaces in the *standard model* of $\mathbb{R}^n$. Even more general result is proven in [7, Theorem 1.9].

Theorem 56. *A preimage of a rational surface in the isotropic model is a rational surface.*

2.2.2 One parameter systems of spheres in cyclographic model

In the cyclographic model, oriented spheres in $\mathbb{R}^n$ are mapped to points in $\mathbb{R}^{n+1}$. One parameter systems of spheres can hence be seen as curves in $\mathbb{R}^{n+1}$.

Before we continue with one parameter systems of spheres in $\mathbb{R}^3$, let us begin with a simple planar example.

Example 57. Consider the system $\mathcal{F}^1_t$

$$\mathcal{F}^1_t = \{(x,y) \in \mathbb{R}^2, \left(x-t^3\right)^2 + \left(y-t^2\right)^2 - \frac{1}{4} = 0,\ t \in \mathbb{R}\}$$

of circles with constant radius $\frac{1}{2}$, whose centres lie on a planar curve $\mathbf{c}(t) = (t^3, t^2),\ t \in \mathbb{R}$.

The curve $\mathbf{c}_1 \subset \mathbb{R}^3$ corresponding to the system $\mathcal{F}_t^1$ in the cyclographic model is the curve $\mathbf{c}_1(t) = (t^3, t^2, \frac{1}{2}),\ t \in \mathbb{R}$.

A system $\mathcal{F}_t^2$

$$\mathcal{F}_t^2 = \{(x, y) \in \mathbb{R}^2, \left(x - t^3\right)^2 + \left(y - t^2\right)^2 - \frac{t^2}{4} = 0,\ t \in \mathbb{R}\}$$

is a one parameter system of circles whose radii are not constant and whose centres lie on the same curve $\mathbf{c}(t)$ as in the previous case. The spheres corresponding to $t < 0$ are negatively oriented and the orientation changes for circles corresponding to $t > 0$. For $t = 0$ the element $F_0 \in \mathcal{F}_t$ is point $[0, 0] \in \mathbf{c}(t)$, which is a sphere with zero radius and no orientation.

A curve in $\mathbb{R}^3$ corresponding to this system in the cyclographic model is the curve $\mathbf{c}_2(t) = (t^3, t^2, \frac{t}{2}),\ t \in \mathbb{R}$.

The curve $\mathbf{c}(t)$ is the orthogonal projection of both the curves $\mathbf{c}_1$ and $\mathbf{c}_2$ and is often called the *medial axis* of the one parameter system.

See Figure 2.10 for the curves $\mathbf{c}_1, \mathbf{c}_2 \subset \mathbb{R}^3$, one parameter systems $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ of planar circles and their medial axes.

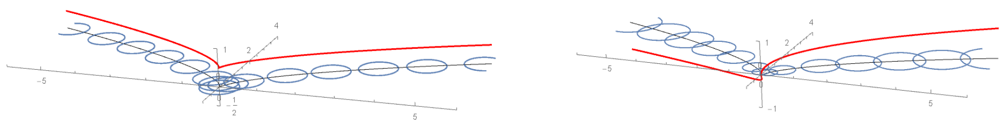

Figure 2.10: One parameter systems of circles with constant (left) and non-constant (right) radii. the red curves are the corresponding curves in the cyclographic models.

In the same manner we can define one parameter system of spheres in $\mathbb{R}^3$ as images of curves in $\mathbb{R}^4$. Let us illustrate it in two simple examples, where the curves are straight lines.

Example 58. Let us examine one parameter system $\mathcal{F}_t$ of spheres in $\mathbb{R}^3$ with constant radii,

$$\mathcal{F}_t = \{(x, y, z) \in \mathbb{R}^3, (x - t)^2 + (y - t)^2 + (z - t)^2 - 1 = 0,\ t \in \mathbb{R}\},$$

see Figure 2.11. The one parameter system $\mathcal{F}_t$ is the cyclographic image of the curve $(t, t, t, 1) \subset \mathbb{R}^4,\ t \in \mathbb{R}$.

Using the characterisation 6 we can compute envelope of the system by eliminating the variable t from the following equations

$$f(x, y, z, t) = (x - t)^2 + (y - t)^2 + (z - t)^2 - 1 = 0,$$
$$\frac{\partial f}{\partial t}(x, y, z, t) = 2(x + y + z) - 6t = 0.$$

Since the second equation is linear, the task is trivial and we obtain the following equation of the envelope,

$$\chi(x, y, z) = 2x^2 - 2x(y + z) + 2y^2 - 2yz + 2z^2 - 3 = 0,$$

which defines a cylinder in $\mathbb{R}^3$. In general, a cylinder of revolution can be described as the envelope of a one parameter system of spheres with constant radii.

A characteristic curve χ_0 is defined as the intersection of the sphere given by the equation $f(x,y,z,0) = x^2+y^2+z^2-1 = 0$ and the tangent plane $\frac{\partial f}{\partial t}(x,y,z,0) = x+y+z = 0$.

The envelope is shown in Figure 2.11 together with the tangent plane $x+y+z=0$ for $t=0$ and the characteristic curve χ_0.

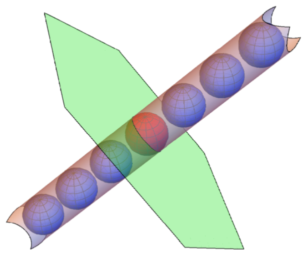

Figure 2.11: A cylinder of a revolution as an envelope of a one parameter system of spheres.

Example 59. Consider once again the simple translation of a sphere, in this case the spheres are moving along the z axis, but now with non-constant radius function. The one parameter system is then defined as

$$\mathcal{F}_t = \{(x,y,z) \in \mathbb{R}^3,\ x^2+y^2+(z-t)^2 - \frac{t^2}{26} = 0,\ t \in \mathbb{R}\}.$$

We have already seen this one parameter system in Example 9 where we concluded that the envelope surface of this system is a cone of revolution with implicit equation

$$x^2+y^2-\frac{1}{25}z^2 = 0.$$

The curve in $\mathbb{R}^4$ corresponding to this system is the parametric curve $c(t) = (0,0,t,\frac{t}{\sqrt{26}})$.

Remark. Generally, one can define one parameter systems of cones and cylinders of revolution two parameter systems of spheres. We use this fact in Section 2.2.3 when proving rationality of the envelope of these systems. The two parameter systems of spheres and their envelopes are discussed in detail for example in [16] or [17].

In the previous examples we have seen that the image of a line $\mathbf{c}(t) = (c_1,\dots,c_{n+1}) \subset \mathbb{R}^{n+1}$ under the cyclographic mapping is a cone or a cylinder of revolution, depending on whether there is a value of t such that $c_{n+1}(t) = 0$.

In case when the curve in $\mathbb{R}^4$ is of a higher degree, the envelope of the corresponding one parameter system of spheres gets also more complicated or the system might even not posses a real envelope. In fact, even if the curve is a line, the one parameter system does not always have a real envelope.

Example 60. Consider a one parameter system of spheres given as the cyclographic image of the curve $\mathbf{c}(t) = (\frac{t}{2}, t, -t, \frac{5t}{2}) \subset \mathbb{R}^4$. Centres of the spheres lie on the line $(\frac{t}{2}, t, -t) \subset \mathbb{R}^3$. Few spheres of the system together with the medial axis are shown in Figure 2.13. To find the envelope surface of this system, we first write down the implicit equation of the system:

$$f(x, y, z, t) = (x - \frac{t}{2})^2 + (y - t)^2 + (z + t)^2 - \frac{25t^2}{4} = 0.$$

From the partial derivative of F with respect to t,

$$\frac{\partial f}{\partial t}(x, y, z, t) = -x - 2y + 2z - 8t = 0,$$

we obtain t and if we put its value into the equation $f(x, y, z, t) = 0$, we obtain the implicit equation

$$\chi(x, y, z) = 17x^2 + 4xy - 4xz + 20y^2 - 8yz + 20z^2.$$

One can verify that apart from the point $(0, 0, 0)$ there are no real solutions satisfying this equation, thus we can say, that the system does not have a real envelope.

Let us discuss whether a one parameter system of spheres defined as the cyclographic image of a curve $\mathbf{c}(t) \subset \mathbb{R}^{n+1}$ posses a real envelope surface. Let us begin with the case we have seen in the previous examples, where the curve $\mathbf{c}(t)$ is a line with the directional vector $\mathbf{a} \in \mathbb{R}^{n+1}$. Then the corresponding one parameter system possesses a real envelope if the vector is space-like. We will demonstrate the derivation of this fact in case $n = 2$.

Consider a line $\mathbf{c} \subset R^3$, $\mathbf{c}(t) = A + t\mathbf{a}$, where

$$A = [A_1, A_2, A_3],$$
$$\mathbf{a} = (a_1, a_2, a_3).$$

The one parameter system is then defined by the equation

$$(x - (A_1 + ta_1))^2 + (y - (A_2 + ta_2))^2 - (A_3 + ta_3)^2 = 0. \quad (2.2)$$

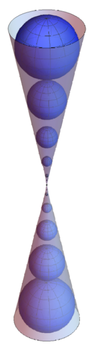

Figure 2.12: A cone of revolution as the envelope of one parameter system of spheres.

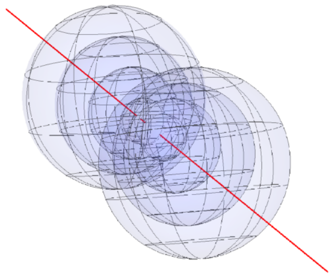

Figure 2.13: One parameter system of spheres with no real envelope.

Differentiating the equation with respect to t, we obtain

$$-2a_1(x-(A_1+ta_1))-2a_2(y-(A_2+ta_2))+2a_3(A_3+ta_3)=0.$$

We can eliminate t from the second equation and we get

$$t=\frac{1}{a_1^2+a_2^2-a_3^2}(-a_1A_1+a_1x-a_2A_2+a_2y+a_3A_3). \tag{2.3}$$

Directly we can see that for the system to have a real envelope, it must hold $\langle \mathbf{a},\mathbf{a}\rangle \neq 0$. Thus assume this holds and put the expression 2.3 into 2.2. We obtain the implicit equation of the envelope of the system,

$$\begin{aligned}\chi = &\left(x-A_1-\frac{a_1(-a_1A_1+a_1x-a_2A_2+a_2y+a_3A_3)}{\langle \mathbf{a},\mathbf{a}\rangle}\right)^2\\ &+\left(y-A_2-\frac{a_2(a_1A_1+a_1x-a_2A_2+a_2y+a_3A_3)}{\langle \mathbf{a},\mathbf{a}\rangle}\right)^2\\ &-\left(\mathrm{A3}+-\frac{a_3(a_1A_1+a_1x-a_2A_2+a_2y+a_3A_3)}{\langle \mathbf{a},\mathbf{a}\rangle}\right)^2=0.\end{aligned}$$

For a time-like vector $\mathbf{a}$, there are no real points $(x,y)\in\mathbb{R}^2$ satisfying the equation $\chi=0$, thus we conclude, that only systems of spheres that are images of hyperbolic lines possesses real envelopes.

For one parameter system of spheres in higher dimensions and for those that corresponds to a curves of higher degree, the following result holds. One can find it in [15, Section 2.1].

Theorem 61. *Cyclographic image of a rational curve $\mathbf{c}(t)\subset\mathbb{R}^{n+1}$ possesses a real envelope if and only if $\langle c'(t),c'(t)\rangle\geq 0$ and the equality holds only for isolated values of t.*

Definition 62. *A one parameter system of spheres in $\mathbb{R}^n$ is called* rational, *if it is the cyclographic image of a rational curve in $\mathbb{R}^{n+1}$. We say a rational one parameter system of spheres is* real *if it possesses a real envelope.*

Example 63. Consider a curve

$$\mathbf{c}(t) = \left(\frac{26t}{t^2+1}, \frac{12\,(1-t^2)}{t^2+1}, 0, 6 - \frac{10t}{t^2+1} \right) \subset \mathbb{R}^4.$$

The curve defines a one parameter system of spheres

$$\mathcal{F}_t = \{(x,y,z) \in \mathbb{R}^3, f(x,y,z,t) = 0,\ t \in \mathbb{R}\},$$

where

$$f(x,y,z,t) = \left(x - \frac{26t}{t^2+1}\right)^2 + \left(y - \frac{12\,(1-t^2)}{t^2+1}\right)^2 + z^2 - \left(6 - \frac{10t}{t^2+1}\right)^2 = 0.$$

Some elements of the system are shown in Figure 2.14.

We compute tangent vectors $\mathbf{t}(t) = \mathbf{c}'(t)$ of the curve $\mathbf{c}(t)$ and the inner product $\langle \mathbf{t}, \mathbf{t} \rangle$, we get

$$\langle \mathbf{t}(t), \mathbf{t}(t) \rangle = \frac{576}{(t^2+1)^2},$$

which is positive for every $t \in \mathbb{R}$. Thus we conclude that the system $\mathcal{F}_t$ possesses a real envelope.

Next we show that the envelope of this system is a Dupin cyclide. The partial derivative of the equation above representing the tangent surfaces of the system, reads as follows

$$\frac{\partial f}{\partial t}(x,y,z,t) = \frac{4\,(t^2(13x-30) + 24ty - 13x + 30)}{(t^2+1)^2} = 0.$$

Thanks to the Lemma 7 we can get rid of the denominators to simplify the expressions. Once again it is easy to eliminate the variable t from the equations, since the second one is of degree two. One of the solutions leads to an implicit equation representing a Dupin cyclide, see Figure 2.14.

Let us study the characteristic curves of the system. For each element of the system the characteristic curve is defined as the intersection of the sphere $F(x,y,z,t) = 0$ and $\frac{\partial f}{\partial t}(x,y,z,t) = 0$ at the corresponding value of t. For instance, for $t = 0$ we have

$$f(x,y,z,0) = x^2 + (y-12)^2 + z^2 - 36 = 0,$$

$$\frac{\partial f}{\partial t}(x,y,z,0) = 30 - 13x = 0.$$

The characteristic curve at $t = 0$ is then simply given as

$$\chi_0 = \{(x,y,z) \in \mathbb{R}^3, x = \frac{30}{13} \wedge (y-12)^2 + z^2 - \frac{5184}{169} = 0\}.$$

The sphere $f(x,y,z,0) = 0$, the plane $\frac{\partial f}{\partial t}(x,y,z,0) = 0$ and the characteristic curve χ_0 are shown in Figure 2.14.

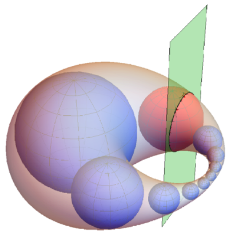

Figure 2.14: Envelope of the one parameter system of spheres with non-constant radius $\mathcal{F}_t$ is a Dupin cyclide. A sphere of the system for $t = 0$ (red), the tangent plane of the system at $t = 0$ (green) and the characteristic curve χ_0 (black).

2.2.3 Rational parameterization of envelopes

In this section, we finally present the proof of the fact that envelopes of one parameter systems of *natural quadrics* posses rational envelopes. The proof can be found in [7, Thm 3.1].

Definition 64. Natural quadrics *in* $\mathbb{R}^3$ *are spheres, cylinders and cones of revolution.*

In the cyclographic model of Laguerre geometry, oriented spheres and points in $\mathbb{R}^3$ are mapped into points in $\mathbb{R}^4$. We can represent cones and cylinders of revolution as one parameter systems of spheres (either with changing or constant radii) and then the images of these quadrics in the cyclographic model are lines whose points correspond to the spheres of the systems (see examples 58 and 59).

One parameter systems of cones or cylinders of revolution in $\mathbb{R}^3$ are then two parameter systems of spheres and their images in the cyclographic model are ruled surfaces in $\mathbb{R}^4$. The rulings of the ruled surface correspond to the cones or cylinders of revolution - the elements of the one parameter systems.

Example 65. Let us again start with a two dimensional example. In $\mathbb{R}^2$ by a cone we mean a set of two lines which intersect in one point. Consider a ruled surface

$$\mathbf{f}(u, v) = (1 - v)(u + 1, 2u + 1, \frac{1}{3}u + 1) + v(u - 5, u + 5, u + 5) \subset \mathbb{R}^3.$$

For a fixed u, $\mathbf{f}(u, v)$ is a line. Its cyclographic image is the one parameter set of spheres with a cone as its envelope. In Figure 2.15 there is the ruled surface and envelopes of the cyclographic images of $\mathbf{f}(-5, v)$, $\mathbf{f}(-3, v)$, $\mathbf{f}(0, v)$ and $\mathbf{f}(2, v)$ and few elements of the corresponding one parameter systems.

Equivalently, a cone or cylinder of revolution F can be also defined as the common tangent cone of two spheres S_1, S_2. This way, we can construct a one parameter system of natural quadrics as the system of common tangent quadrics (cones, cylinders or spheres) of two one parameter systems of spheres

$$\mathcal{F}_t^1 = \{(x, y, z) \in \mathbb{R}^3, S_1(t) = 0,\ t \in I\},$$
$$\mathcal{F}_t^2 = \{(x, y, z) \in \mathbb{R}^3, S_2(t) = 0,\ t \in I\}$$

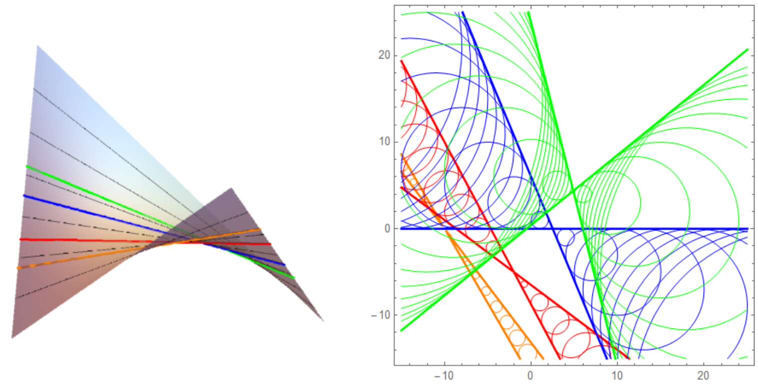

Figure 2.15: A ruled surface and four elements of the one parameter system of cones defined as cyclographic image of the surface.

both depending on the same parameter $t \in I \subseteq \mathbb{R}$.

For fixed t, the corresponding ruling of the ruled surface in $\mathbb{R}^4$ is the line

$$f(u,t) = (1-u)\zeta(s_1) + u\zeta(s_2),\ u \in \mathbb{R},$$

where ζ is the cyclographic map from 42.

Definition 66. *A one parameter system of natural quadrics is called* real rational system *if it is a cyclographic image of a rational ruled surface with only hyperbolic and isolated parabolic generators.*

The one parameter systems from the definition above posses rational envelope surfaces.

Theorem 67. *[7, Thm 3.1] The envelope of real rational one parameter system of cones and cylinders of revolution is rational surface.*

Proof. Every one parameter system $\mathcal{F}_t$ of cones or cylinders of revolution is determined by two one parameter sets $\mathcal{S}_t^1$ and $\mathcal{S}_t^2$ of spheres, both dependent of the same parameter t. For a cylinder, the spheres differ in their centres, but their radii are equal and they coincide with the radius of the cylinder. A cone is determined by two distinct spheres, such that the cone is tangent to both of them. We prove the theorem for the most general case when the elements of $\mathcal{F}$ can be spheres, cylinders or cones, but for simplicity, we refer to the elements only as *cones*. We suppose the spheres do not coincide.

Consider a one parameter system of cones $\mathcal{F}_t$ generated by two real rational sets of oriented spheres $\mathcal{S}_t^1$, $\mathcal{S}_t^2$ defined by implicit equations $s_1(t)$ and $s_2(t)$:

$$s_i(t) : (x - c_{i,1}(t))^2 + (y - c_{i,2}(t))^2 + (z - c_{i,3}(t))^2 - r_i^2(t) = 0,\ i \in \{1,2\}.$$

For a fixed t, denote $s_i(t) \in \mathcal{S}_t^i$ the sphere in $\mathcal{S}_t^i$. We can write the set of all tangent planes to $s_i(t)$ at points $P = [p_1, p_2, p_3] \in s_i(t)$ as

$$T_i = \{t_i(P),\ P \in s_i(t)\},$$

where

$$t_i(P) : \frac{1}{r_i}\left((p_1 - c_{i,1})(x - p_1) + (p_2 - c_{i,2})(y - p_2) + (p_3 - c_{i,3})(z - p_3)\right) = 0.$$

The image of the set of tangent planes T_i in the isotropic model is a paraboloid of revolution (see Theorem 54) given by implicit equation

$$\Psi_i(t) = 2y_3 + (y_1^2 + y_2^2)(r_i(t) + c_{i,3}(t)) + 2y_1 c_{i,1}(t) + 2y_2 c_{i,2}(t) + r_i(t) - c_{i,3}(t) = 0. \tag{2.4}$$

Denote $\iota(t) = \Psi(t) \cap \Psi(t)$ the intersection of the paraboloid of revolution. Then $\iota(t)$ is the isotropic image of the planes that are tangent to both $s_1(t)$ and $s_2(t)$. These are exactly tangent hyperplanes of the cone $F(t)$ which is tangent to $s_1(t)$ and $s_2(t)$.

Denote $R(t) = r_1 - r_2$, $C_i(t) = c_{1,i} - c_{2,i}$ for $i = 1, 2, 3$, and project the intersection onto the plane $y_3 = 0$:

$$p(t) = \pi(\iota(t)) : (y_1^2 + y_2^2)(R + C_3) + 2y_1 C_1 + 2y_2 C_2 + R - C_3 = 0. \tag{2.5}$$

If $R + C_3 \neq 0$, $p(t)$ is an euclidean circle and if we rewrite it as

$$p(t) : \left(y_1 + \frac{C_1}{R + C_3}\right)^2 + \left(y_2 + \frac{C_2}{R + C_3}\right)^2 = \frac{1}{(R + C_3)^2}(C_1^2 + C_2^2 + C_3^2 - R^2) \tag{2.6}$$

we can see that centre of the circle is the point

$$(n_1(t), n_2(t)) = \left(-\frac{C_1}{R + C_3}, -\frac{C_2}{R + C_3}\right)$$

and the radius is generally a non-rational function satisfying

$$\rho^2 = \frac{1}{(R + C_3)^2}(C_1^2 + C_2^2 + C_3^2 - R^2).$$

We want to provide rational parameterization of the euclidean circles $p(t)$. We assume the envelope of the system is real, thus $\rho(t) \geq 0$ (and the expression equals zero only for finitely many values of t), therefore we can find two rational functions $\rho_1(t)$, $\rho_2(t)$ satisfying $\rho_1^2(t) + \rho_2^2(t) = \rho^2(t)$. The proof of this fact together with an algorithm for computing ρ_1 and ρ_2 can be found in [18]. Having these two rational functions, we can form a planar rational curve $f(t) = (n_1 + \rho_1, n_2 + \rho(2))$. Points of this curve satisfy the equation 2.6, thus they lie a on the circles $p(t)$.

For a fixed t, $f(t)$ is a point on the circle $p(t)$. We now parameterize the circle $p(t)$ using stereographic projection. Write $p(t)$ as

$$p(t) = (x - n_1(t))^2 + (y - n_2(t))^2 = \rho^2(t). \tag{2.7}$$

Consider a point $P = [u, 1]$. Then the line l connecting $f(t)$ and P intersects the circle $p(t)$ at two points, $f(t)$ and $z(t)$. To obtain all the points $z(t) \in p(t)$, we assume $u \in \mathbb{P}^1$. To compute the points $z(t)$, we parameterize the line $l(s)$ as

$$l(s) = f(t) + s \cdot (f(t) - P),$$

and plug the parameterization into equation 2.7 of the circle $p(t)$. We obtain the following quadratic equation in s

$$(s(n_1 - \rho_1 + u) + \rho_1)^2 + (s(-n_2 - \rho_2 + 1) + \rho_2)^2 = \rho^2,$$

which we can solve and we get two solutions; $s = 0$, corresponding to the point $f(t) \in p(t)$ and

$$s = \frac{(2(n_1\rho_1 + \rho_1^2 + \rho_2(-1 + n_2 + \rho_2) - \rho_1 u))}{(1 + n_1^2 + n_2^2 + \rho_1^2 + 2n_2(-1 + \rho_2) - 2\rho_2 + \rho_2^2 + 2n_1(\rho_1 - u) - 2\rho_1 u + u^2)}.$$

This value of s corresponds to the point $z(t, u) \in p(t)$ and we have

$$z(t,u) = \begin{pmatrix} z_1(t,u) \\ z_2(t,u) \end{pmatrix} = \begin{pmatrix} f_1 - \frac{2(f_1-u)(r_1(f_1-u)+r_2(f_2-1))}{(f_1-u)^2+(f_2-1)^2} \\ f_2 - \frac{2(f_2-1)(r_1(f_1-u)+r_2(f_2-1))}{(f_1-u)^2+(f_2-1)^2} \end{pmatrix} \tag{2.8}$$

$$= 2\,\frac{\langle \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \begin{pmatrix} f_1 - u \\ f_2 - 1 \end{pmatrix} \rangle}{\left\| \begin{pmatrix} f_1 - u \\ f_2 - 1 \end{pmatrix} \right\|^2} \begin{pmatrix} f_1 - u \\ f_2 - 1 \end{pmatrix} \tag{2.9}$$

Hence we have a rational parameterization of the circle $p(t)$.

The case when $R + C_3 = 0$ the projection $p(t)$ from the equation 2.5 collapses into a line

$$p(t) : 2y_1C_1 + 2y_2C_2 + R - C_3 = 0,$$

which can be rationally parameterized as follows

$$z(t,u) = \begin{pmatrix} z_1(t,u) \\ z_2(t,u) \end{pmatrix} = Q + u \begin{pmatrix} -C_2 \\ C_1 \end{pmatrix},$$

where $Q \in p(t)$ is any point on the line $p(t)$.

By plugging $y_1 = z_1$ and $y_2 = z_2$ into equation 2.4 we can compute y_3, hence we found rational parameterization of the set of all tangent planes of the one parameter system $\mathcal{F}_t$ in the isotropic model. Hence, from Theorem 56, the tangent planes to $\mathcal{F}_t$ are rational and from the discussion at the end of the previous section the envelope surface of $\mathcal{F}_t$ is a rational surface. □

Remark. In fact one parameter systems of all natural quadrics posses rational envelopes. In the case of one parameter system of spheres, however, the proof has to be done differently, since the two paraboloids from 2.4 coincide. The projection of their intersection is thus the whole plane.

The envelopes of one parameter systems are called canal surfaces (if the radii of the spheres vary) of pipe surfaces, if the radii are constant. The fact that canal (and pipe) surfaces are rational is proven for example in [2], where also the parameterization of the surface is constructed.

Example 68. Let us recall the one parameter system of cones from Example 3. We can construct two one parameter systems of spheres $\mathcal{S}_t^1$, $\mathcal{S}_t^2$ defined by implicit equations

$$s_1(t):\left(-\frac{16(2t+1)(12t-29)}{1108t^2+788t+517}-t+x-\frac{3}{2}\right)^2$$
$$+\left(\frac{4(2t+1)(46t-57)}{1108t^2+788t+517}-t+y-\frac{1}{2}\right)^2$$
$$+\left(\frac{972t^2+1292t+3}{1108t^2+788t+517}-3t+z-\frac{3}{2}\right)^2-\frac{1}{26}=0,$$
$$s_2(t):\quad\left(\frac{32(2t+1)(12t-29)}{1108t^2+788t+517}-t+x-\frac{3}{2}\right)^2$$
$$+\left(-\frac{8(2t+1)(46t-57)}{1108t^2+788t+517}-t+y-\frac{1}{2}\right)^2$$
$$+\left(-\frac{2\left(972t^2+1292t+3\right)}{1108t^2+788t+517}-3t+z-\frac{3}{2}\right)^2-\frac{2}{13}=0.$$

Then every cone $F_t = F(t)$ of the system $\mathcal{F}_t$ is tangent to the pair of spheres $s_1(t)$ and $s_2(t)$, see Figure 2.16.

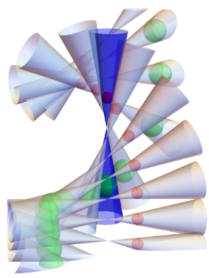

Figure 2.16: One parameter system of cone $\mathcal{F}_t$ defined by two one parameter systems of spheres with constant radii. Cone $F(-\frac{1}{2})$ is highlighted in blue

.

For $t = -\frac{1}{2}$ we have

$$s_1(-\frac{1}{2}):(x-1)^2+y^2+(z-1)^2-\frac{1}{26}=0,$$
$$s_2(-\frac{1}{2}):(x-1)^2+y^2+(z+2)^2-\frac{2}{13}=0$$

and their common tangent cone

$$F(-\frac{1}{2}):x^2-2x+y^2-\frac{z^2}{25}+1=0.$$

The sets of all tangent planes of $s_1(-\frac{1}{2})$ and $s_2(-\frac{1}{2})$ in the isotropic model are the paraboloids of revolution $\Psi_1(-\frac{1}{2})$ and $\Psi_2(-\frac{1}{2})$, see Figure 2.17:

$$\begin{aligned}\Psi_1(-\frac{1}{2}):&\quad \left(1+\frac{1}{\sqrt{26}}\right)\left(y_1^2+y_2^2\right)+2y_1+2y_3+\frac{1}{\sqrt{26}}-1=0 \\ \Psi_2(-\frac{1}{2}):&\quad \left(\sqrt{\frac{2}{13}}-2\right)\left(y_1^2+y_2^2\right)+2y_1+2y_3+\sqrt{\frac{2}{13}}+2=0.\end{aligned} \tag{2.10}$$

And their intersection $\iota(-\frac{1}{2})$ corresponds to the isotropic image of the tangent planes of the cone $F(-\frac{1}{2})$.

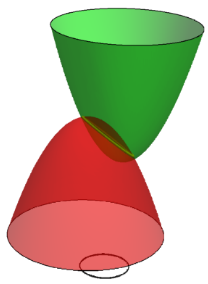

Figure 2.17: Images of spheres $s_1(-\frac{1}{2})$ and $s_2(-\frac{1}{2})$ in the isotropic model and the projection of their intersection $p(-\frac{1}{2})$.

Projecting $\iota(-\frac{1}{2})$ to the plane $y_3=0$, we obtain circle

$$p(-\frac{1}{2}):\left(y_1^2+y_2^2\right)-1=0,$$

which is a circle centred at the point $[0,0]$ with radius 1.

For general t, in this case, the projection to $y_3=0$ of the intersection of the two paraboloids of revolution $\Psi_1(t)$ and $\Psi_2(t)$ is

$$\begin{aligned}p(t):-\frac{1}{1108t^2+788t+517}&\left(-2916t^2x^2+2304t^2x-2916t^2y^2-2208t^2y\right.\\ &+2916t^2-3876tx^2-4416tx-3876ty^2+1632ty+3876t-9x^2-2784x\\ &\left.-9y^2+1368y+9\right)=0,\end{aligned}$$

a circle centred at the point

$$[n_1(t),n_2(t)]=\left[\frac{16(2t+1)(12t-29)}{972t^2+1292t+3},-\frac{4(2t+1)(46t-57)}{972t^2+1292t+3}\right]$$

with radii $\rho(t)$ satisfying

$$\rho^2(t)=\frac{\left(1108t^2+788t+517\right)^2}{\left(972t^2+1292t+3\right)^2}.$$

One can thus choose

$$\rho_1 = \rho = \frac{(1108t^2 + 788t + 517)}{972t^2 + 1292t + 3},$$
$$\rho_2 = 0.$$

We can thus rationally parameterise all the circles $p(t)$ as

$$z(t,u) = \begin{pmatrix} z_1(t,u) \\ z_2(t,u) \end{pmatrix} = \frac{1}{c}\begin{pmatrix} a \\ b \end{pmatrix},$$

where

$$\begin{aligned} a = {} & 16t^4\left(43983u^2 - 135026u - 68632\right) + 32t^3\left(75523u^2 - 144466u + 24168\right) \\ & + 8t^2\left(365589u^2 - 395318u + 83704\right) + 8t\left(159003u^2 - 32946u + 6568\right) \\ & + 3\left(981u^2 - 34662u + 8056\right), \end{aligned}$$

$$\begin{aligned} b = {} & -96t^4\left(3726u^2 + 19493u - 31619\right) - 64t^3\left(3298u^2 + 80399u - 44797\right) \\ & + 16t^2\left(35746u^2 - 197757u + 120611\right) + 16t\left(18462u^2 - 47039u - 6923\right) \\ & + 684u^2 + 208482u - 49166, \end{aligned}$$

and

$$\begin{aligned} c = {} & 16t^4\left(59049u^2 - 181278u + 251354\right) + 32t^3\left(78489u^2 - 123638u + 90274\right) \\ & + 8t^2\left(209387u^2 - 30794u + 74782\right) + 8t\left(969u^2 - 17158u - 56686\right) \\ & + 9u^2 - 318u + 53434. \end{aligned}$$

From Ψ_1 or Ψ_2 from 2.10 we compute $z_3(t,u) = y_3$,

$$z_3(t,u) = \frac{d}{e},$$

where

$$\begin{aligned} d = {} & 38719200t^5u^2 - 129351040t^5u - 51671360t^5 \\ & + \left(125677968 - 802192\sqrt{26}\right)t^4u^2 - \left(71275776 - 1338464\sqrt{26}\right)t^4u \\ & - \left(3306272\sqrt{26} + 75893792\right)t^4 - \left(2259104\sqrt{26} - 91026416\right)t^3u^2 \\ & + \left(166315968 + 4417728\sqrt{26}\right)t^3u - \left(2466624\sqrt{26} + 112631584\right)t^3 \\ & - \left(2662168\sqrt{26} + 27623128\right)t^2u^2 + \left(205182016 + 3601296\sqrt{26}\right)t^2u \\ & - \left(1742128\sqrt{26} + 85417488\right)t^2 - \left(1560936\sqrt{26} + 54305706\right)tu^2 \\ & + \left(81901352 + 1981232\sqrt{26}\right)tu - \left(137296\sqrt{26} + 17751396\right)t \\ & - \left(507177\sqrt{26} + 19673667\right)u^2 + \left(2537184 + 238854\sqrt{26}\right)u \\ & - 54802\sqrt{26} + 1528358 \end{aligned}$$

$$
\begin{aligned}
e &= 416t^4 \left(59049u^2 - 181278u + 251354\right) + 832t^3 \left(78489u^2 - 123638u + 90274\right) \\
&+ 208t^2 \left(209387u^2 - 30794u + 74782\right) + 208t \left(969u^2 - 17158u - 56686\right) \\
&+ 234u^2 - 8268u + 1389284
\end{aligned}
$$

Hence we have the parameterization of all tangent planes of the system $\mathcal{F}_t$ and thus the parameterization of the envelope surface of $\mathcal{F}_t$ in the isotropic model. From Theorem 56 we know that then also the envelope in the standard model possesses a rational parameterization.

3. Kinematic approach to envelope surfaces

A way to understand one parameter systems of surfaces in $\mathbb{R}^n$ (we will mostly work with $n = 3$) is to look at them as at the set of all transformations of a given surface $\overline{F}$. The surface $\overline{F}$ is transformed into other elements of the system by via elements of a suitable group of transformations. Such groups have, beside the structure of a group, also the structure of a smooth manifold. We call these groups *Lie groups*. In this chapter we use this approach to describe characteristic curves on one parameter systems of surfaces and we provide their parameterization in special cases.

3.1 Examples

Example 69. Let us illustrate the concept on the simple planar example 2. We choose a line $l : x = 1$ and transform it via a map g_t :

$$g_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2(t+1)}{(t+1)^2+1} & \frac{(t+1)^2-1}{(t+1)^2+1} \\ \frac{1-(t+1)^2}{(t+1)^2+1} & \frac{2(t+1)}{(t+1)^2+1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For each $t \in \mathbb{R}$, g_t is a rotation. The group of all rotations is denoted by $\mathbb{SO}(2)$ and it is called the special orthogonal group.

For $t = 0$ we obtain the line l. As another example, for $t = -\frac{1}{2}$ we obtain the line $\frac{4}{5}x + \frac{3}{5}y - 1 = 0$ (the yellow and red lines in Figure 1.1).

Example 70. Let $\mathcal{F}_t$ be the one parameter system from our running example 3. Let $\overline{F} : x^2 + y^2 - \frac{1}{25}z^2 = 0$ be a cone of revolution. We define the one parameter system $\mathcal{F}_t$ to be the set of all transformations F_t of $\overline{F}$ via a rational map g_t for $t \in \mathbb{R}$. The map g_t is defined as follows

$$g_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3.1}$$
$$= \begin{pmatrix} -\frac{468t^2+788t-123}{1108t^2+788t+517} & \frac{16(2t+1)(29t+12)}{1108t^2+788t+517} & \frac{16(2t+1)(12t-29)}{1108t^2+788t+517} \\ \frac{16(2t+1)(27t+16)}{1108t^2+788t+517} & \frac{588t^2+268t+387}{1108t^2+788t+517} & -\frac{4(2t+1)(46t-57)}{1108t^2+788t+517} \\ -\frac{16(2t+1)(16t-27)}{1108t^2+788t+517} & \frac{4(2t+1)(18t-71)}{1108t^2+788t+517} & -\frac{972t^2+1292t+3}{1108t^2+788t+517} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{3}{2}+t \\ \frac{1}{2}+t \\ \frac{3}{2}+3t \end{pmatrix}.$$

For every $t \in \mathbb{R}$ the map g_t is an isomorphism in $\mathbb{R}^3$. The set of all such transforms form a group which is called the special Euclidean group and is denoted by $\mathbb{SE}(3)$.

For $t = -\frac{4}{5}$ the isometry $g_{-\frac{4}{5}}$ is

$$g_{-\frac{4}{5}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{11347}{14893} & \frac{2688}{14893} & \frac{9264}{14893} \\ \frac{1344}{14893} & \frac{13723}{14893} & -\frac{5628}{14893} \\ -\frac{9552}{14893} & \frac{5124}{14893} & \frac{10213}{14893} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \frac{7}{10} \\ -\frac{3}{10} \\ -\frac{9}{10} \end{pmatrix}$$

and the corresponding surface $F_{-\frac{4}{5}} \in \mathcal{F}_t$ is a cone whose implicit equation is

$$f(x,y,z,-\frac{4}{5}) = \big(13254688516x^2 + 10844660736xy - 19679552256xz \\ -33014762732x + 18886008964y^2 + 11955582912yz \\ +14500367484y + 11332386524z^2 + 37760657196z \\ +30722517817\big) = 0.$$

Some elements of the one parameter system $\mathcal{F}_t$ for $t \in [-2, \frac{2}{5}]$ is displayed in Figure 3.1. The cone $\overline{F}$ and the its image under the map $g_{-\frac{4}{5}}$ are highlighted by green and red colour, respectively. Notice that the cone $\overline{F}$ does not belong to the system $\mathcal{F}_t$.

The elements of the system are infinite cones, but we trim them for clearer plots.

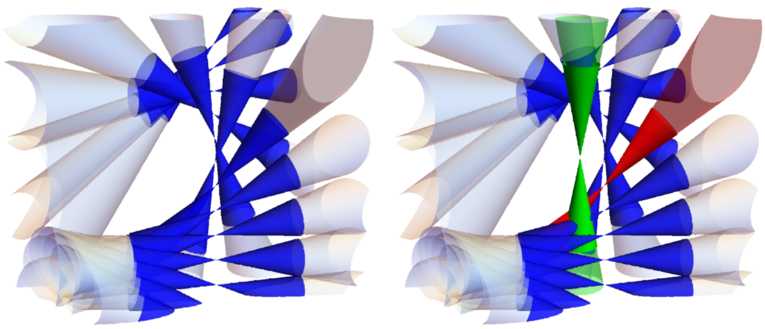

Figure 3.1: Illustration of the one parameter system $\mathcal{F}_t$ from Example 70, some elements of the system for $t \in [-2, \frac{2}{5}]$ (left). The cone $\overline{F}$ (green) and its image under $g_{-\frac{4}{5}}$ (red), (right).

3.2 Lie groups and homogeneous spaces

The transformations from the examples above are elements of certain Lie groups. In this section, we define Lie groups and their action on sets of quadrics in $\mathbb{R}^3$.

Let us begin with a definition of Lie groups. For more details we refer to any introductory textbook on Lie groups and Lie algebras, for instance [19], [20] and [21].

Definition 71. *A* Lie group *is a differentiable manifold G which is also a group such that the group binary operation $G \times G \to G$ and the inverse map $g \mapsto g^{-1}$ are smooth.*

Remark. The dimension of a Lie group G is its dimension as a smooth manifold.

Remark. Typical examples of Lie groups are closed subgroups of $GL(n, \mathbb{R})$. Such Lie groups are called *matrix Lie groups*. A closed subgroup G in $GL(n, \mathbb{R})$ is a subgroup satisfying the following property. If A_k is a sequence of matrices in G

and A_k converges to a matrix A, then either $A \in G$ or A is not invertible. We say a sequence of matrices converges to a matrix A if for every $1 \leq i, j \leq n: \ (A_k)_{ij}$ converges to A_{ij}. A Lie group G is a group of matrices of dimension $n \in \mathbb{N}$, thus we can choose any metric in the definition of the limit.

Let us provide some basic examples of (matrix) Lie groups.

Example 72. The simplest example of a matrix Lie group is the general linear group $GL(n, \mathbb{R})$. It is a subgroup of $GL(n, \mathbb{R})$ and if A_k is a converging sequence of matrices in $GL(n, \mathbb{R})$, then either the limit A is an element of $GL(n, \mathbb{R})$ or it is not an invertible matrix.

Example 73. The special orthogonal group $\mathbb{SO}(n)$ is a Lie group. It is easy to show that it is a subgroup of $GL(n, \mathbb{R})$. To show that it is also a Lie group, let $A_k \subset \mathbb{SO}(n)$ be a sequence of matrices converging to a matrix A. Then for every k it holds $A_k^T A_k = I$, thus also $A^T A = I$ and also for every k, $detA_k = 1$ hence also $detA = 1$ and $A \in \mathbb{SO}(n)$. We saw an example of elements of the special orthogonal group in Example 69.

Example 74. The transformations in Example 70 are elements of the special Euclidean group $\mathbb{SE}(3)$. It is the group of all rigid motions, or in other words, of all isometries in $\mathbb{R}^3$. This matrix group can be defined as follows

$$\mathbb{SE}(3) = \left\{ \begin{pmatrix} A & \overrightarrow{p} \\ 0 & 1 \end{pmatrix}, A \in \mathbb{R}^{3 \times 3}, A^T A = AA^T = I_3, det(A) = 1, \overrightarrow{p} \in \mathbb{R}^3 \right\}.$$

It is easy to show that $\mathbb{SE}(3) \subset GL(n, 4)$ and that it is a closed subgroup.

The following matrices generate one parameter subgroups in $\mathbb{SE}(3)$,

$$g_1 = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} \cos(t) & 0 & -\sin(t) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) & 0 \\ 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, g_4 = \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, g_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first three of them correspond to rotations around x, y and $z-$axis and the last three ones to translations in each direction. These matrices generate the whole Lie group $\mathbb{SE}(3)$. The dimension of the group is the dimension of the smooth manifold and in this case it is 6.

Example 75. Other well known examples of matrix Lie groups are orthogonal group, unitary and special unitary groups.

Later in this chapter, we use tangent space to Lie groups at the identity element of the group.

Definition 76. *Let G be a Lie group of dimension n. A* tangent vector *at the identity element $\mathbf{1} \in G$ of the Lie group G is a vector $X \in \mathbb{R}^n$ of the form*

$$X = c'(0), \tag{3.2}$$

where $c : \mathbb{R} \to G$ is a curve in G satisfying $c(0) = \mathbf{1}$. The tangent space *to G at $\mathbf{1}$ is the set of all tangent vectors at $\mathbf{1}$ and is denoted by $T_{\mathbf{1}}G$.*

A tangent vector of a Lie group represented by a $n \times n$ matrix can be also represented by a $n \times n$ matrix.

Example 77. Let us find the tangent space of $\mathbb{SE}(3)$ at its identity element. Let $c : \mathbb{R} \to \mathbb{SE}(3)$ be a curve such that $c(0) = \mathbf{1}$. Then

$$c(t) = \begin{pmatrix} A(t) & \overrightarrow{p}(t) \\ \overrightarrow{0} & 1 \end{pmatrix},$$

where $A(t) \in \mathbb{R}^{3\times 3}$, $A^T(t)A(t) = A(t)A^T(t) = I_3$, $det(A(t)) = 1$ and $\overrightarrow{p}(t) \in \mathbb{R}^3$ for every $t \in \mathbb{R}$.

Taking the derivative with respect to t of $A(t)A^T(t) = I_3$, we obtain

$$A'(t)A^T(t) + A(t)(A^T)'(t) = 0$$

Then since $c(0) = \mathbf{1}$, we have $A(0) = I_3$, thus

$$A'(0) = -(A^T)'(0) = -(A')^T(0).$$

Hence a tangent vector of $\mathbb{SE}(3)$ at $\mathbf{1}$ is a matrix

$$X = c'(0) = \begin{pmatrix} A & \overrightarrow{p} \\ \overrightarrow{0} & 0 \end{pmatrix},$$

where $A \in \mathbb{R}^{3\times 3}$ satisfies $A = -A^T$ and $p \in \mathbb{R}^3$. Hence

$$X = \begin{pmatrix} 0 & x_1 & x_2 & p_1 \\ -x_1 & 0 & x_3 & p_2 \\ -x_2 & -x_3 & 0 & p_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

where $x_1, x_2, x_3, p_1, p_2, p_3 \in \mathbb{R}$. Hence the tangent space of G at $\mathbf{1}$ is the set

$$T_{\mathbf{1}}\mathbb{SE}(3) = \left\{ \begin{pmatrix} A & \overrightarrow{p} \\ \overrightarrow{0} & 0 \end{pmatrix}, A \in \mathbb{R}^{3\times 3}, A = -A^T, p \in \mathbb{R}^3 \right\}.$$

This space is a six dimensional real vector space generated by the following matrices

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \gamma_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Important tools when studying Lie groups are the Lie algebras. Again let us first define it generally and then specially for the matrix Lie groups.

Definition 78. *A finite dimensional* Lie algebra *is a finite dimensional real vector space* $\mathfrak{g}$ *together with a map* $[] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *which is called* Lie bracket *and satisfies the following:*

1. $[]$ *is bilinear,*
2. *for all* $X, Y \in \mathfrak{g}, \ [X, Y] = -[Y, X]$,
3. *for all* $X, Y, Z \in \mathfrak{g}, \ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$

Remark. For a matrix Lie group $G \subset GL(n, \mathbb{R})$ the Lie bracket is defined by $[X, Y] = XY - YX$, $X, Y \in \mathbb{R}^{n \times n}$. The conditions 1 and 2 are easy to see. The third condition can be verified simply by writing down the equation.

Remark. For a matrix Lie group we can equivalently define its Lie algebra using an exponential map. For a real $n \times n$ matrix X define an exponential map e^X by the power series

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}.$$

Then it can be shown (see [19, Prop. 3.1]) that for any $X \in \mathbb{R}^{n \times n}$ the power series converges and the matrix exponential is a continuous map.

For a matrix Lie group $G \subset GL(n, \mathbb{R})$ its Lie algebra $\mathfrak{g}$ is the set of all matrices $X \in \mathbb{R}^{n \times n}$ such that e^{tX} is in G for all $t \in \mathbb{R}$.

Example 79. Let $G = \mathbb{SE}(3)$ be the special Euclidean group. Let $X \in \mathbb{R}^{4 \times 4}$ be a matrix in the Lie algebra $\mathfrak{se}(3)$, that is, $e^X \in \mathbb{SE}(3)$. Since $X = \frac{d}{dt}|_{t=0} e^{tX}$, we have

$$X = \begin{pmatrix} A & \overrightarrow{p} \\ \overrightarrow{0} & 0 \end{pmatrix}.$$

For such matrix X and any $n \in \mathbb{N}$ it holds

$$X^n = \begin{pmatrix} A & \overrightarrow{p} \\ \overrightarrow{0} & 0 \end{pmatrix}^n = \begin{pmatrix} A^n & A^{n-1}\overrightarrow{p} \\ \overrightarrow{0} & 0 \end{pmatrix},$$

thus

$$e^{tX} = \begin{pmatrix} e^{tA} & \overrightarrow{v} \\ \overrightarrow{0} & 1 \end{pmatrix}$$

for some $\overrightarrow{v} \in \mathbb{R}^3$. For this matrix to be in $\mathbb{SE}(3)$ we need $e^{tA} \in \mathbb{SO}(3)$, that is, $(e^{tA})^{-1} = (e^{tA})^T$ and $det(e^{tA}) = 1$. From the first condition we see

$$(e^{tA})^{-1} = e^{-tA} = (e^{tA})^T = e^{tA^T},$$

hence $-A = A^T$. To conclude, $X \in \mathfrak{se}(3)$ if and only if

$$X = \begin{pmatrix} A & \overrightarrow{p} \\ \overrightarrow{0} & 0 \end{pmatrix},$$

where $A \in \mathbb{R}^{3 \times 3}$ satisfies $A^T = -A$ and $\overrightarrow{p} \in \mathbb{R}^3$, hence X is exactly a anti-symmetric matrix from Example 77, equation 3.3 and

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} A & \overrightarrow{p} \\ \overrightarrow{0} & 0 \end{pmatrix}, A \in \mathbb{R}^{3 \times 3}, A = -A^T, p \in \mathbb{R}^3 \right\} \cong T_{\mathbf{1}}\mathbb{SE}(3).$$

The conclusion from the previous example is true for any matrix Lie group G.

Theorem 80. *[21, Thm. 8.37] Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then $T_{\mathit{1}}G$ and $\mathfrak{g}$ are isomorphic as linear spaces. Thus $\mathfrak{g}$ is finite-dimensional with dimension equal to the dimension of G.*

Next let us recall some basic definitions from the theory of groups. One can find more details in any book on group theory, for instance [22].

Definition 81. *Let $(G, \cdot, id)$ be a group, X a set. Then the (left) group action of G on X is a mapping $p : G \times X \to X, (g, x) \mapsto gx$ such that for every $x \in X$ and every $g, h \in G$ the following holds*

1. $p(id, x) = idx = x$,
2. $p(g, p(hx)) = p(g, hx) = p(g \cdot h, x)$.

If $p(g, x) = gx = y$ for some $g \in G$, we write $x \sim_G y$.

Lemma 82. *The relation $x \sim_G y$ is an equivalence class on X.*

Proof. The relation is reflexive, since $idx = x$, symmetric, since if for some $g \in G$ $gx = y$ then $x = g^{-1}y$, hence $x \sim_G y \iff y \sim_G x$ and transitive; let $gx = y$ and $hy = z$ for some $g, h \in G$, then $(h \cdot g)x = h(gx) = h(y) = z$, hence $x \sim_G z$. □

Example 83. Let $G = \mathbb{SE}(3)$ acting on a set X, the set of all quadrics in $\mathbb{R}^3$. We can write

$$X = \{(x, y, z) \in \mathbb{R}^3, F(x, y, z) = 0\},$$

where

$$F(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4xy + a_5xz + a_6yz + a_7x^2 + a_8y^2 + a_9z^2,$$

and $a_0, a_1, \ldots, a_9 \in \mathbb{R}$.

Let $F_1 : x^2 + y^2 + z^2 - 1 \in X$ be a unit sphere and $F_2 : x^2 + y^2 - \frac{1}{25}z^3 \in X$ be a cone of revolution from our running example, whose axis is the $z-$axis. Let $g \in G$ be a rotation around the $x-$axis by $\frac{\pi}{2}$,

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$gF_1 = F_1(g^{-1})(x, y, z) : x^2 + y^2 + z^2 - 1$$

is again the unit sphere F_1, and

$$gF_2 = F_2(g^{-1})(x, y, z) : x^2 + z^2 - \frac{1}{25}y^2$$

is again a cone, but its axis is the y-axis.

Definition 84. *Let G act on X, $x \in X$.*

- *The* stabilizer *of x, denoted by G_x or $Stab_G(x)$ is the set $\{g \in G;\ gx = x\} \subset G$.*
- *The* orbit *of x is the set $G_x = \{gx; g \in G\} \subset X$.*

Example 85. Let $G = \mathbb{SE}(3)$ and X be the set of all quadrics in $\mathbb{R}^3$. We saw that the map $g \in G$, which a rotation around the $x-$axis by $\frac{\pi}{2}$, acts on the unit sphere F_1 as

$$gF_1 = F_1,$$

hence $g \in Stab_G(F_1)$. In fact, any pure rotation (e.i. the vector of translation is zero) $h \in \mathbb{SE}(3)$ does not change the sphere. Hence

$$Stab_G(F_1) = \left\{ \begin{pmatrix} A & \overrightarrow{0} \\ \overrightarrow{0} & 1 \end{pmatrix}, A \in \mathbb{R}^{3\times 3}, A^T A = I_3, det(A) = 1 \right\} \cong \mathbb{SO}(3).$$

Let us describe the stabilizer of the cone $F_2 = x^2 + y^2 - \frac{1}{25}z^2$. Since it is a cone of revolution, its stabilizer in $\mathbb{SE}(3)$ is the set of all rotations around its axis, which is the $z-$axis. Hence

$$Stab_G(F_2) = \left\{ \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \phi \in \mathbb{R} \right\} \cong \mathbb{SO}(2).$$

Definition 86. *Let G be a group acting on X. The action is called* transitive, *if for every $x, y \in X$ there exists a $g \in G$ such that $gx = y$, that is, if there is only one orbit. The set X is then called the* homogeneous set*.*

If G is a Lie group acting transitively on a sex X, we say that X is a homogeneous space*.*

Example 87. Let again $G = \mathbb{SE}(3)$ and let X be the set of all cones of revolution in $\mathbb{R}^3$, such that the angle between their axis and any line on the cone (*a generator*) is α. For any cone $F \in X$ it holds $F = gF_0$, where

$$F_0 = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 - \frac{1}{\arctan^2(\alpha)} z^2 = 0\}$$

and $g \in G$. Thus the action of G on X is transitive and X is said to be homogeneous.

Let $H = \mathbb{SO}(3)$ be the group of all rotations in $\mathbb{R}^3$. The action of H on the set X is not transitive and X is not homogeneous with respect to H, since there is no $h \in H$ that would transform F_0 (or any other cone with vertex in the origin) to a cone, whose vertex is a point $P \neq \overrightarrow{0}$. For a cone $F \in X$, its orbit is then the set of all cones with the same vertex as F.

We used the group structure of a Lie group to describe how the group transforms a given surface. Next we exploit the structure of a smooth variety. This will later allow us to describe a one parameter system of surfaces solely in the terms of a Lie group.

Definition 88. *Let E, B, F be smooth varieties, $p : E \to B$ be a smooth surjective map such that $\forall b \in B$ the exists a neighbourhood U of b in B such that $p^{-1}(U)$ is isomorphic to $U \times F$.*

$$\begin{array}{ccc} p^{-1}(U) & \cong & U \times F \\ \big\downarrow{\scriptstyle p} & & \\ U & & \end{array}$$

The quadruple (E, B, p, F) is then called a fibre bundle. *The space B is called the* base space*, E the* total space *and F the* fibre*.*

The map $p : E \to B$ is then the projection from $U \times F$ onto the first coordinate.

Example 89. Let $G = \mathbb{SE}(3)$ be the special orthogonal group and X be the set of all cones of revolution in $\mathbb{R}^3$ such that the angle between their axis and any generator is α. Then the quadruple (G, X, p, S) is a fibre bundle, where $p : G \to X$ is the action of G on X and $S = Stab_G(F) \cong \mathbb{SO}(2)$, where F is any cone from the set X.

In the next section we generalise the example above to Lie groups that act transitively on a set. Thanks to the transitivity all the fibres are isomorphic to the stabilizer of a single element of the set.

3.3 Systems of surfaces in homogeneous spaces

In this section we define a one parameter system of surfaces as a subset of the set of all transformations of a fixed surface $\overline{F}$ via a Lie group G.

Definition 90. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a set of surfaces in $\mathbb{R}^3$. Let G be a Lie group acting smoothly on $\mathbb{R}^3$. If there exists an element $\overline{F} \in \mathcal{M}$, such that*

$$\mathcal{M} = \{g\overline{F}; g \in G\}$$

, then we say that $\mathcal{M}$ is a model of transformations *of $\overline{F}$ by G. We say that a one parameter system $\mathcal{F}_t \subset \mathbb{R}^3$ of surfaces is* contained in the model $\mathcal{M}$ *if $\mathcal{F}_t \in \mathcal{M}$ $\forall t \in I$.*

In other words, $\mathcal{M}$ is a model of transformations of a surface $\overline{F}$ if the Lie group G acts transitively on $\mathcal{M}$.

Example 91. Let $\mathcal{M}$ be the set of all cones of revolution in $\mathbb{R}^3$ such that the angle between their axis and their generators is $\arctan(\frac{1}{5})$. Let

$$\overline{F} = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 - \frac{1}{25}z^2 = 0\} \in \mathcal{M}$$

and let $G = \mathbb{SE}(3)$. Then $\mathcal{M}$ is the model of transformations of $\overline{F}$ by G, since G acts transitively on $\mathcal{M}$. The one parameter system $\mathcal{F}_t$ from our running example 70 is contained in the model $\mathcal{M}$, since it is clearly a subset of $\mathcal{M}$.

Next let us formalise Example 89 into a simple corollary.

Corollary. Let $\mathcal{M}$ be a model of transformations of a surface $\overline{F}$ by a Lie group G. Let p be the action of G on $\mathcal{M}$ and let $S = p^{-1}(\overline{F})$. Then $(G, \mathcal{M}, p, S)$ is a fibre bundle.

By definition, a Lie group G acts transitively on the model of transformations $\mathcal{M}$ of a surface $\overline{F}$ by G, hence $\mathcal{M}$ is a homogeneous space. A fibre corresponding to an element $F \in \mathcal{M}$ is the stabilizer $Stab_G(F)$ and because of the transitivity, all the fibres are isomorphic. We can thus for any $F \in \mathcal{M}$ write

$$S = p^{-1}(F) = Stab_G(\overline{F})$$

and

$$\mathcal{M} \cong G/Stab_G(\overline{F}).$$

Example 92. To continue with the examples 89 and 91, let $\mathcal{M}$ be again the model of transformations of $\overline{F}$ by $\mathbb{SE}(3)$. $\mathbb{SE}(3)$ acts transitively on $\mathcal{M}$, hence the fibres over every $F \in \mathcal{M}$ are isomorphic, namely isomorphic to $Stab_G(\overline{F}) \cong \mathbb{SO}(2)$. Hence

$$\mathcal{M} \cong \mathbb{SE}(3)/\mathbb{SO}(2).$$

The space $\mathcal{M}$ is a homogeneous space of dimension 5.

For a curve in $\mathcal{M}$ we define its *lift to G*.

Definition 93. *Let $(G, \mathcal{M}, p, S)$ be a fibre bundle. Let $\phi : I \to \mathcal{M}$ be a curve in $\mathcal{M}$, where $I \subseteq \mathbb{R}$ is an interval. A curve $\Phi : I \to G$ such that $p \circ \Phi = \phi$ is called a* lift *of ϕ, that is, the following diagram commutes.*

$$\begin{array}{ccc} I & \xrightarrow{\Phi} & G \\ & \searrow^{\phi} & \downarrow p \\ & & \mathcal{M} \end{array}$$

Let $g(t) : I \to G$ be a rational curve in G and denote $g_t = g(t)$ for $t \in I$. We construct a curve $\phi = p \circ g$,

$$\begin{aligned} \phi : I &\to \mathcal{M} \\ t &\mapsto g_t\overline{F}, \end{aligned}$$

The curve g_t is clearly a lift of ϕ and g_t defines a one parameter system of surfaces $\mathcal{F}_t$ contained in $\mathcal{M}$, $\mathcal{F}_t = \mathrm{im}(\phi) = \mathrm{im}(g_t \circ p) = \{g_t\overline{F},\ t \in I\}$.

Example 94. Let $G = \mathbb{SE}(3)$, $\overline{F} = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 - \frac{1}{25}z^2\}$ be a cone in $\mathbb{R}^3$ and $\mathcal{M} = \{g\overline{F}, g \in G\}$ be the model of transformations of $\overline{F}$ by G. $\mathcal{M}$ is then the set of all cones of revolution in $\mathbb{R}^3$ that are isomorphic to $\overline{F}$. Let $g_t : I \to G$ be a curve defined as follows,

$$g_t = \begin{pmatrix} -\frac{468t^2+788t-123}{1108t^2+788t+517} & \frac{16(2t+1)(29t+12)}{1108t^2+788t+517} & \frac{16(2t+1)(12t-29)}{1108t^2+788t+517} & \frac{3}{2}+t \\ \frac{16(2t+1)(27t+16)}{1108t^2+788t+517} & \frac{588t^2+268t+387}{1108t^2+788t+517} & -\frac{4(2t+1)(46t-57)}{1108t^2+788t+517} & \frac{1}{2}+t \\ -\frac{16(2t+1)(16t-27)}{1108t^2+788t+517} & \frac{4(2t+1)(18t-71)}{1108t^2+788t+517} & -\frac{972t^2+1292t+3}{1108t^2+788t+517} & \frac{3}{2}+3t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then let $\phi : I \to \mathcal{M}$ be a curve in $\mathcal{M}$, $\phi(t) = g_t \circ p = g_t\overline{F}$. The set $\mathrm{im}(\phi)$ is the one parameter system of cones $\mathcal{F}_t$ from our running example 70,

$$\mathcal{F}_t = \mathrm{im}(\phi) = \{g_t\overline{F}, t \in \mathbb{R}\}.$$

One can check that in this case, $\overline{F} \notin \mathcal{F}_t$.

A lift Φ of ϕ is every curve $\Phi : I \to \mathbb{SE}(3)$, such that $\Phi(t)\overline{F} = g_t\overline{F} = F_t \in \mathcal{F}$. In particular, the curve

$$\begin{aligned} g_t : I &\to G \\ t &\mapsto g_t \end{aligned}$$

is a lift of ϕ.

For a given curve $\phi : I \to \mathcal{M}$ there does not always have to exist a global lift $\Phi : I \to G$. Even if there is such a lift, it does not have to be unique and, more importantly, rational. Luckily, since we consider a one parameter system to be defined as an image of a rational curve ϕ, we also know its rational lift Φ.

For a model of transformations of a surface in $\mathbb{R}^3$ we define its *implicit representation space* as follows.

Definition 95. *Let $\mathcal{M}$ be a model of transformations of a surface $\overline{F}$ by a Lie group G. Let $\overline{f}$ be the implicit equation of $\overline{F}$, i. e.*

$$\overline{F} = \{(x, y, z) \in \mathbb{R}^3, \overline{f}(x, y, z) = 0\}.$$

Let Q be the linear space of the functions in three variables,

$$Q = span\{f(x, y, z) = \overline{f}(g^{-1}(x, y, z)), g \in G\}.$$

We call Q to be the implicit representation space *of $\mathcal{M}$. The mapping*

$$\begin{aligned} \psi : G &\to Q \\ g &\mapsto \overline{f} \circ g^{-1} \end{aligned}$$

is called the implicit representation mapping.

The linear space Q contains all the implicit equations of surfaces in $\mathcal{M}$,

$$(x, y, z) \in \mathcal{M} \iff (x, y, z) \in g\overline{F} \iff \overline{f}(g^{-1}(x, y, z)) = 0.$$

Remark. Every element of a one parameter system of surfaces in $\mathbb{R}^3$ contained in $\mathcal{M}$ has its implicit equation contained in Q.

Example 96. Let $G = \mathbb{SE}(3)$ and $\mathcal{M} = \{g\overline{F}, g \in G\}$, where $\overline{F}x^2 + y^2 - \frac{1}{25}z^2 = 0$ and $\overline{f} = x^2 + y^2 - \frac{1}{25}z^2$ is its implicit equation. Then the implicit representation space Q of $\mathcal{M}$ is the space

$$Q = \text{span}\{\overline{f}(g^{-1}(x, y, z)), g \in \mathbb{SE}(3)\}.$$

Let $g \in \mathbb{SE}(3)$,

$$g = \begin{pmatrix} a_{11} & a_{12} & a_{13} & p_1 \\ a_{21} & a_{22} & a_{23} & p_2 \\ a_{31} & a_{32} & a_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ a_{ij}, p_i \in \mathbb{R}.$$

Denote by

$$g^{-1} = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ g_{ij} \in \mathbb{R},$$

the inverse of g.

Then

$$\begin{aligned}\overline{f}(g^{-1}(x,y,z)) &= g_{11}^2x^2 + g_{21}^2x^2 - \frac{1}{25}g_{31}^2x^2 + 2g_{12}g_{11}xy + 2g_{21}g_{22}xy - \frac{2}{25}g_{31}g_{32}xy \\ &+ 2g_{13}g_{11}xz + 2g_{21}g_{23}xz - \frac{2}{25}g_{31}g_{33}xz + 2g_{14}g_{11}x + 2g_{21}g_{24}x \\ &- \frac{2}{25}g_{31}g_{34}x + g_{12}^2y^2 + g_{22}^2y^2 - \frac{1}{25}g_{32}^2y^2 + 2g_{12}g_{13}yz + 2g_{22}g_{23}yz \\ &- \frac{2}{25}g_{32}g_{33}yz + 2g_{12}g_{14}y + 2g_{22}g_{24}y - \frac{2}{25}g_{32}g_{34}y + g_{13}^2z^2 + g_{23}^2z^2 \\ &- \frac{1}{25}g_{33}^2z^2 + 2g_{13}g_{14}z + 2g_{23}g_{24}z - \frac{2}{25}g_{33}g_{34}z + g_{14}^2 + g_{24}^2 - \frac{g_{34}^2}{25}.\end{aligned}$$

And thus

$$Q = \text{span}\{\overline{f}(g^{-1}(x,y,z)), g \in \mathbb{SE}(3)\} = \text{span}\{1, x, y, z, xy, xz, yz, x^2, y^2, z^2\}.$$

It is a linear space of dimension 10.

In many cases, the implicit representation space is a finite dimensional linear space.

Corollary. Let $\overline{f} \in \mathbb{R}[x,y,z]$ be a polynomial and let

$$\overline{F} = \{(x,y,z) \in \mathbb{R}^3, \overline{f}(x,y,z) = 0\}$$

be a surface. Let $\mathcal{M}$ be the model of transformation by a group Lie group G. Then the implicit representation space Q of $\mathcal{M}$ is a finite-dimensional linear space.

Next let us first study the *tangent mapping* of the map ψ at a given element $g \in G$.

Definition 97. *Let $\psi : G \to Q$ be the implicit representation mapping and let $g \in G$. The tangent map $d_g\phi$ is a map*

$$\begin{aligned} d\psi_g : T_gG &\to T_{\psi(g)}Q \\ \gamma &\mapsto (\psi \circ \Gamma)'(0), \end{aligned}$$

where Γ is a smooth curve in G satisfying $\Gamma(0) = g$ and $\Gamma'(0) = \gamma$.

In particular, we are interested in the case $g = \mathbf{1}$, the identity element in G. Then $\psi(\mathbf{1}) = (\overline{f} \circ \mathbf{1}^{-1}) = \overline{f}$, hence the codomain of $d\psi_{\mathbf{1}}$ is $T_{\overline{f}}Q$.

By Theorem 80 we know $T_{\mathbf{1}}G \cong \mathfrak{g}$. Furthermore, the following lemma holds.

Lemma 98. *Let Q be the implicit representation space of a model of transformations $\mathcal{M}$ for a surface $\overline{F}$ by a Lie group G. Then $T_{\overline{f}}Q \cong Q$.*

Proof. Let $\overline{F} = \{(x,y,z) \in \mathbb{R}^3, \overline{f}(x,y,z) = 0\}$. The tangent space of Q at $\overline{f}$ is the set of all tangent vectors to $\mathcal{M}$ at $\overline{f}$. Let $c : \mathbb{R} \to Q$ be a curve in Q such that $c(0) = \overline{f}$. Then $c'(0)$ is a tangent vector to Q at $\overline{f}$. Clearly $c'(0) \in Q$ and hence

$$T_{\overline{f}}Q = \{c'(0),\ c : \mathbb{R} \to Q, c(0) = \overline{f}\} = Q.$$

□

We can hence describe the tangent map $d\psi_{\mathbf{1}}$ as follows.

Corollary. The tangent map $d\psi_{\mathbf{1}}$ is the map

$$\begin{aligned} d\psi_{\mathbf{1}} : \mathfrak{g} &\to Q \\ \gamma &\mapsto (\psi \circ \Gamma)'(0), \end{aligned}$$

where $\Gamma : I \to G$ is a smooth curve in G such that $\Gamma(0) = \mathbf{1}$ and $\Gamma'(0) = \gamma$.

Remark. We can choose any other value then 0, say $a \in I$, then if $\Gamma(a) = \mathbf{1}$ and $\Gamma'(a) = \gamma$, we evaluate $(\psi \circ \Gamma)'$ in a.

Example 99. For the Lie group $G = \mathbb{SE}(3)$ and Q being the implicit representation space of a model of transformations of a quadric $\overline{F}$ by $\mathbb{SE}(3)$, the Lie algebra $\mathfrak{g} = \mathfrak{se}(3) \subset \mathbb{R}^{4\times 4}$ and $Q = \operatorname{span}\{1, x, y, z, xy, xz, yz, x^2, y^2, z^2\} \cong \mathbb{R}^{10}$. The image of a matrix $\gamma \in \mathfrak{se}(3)$ under $d\psi_{\mathbf{1}}$ is the implicit equation

$$(\overline{f} \circ \Gamma^{-1})'(0) \in \operatorname{span}\{1, x, y, z, xy, xz, yz, x^2, y^2, z^2\},$$

where $\Gamma(0) = \mathbf{1}$ and $\Gamma'(0) = \gamma$.

Let us briefly summarise the situation. Let G be a Lie group and let

$$\overline{F} = \{(x, y, z) \in \mathbb{R}^3, \overline{f}(x, y, z) = 0\}$$

be a surface in $\mathbb{R}^3$. We defined the model of transformations $\mathcal{M}$ of $\overline{F}$ by G. A curve $g_t : I \to G$ defines a one parameter system $\mathcal{F}_t = g_t\overline{F}$, which is included in $\mathcal{M}$.

The following theorem describes characteristic curves of a system $\mathcal{F}_t$ as images of certain curves on a surface $\overline{F}$.

Theorem 100. *Let $\{g_t\overline{F}, t \in I\}$ be a one parametric system of surfaces in a model $\mathcal{M}$. Then for each $t \in I$ there exists an element $\gamma_t \in \mathfrak{g}$ so that*

$$\chi_t = g_t \chi_{\gamma_t} \tag{3.4}$$

where

$$\chi_{\gamma_t} = \{(x, y, z) \in \mathbb{R}^3, \overline{f}(x, y, z) = 0 \wedge d\psi_{\mathbf{1}}(\gamma_t)(x, y, z) = 0\}. \tag{3.5}$$

Proof. Let us denote $h(x, y, z, t) := \bar{f}(g_t^{-1}(x, y, z))$. Then from the definition we have

$$g_t\bar{F} = \{(x, y, z) \in \mathbb{R}^3, h(x, y, z, t) = 0\}$$

and by Theorem 6

$$\chi_t = \{(x, y, z) \in \mathbb{R}^3, h(x, y, z, t) = 0 \wedge h'(x, y, z, t) = 0\}, \tag{3.6}$$

where $'$ denotes the partial derivative $\frac{d}{dt}$.

Let us fix a value $t = t_0$ and define $\Gamma(t) = g_{t_0}^{-1} \circ g_t$ and $\gamma_{t_0} = \Gamma'(t_0)$. Γ is a smooth curve in G satisfying $\Gamma(t_0) = \mathbf{1}$ and therefore $\gamma_{t_0} \in \mathfrak{g}$. We claim that γ_{t_0} satisfies (3.4) for $t = t_0$.

Comparing (3.5) and (3.6) for $t = t_0$ it is sufficient to show that for any $(x, y, z) \in \mathbb{R}^3$ we have

$$\bar{f}(g_{t_0}^{-1}(x, y, z)) = h(x, y, z, t_0) \quad \text{and} \quad d\psi_{\mathbf{1}}(\gamma_{t_0})(g_{t_0}^{-1}(x, y, z)) = h'(x, y, z, t_0).$$

The first equation is directly the definition of h. For the second one we have by definition

$$d\psi_{\mathbf{1}}(\gamma_{t_0})(x,y,z) = \frac{d}{dt}\bar{f}(g_t^{-1}(g_{t_0}(x,y,z)))\Big|_{t=t_0}$$

and because g_{t_0} is a constant mapping which commutes with the derivative we get

$$d\psi_{\mathbf{1}}(\gamma_{t_0})(g_{t_0}^{-1}(x,y,z)) = \frac{d}{dt}\bar{f}(g_t^{-1}(x,y,z))\Big|_{t=t_0} = h'(x,y,z,t_0).$$

□

Generally, in our case, G is a matrix Lie group, thus the composition in $\Lambda(t) = g_{t_0}^{-1} \circ g_t$ may be understood as matrix multiplication.

In the next section, we use this description to find rational parameterization of the characteristic curves for some one parameter systems. Since $d\psi_{\mathbf{1}}$ is a linear map between linear spaces $\mathfrak{g}$ and Q, we can evaluate the mapping only on the basis of $\mathfrak{g}$.

3.4 Parameterization of envelopes of quadrics

In this section we apply the theory that we built earlier in this chapter to parameterization of envelopes of quadratic surfaces. We show that the one parameter system of cones from our running example (and in fact any one parameter system of cones with constant angle between axes and its generators) posses a rational parameterization. We compute the explicit parameterization for several examples.

Theorem 101. *Let $\mathcal{F}_t$ be a one parameter system of cones of revolution with constant angle* $\arctan(r)$ *between their axes and generators,*

$$\mathcal{F}_t = \{g_t\overline{F}, g_t \in \mathbb{SE}(3), t \in I\},$$

, where $\overline{F} = \{(x,y,z) \in \mathbb{R}^3, x^2+y^2-r^2z^2 = 0\}$. Then any characteristic curve χ_t is isometric to a curve

$$\tau = \{(x,y,z) \in \mathbb{R}^3, x^2+y^2-r^2z^2 = 0 \wedge h(x,y,z) = 0\},$$

where $h(x,y,z) \in span\{x,y,z,xz,yz\}$.

From this statement it is easy to conclude that the characteristic curves are rational and if the system is rational, then its envelope is also a rational surface.

Proof. Let us first denote $\overline{f} = x^2+y^2-r^2z^2$, the implicit equation of $\overline{F}$. The Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ is spanned by the following matrices

$$\gamma_1 = \begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0&0&-1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \gamma_6 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}.$$

Let Γ_i be an element in $\mathbb{SE}(3)$ corresponding to γ_i such that $\Gamma_i'(0) = \gamma_i$. One can choose for instance the following six matrices

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) & 0 \\ 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Gamma_2 = \begin{pmatrix} \cos(t) & 0 & -\sin(t) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(t) & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma_3 = \begin{pmatrix} \cos(t) & -\sin(t) & 0 & 0 \\ \sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Gamma_4 = \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Gamma_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we evaluate $d\psi_1(\gamma_i)) = (\psi \circ \Gamma_i)'(0)$. We get

$$\begin{aligned} d\psi_1(\gamma_1) &= \frac{\partial}{\partial t}\Big|_{t=0} (-r^2(z\cos(t) - y\sin(t))^2 + (y\cos(t) + z\sin(t))^2 + x^2) \\ &= -2\left(r^2+1\right)(y\sin(t) - z\cos(t))(y\cos(t) + z\sin(t))\Big|_{t=0} \\ &= -2\left(r^2+1\right)yz, \end{aligned}$$

$$\begin{aligned} d\psi_1(\gamma_2) &= \frac{\partial}{\partial t}\Big|_{t=0} (-r^2(z\cos(t) - x\sin(t))^2 + (x\cos(t) + z\sin(t))^2 + y^2) \\ &= \left(-2\left(r^2+1\right)(x\sin(t) - z\cos(t))(x\cos(t) + z\sin(t))\right)\Big|_{t=0} \\ &= 2\left(r^2+1\right)xz, \end{aligned}$$

$$\begin{aligned} d\psi_1(\gamma_3) &= \frac{\partial}{\partial t}\Big|_{t=0} (-r^2z^2 + (x\cos(t) + y\sin(t))^2 + (y\cos(t) - x\sin(t))^2) \\ &= \frac{\partial}{\partial t}\Big|_{t=0} (-r^2z^2 + x^2 + y^2) \\ &= 0, \end{aligned}$$

$$\begin{aligned} d\psi_1(\gamma_4) &= \frac{\partial}{\partial t}\Big|_{t=0} (-r^2z^2 + (t-x)^2 + y^2) \\ &= (2(t-x))|_{t=0} \\ &= -2x, \end{aligned}$$

$$\begin{aligned} d\psi_1(\gamma_5) &= \frac{\partial}{\partial t}\Big|_{t=0} (-r^2z^2 + (t-y)^2 + x^2) \\ &= (2(t-y)))|_{t=0} \\ &= -2y, \end{aligned}$$

$$
\begin{aligned}
d\psi_1(\gamma_6) &= \frac{\partial}{\partial t}\bigg|_{t=0} (x^2 + y^2 - r^2(t-z)^2) \\
&= \left(2r^2(t-z)\right)\Big|_{t=0} \\
&= 2r^2 z.
\end{aligned}
$$

The equation $d\psi_1(\gamma_3) = 0$ corresponds to the fact that the cone $\overline{F}$ is invariant to rotations around the $z-$axis.

The map $d\psi_1$ is linear, hence $\mathrm{im}(d\psi_1) = \mathrm{span}\{xz, yz, x, y, z\}$. Hence, from Theorem 100, for any $t \in I$ it holds χ_t is isometric to a curve

$$\tau = \{(x, y, z) \in \mathbb{R}^3, \overline{f} = 0 \wedge h(x, y, z) = 0\},$$

where $h \in \mathrm{im}(d\phi_1)$.

$\square$

Corollary. In the settings of the previous theorem, any characteristic curve of the one parameter system of cones that depends rationally on a parameter t is rational.

Proof. We need to show that the intersection of the cone

$$\overline{F} = \{(x, y, z) \in \mathbb{R}^3, \overline{f} = 0\}$$

and a surface

$$H = \{(x, y, z) \in \mathbb{R}^3, h(x, y, z) = 0\}$$

for $h(x, y, z) = a_1x + a_2y + a_3z + a_4xz + a_5yz,\ a_1, \ldots, a_5 \in \mathbb{R}$ is rational. This is true thanks to the fact that H passes through the vertex of F, which is the origin. In fact we can compute the explicit parameterization of the intersection τ of $\overline{F}$ and H,

$$\tau(s) = \begin{pmatrix} \frac{-\left(s^2-1\right)\left(r\left(-a_1s^2+a_1+2a_2s\right)+a_3\left(s^2+1\right)\right)}{(s^2+1)(a_4(s^2-1)-2a_5s)} \\ \frac{2s\left(r\left(-a_1s^2+a_1+2a_2s\right)+a_3\left(s^2+1\right)\right)}{(s^2+1)(a_4(s^2-1)-2a_5s)} \\ \frac{1\left(\left(s^2+1\right)\left(r\left(-a_1s^2+a_1+2a_2s\right)+a_3\left(s^2+1\right)\right)\right)}{r(s^2+1)(a_4(s^2-1)-2a_5s)} \end{pmatrix}, \quad s \in \mathbb{R}.$$

Then for a cone $F_t \in \mathcal{F}_t, F_t = g_t\overline{F}$, for some $g_t \in \mathbb{SE}(3)$, its characteristic curve χ_t is an image of a rational curve $\tau(s)$, $\chi_t(s) = g_t\tau(s)$. The isometry g_t is a rational map, hence $g_t(\tau(s))$ is again a rational curve. $\square$

Corollary. In the settings of Theorem 101, if the motion of the cone $\overline{F}$ is rational, that is, if g_t depends rationally t for each $t \in I$, then the envelope of $\mathcal{F}_t$ is a rational surface.

Proof. The statement follows from the fact that the envelope is the union of characteristic curves. $\square$

Example 102. Let $\mathcal{F}_t$ be the one parameter system of isometric cones of revolution from the Example 70. Recall that for every $t \in I$ the cone $F(t) \in \mathcal{F}_t$ is isometric to the cone $\overline{F}$ with implicit equation $\overline{f} = x^2+y^2-\frac{1}{25}z^2$ via the isometry $g_t \in \mathbb{SE}(3)$

from the Equation 3.1. The characteristic curve of the cone F_t is an image of a curve τ_t, that is, $\chi_t = g_t(\tau_t)$, where

$$\tau_t = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 - \frac{1}{25}z^2 = 0 \wedge d\psi_{\mathbf{1}}(\gamma_t) = 0\}.$$

From Theorem 101 we know that the implicit equation $h_t(x, y, z) = d\psi_{\mathbf{1}}(\gamma_t)$ is in the span$\{x, y, z, xz, yz\}$ and

$$h_t(x, y, z) = \frac{1}{5\,(1108t^2 + 788t + 517)}\big(5700t^2x - 9740t^2y - 580t^2z - 9900tx \\ +1860ty - 868tz + 5824xz - 8375x - 3328yz + 1365y - 49z\big)\,.$$

Notice that for any t, the surface $H_t = \{(x, y, z) \in \mathbb{R}^3, h_t(x, y, z) = 0\}$ passes through the origin.

Next, according to Theorem 101, we can find the parameterization of the characteristic curve of F_t as the intersection of the surface H_t and the cone $\overline{F}$,

$$\tau_t(s) = \begin{pmatrix} -\frac{(s^2-1)\left(4(s(70s+487)-215)t^2-4s(356s+93)t-s(862s+273)+556t+813\right)}{416(7s^4+8s^3+8s-7)} \\ \frac{s\left(4(s(70s+487)-215)t^2-4s(356s+93)t-s(862s+273)+556t+813\right)}{208(7s^4+8s^3+8s-7)} \\ \frac{5\left(-4(s(70s+487)-215)t^2+4s(356s+93)t+s(862s+273)-556t-813\right)}{416(s(7s+8)-7)} \end{pmatrix}, \quad s \in \mathbb{R}.$$

Fix for instance $t = -\frac{4}{5}$. The characteristic curve $\chi_{-\frac{4}{5}}$ is the intersection of the cone $\overline{F}$ and the surface $H_{-\frac{4}{5}}\{(x, y, z) \in \mathbb{R}^3, h_{-\frac{4}{5}}(x, y, z) = 0\}$, where

$$h_{-\frac{4}{5}} = \frac{2}{14893}(29120xz + 15965x - 16640yz - 31783y + 1371z).$$

Both the surfaces are in Figure 102.

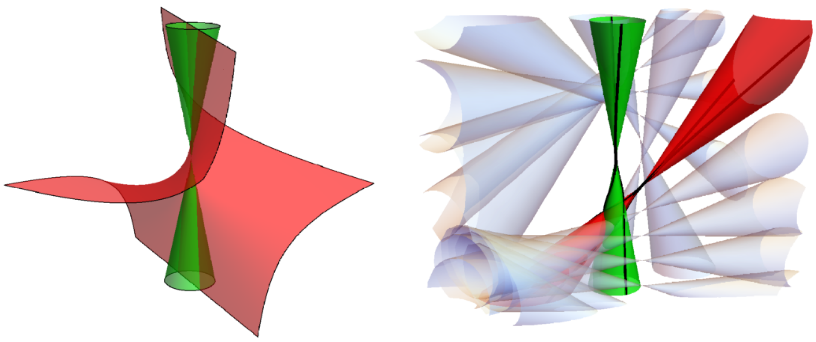

Figure 3.2: Characteristic curve $\chi_{\frac{4}{5}}$ is the image of the intersection of $\overline{F}$ and $H_{-\frac{4}{5}}$ (left). The intersection curve is then mapped to the characteristic curve $\chi_{-\frac{4}{5}}$ by the isometry $g_{-\frac{4}{5}}$ (right).

The characteristic curve $\chi_{-\frac{4}{5}}(s)$ is the image of $\tau_{-\frac{4}{5}}(s)$ under $g_{-\frac{4}{5}}$,

$$\chi_{-\frac{4}{5}}(s) = \begin{pmatrix} -\frac{88575776s^4+509169928s^3-382633007s^2+191442147s-204621224}{154887200(s^2+1)(2s^2-7s-2)} \\ -\frac{6171152s^4-386855603s^3-286172329s^2-34112917s+622886657}{309774400(s^2+1)(2s^2-7s-2)} \\ -\frac{1257933616s^4-3086748361s^3-207911307s^2-2315956609s-1445128099}{309774400(s^2+1)(2s^2-7s-2)} \end{pmatrix}, \quad s \in \mathbb{R}.$$

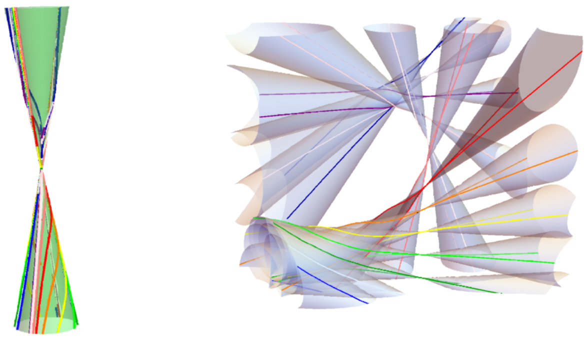

Figure 3.3: Intersection curves of $\overline{F}$ and surfaces H_t (left) and the characteristic curves of the one parameter system $\mathcal{F}_t$ for some values of t (right).

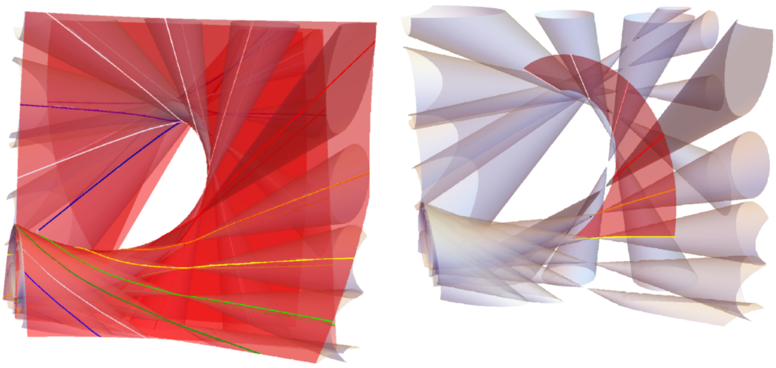

Figure 3.4: Envelope of the one parameter system $\mathcal{F}_t$ (right) and a patch of the envelope (left).

For any $t \in \mathbb{R}$ we are able to compute and parameterize the intersection of $\overline{F}$ and the surface H_t defined by an equation $h_t = 0$. The characteristic curve χ_t is then the image of this curve under the map g_t from 3.1. In Figure 102 we see the curves on the cone $\overline{F}$ and the corresponding characteristic curves for some elements of the system.

Next we can parameterize the whole envelope as

$$\chi(t, s) = g_t \tau_t(s).$$

In Figure 102 we see the whole envelope of the surface and a part of the envelope $\chi(t, s)$ for $t \in (-\frac{6}{5}, -\frac{1}{5})$ and $s \in (-\frac{3}{5}, -\frac{2}{5})$.

Example 103. In the same way one we can prove that a one parameter system of cylinders of revolution is rational. Let us illustrate that on an example. Let $\overline{F} = \{(x, y, z) \in \mathbb{R}^3, \overline{f} = 0\}$,where

$$\overline{f} = x^2 + y^2 - \frac{1}{4},$$

be a cylinder of revolution. Let $\mathcal{M} = \{g\overline{F}, g \in \mathbb{SE}(3)\}$ be the model of $\overline{F}$ by $\mathbb{SE}(3)$.

Let $g : \mathbb{R} \to \mathbb{SE}(3)$ be a curve in $\mathbb{SE}(3)$,

$$g(t) = g_t = \begin{pmatrix} -\frac{180t^2+204t-263}{756t^2+204t+313} & \frac{24(2t+1)(15t-1)}{756t^2+204t+313} & -\frac{24\left(6t^2-31t-7\right)}{756t^2+204t+313} & t+\frac{1}{3} \\ -\frac{24\left(6t^2+29t+7\right)}{756t^2+204t+313} & -\frac{180t^2-564t-7}{756t^2+204t+313} & -\frac{24\left(30t^2+9t-11\right)}{756t^2+204t+313} & 3t \\ -\frac{24\left(30t^2+t+1\right)}{756t^2+204t+313} & -\frac{24\left(6t^2+9t+13\right)}{756t^2+204t+313} & \frac{180t^2+588t-7}{756t^2+204t+313} & 5t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The curve g is a lift of a curve $\phi : \mathbb{R} \to \mathcal{M}$ defined by $\phi(t) = g_t\overline{F}$. Then ϕ defines a one parameter system, see Figure 103,

$$\mathcal{F}_t = im(\phi) = \{g_t\overline{F}, t \in \mathbb{R}\} = \{(x,y,z) \in \mathbb{R}^3, \overline{f}(g_t^{-1}(x,y,z)) = 0, t \in \mathbb{R}\}.$$

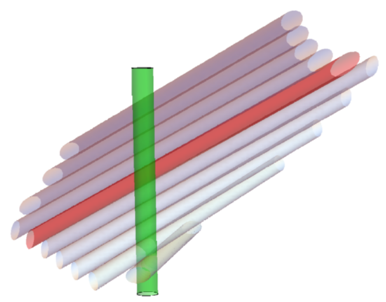

Figure 3.5: One parameter system $\mathcal{F}_t$, surface $\overline{F}$ (green) and F_1 (red).

Similarly as in Theorem 101, we could show that any characteristic curve χ_t of a cylinder $F_t \in \mathcal{F}_t$ is isometric to a curve τ_t, where

$$\tau_t = \{(x,y,z) \in \mathbb{R}^3, \overline{f}(x,y,z) = 0 \wedge h_t(x,y,z) = 0\},$$

where $h_t \in \operatorname{span}\{x, y, xz, yz\}$.

In our example for a $t \in \mathbb{R}$ it holds

$$h_t(x,y,z) = \frac{2}{756t^2+204t+313}\Big(4212t^2x + 540t^2y + 2412tx - 924ty - 696xz + 361x + 648yz + 1563y\Big)$$

and it defines a surface $H_t = \{(x,y,z) \in \mathbb{R}^3, h_t(x,y,z) = 0\}$.

If we fix $t = 1$, we obtain

$$h_1(x,y,z) = -\frac{2(696xz - 6985x - 648yz - 1179y)}{1273}$$

and we can see the intersection of H_1 and $\overline{F}$ in Figure 103 (right).

The curve τ_1 is the intersecting curve of H_1 and $\overline{F}$ and can be parameterized as

$$\tau_1(s) = \begin{pmatrix} \frac{1}{s^2+1} - \frac{1}{2} \\ \frac{s}{s^2+1} \\ \frac{6985}{696} - \frac{37131s}{58(s(29s+54)-29)} \end{pmatrix}, \ s \in \mathbb{R},$$

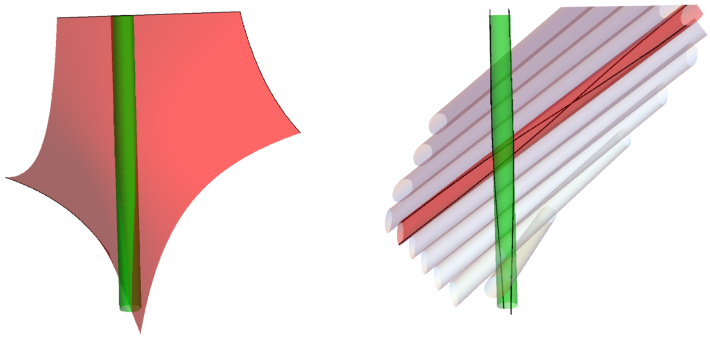

Figure 3.6: Envelope of the one parameter system $\mathcal{F}_t$ (right) and a patch of the envelope (left).

and the corresponding characteristic curve χ_1 is its image under g_1, that is,

$$\chi_1(s) = \begin{pmatrix} \frac{1646983s^4+292194s^3+305538s^2-97794s-1625929}{7638(s^2+1)(29s^2+54s-29)} \\ \frac{-70213s^4+310805s^3-8118s^2+233695s+99445}{1273(s^2+1)(29s^2+54s-29)} \\ \frac{10012889s^4+6484554s^3-1405440s^2+6424650s-9478361}{30552(s^2+1)(29s^2+54s-29)} \end{pmatrix}, \ s \in \mathbb{R}.$$

In figure 103 (right) see the curve τ_1 and the characteristic curve χ_1.
For general $t \in \mathbb{R}$, the parameterization of τ_t is

$$\tau_t(s) = \begin{pmatrix} \frac{1}{s^2+1} - \frac{1}{2} \\ \frac{s}{s^2+1} \\ \frac{s^2(36t(117t+67)+361)-6s(4t(45t-77)+521)-36t(117t+67)-361}{24(s(29s+54)-29)} \end{pmatrix}, \ s \in \mathbb{R}.$$

Notice, that the first two coordinates are the same for every $t \in \mathbb{R}$.
A characteristic curve χ_t of $F_t \in \mathcal{F}_t$ is then parameterized as

$$\chi_t(s) = g_t\tau_t, \ s \in \mathbb{R},$$

and the whole envelope possesses the following parameterization

$$\chi(t,s) = g_t\tau_t(s), \ t \in \mathbb{R}, \ s \in \mathbb{R}.$$

In Figure 103 two pieces of the envelope are displayed.

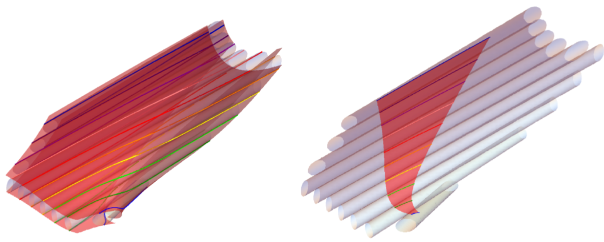

Figure 3.7: Envelope of the one parameter system $\mathcal{F}_t$ (right) and a patch of the envelope for $t \in (0, \frac{9}{5})$ and $s \in (-1, -\frac{3}{5})$ (left).

Example 104. Let us show that the envelope of a rational one parameter system of spheres with constant radii (the pipe surface) is rational. Let

$$\overline{F} = \{(x, y, z) \in \mathbb{R}^3, \overline{f}(x, y, z) = 0\},$$

where $\overline{f} = x^2 + y^2 + z^2 - 1$, be a unit sphere. Let $\mathcal{M} = \{g\overline{F}, g \in \mathbb{SE}(3)\}$ be a model of transformations of $\overline{F}$ by $\mathbb{SE}(3)$. Let $g : \mathbb{R} \to \mathcal{M}$ be a curve,

$$\begin{aligned} g : \mathbb{R} &\to \mathcal{M} \\ t &\mapsto g_t\overline{F}, \end{aligned}$$

where

$$g(t) = g_t = \begin{pmatrix} \frac{9t^4+8t^2-18t+45}{9t^4+82t^2-18t+45} & \frac{6t\left(6t^2+t+11\right)}{9t^4+82t^2-18t+45} & \frac{6(t-4)(t-2)t}{9t^4+82t^2-18t+45} & 5-t \\ -\frac{6t\left(6t^2-t+13\right)}{9t^4+82t^2-18t+45} & \frac{9t^4-8t^2+18t+27}{9t^4+82t^2-18t+45} & -\frac{6\left(3t^3-t^2+6t-6\right)}{9t^4+82t^2-18t+45} & t^2 \\ -\frac{6t\left(t^2+6t-4\right)}{9t^4+82t^2-18t+45} & \frac{6\left(3t^3-5t^2+6t-6\right)}{9t^4+82t^2-18t+45} & \frac{9t^4+62t^2+18t+27}{9t^4+82t^2-18t+45} & 5t+3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The image of g defines a one parameter system $\mathcal{F}_t$ of spheres. Then for every sphere $F_t \in \mathcal{F}_t$ its characteristic curve is image of a curve $\tau_t(s)$ under the isomorphism $g(t)$, where

$$\tau_t = \{(x, y, z) \in \mathbb{R}, \overline{f}(x, y, z) = 0 \wedge h_t(x, y, z) = 0\},$$

and $h(x, y, z)$ is a plane passing though the origin,

$$\begin{aligned} h_t(x, y, z) = -\frac{2}{9t^4 + 82t^2 - 18t + 45}\Big(&- 81t^4x - 18t^3x - 344t^2x + 138tx - 45x \\ &+18t^5y + 38t^3y - 120t^2y + 168ty - 180y \\ &+ 9t^4z + 6t^3z + 274t^2z + 114tz + 135z\Big). \end{aligned}$$

Hence the intersection τ_t of $H_t = \{(x, y, z) \in \mathbb{R}^3, h_t = 0\}$ and $\overline{F}$ is a great circle of $\overline{F}$, which is a rational curve. Some of the curves τ_t, some characteristic curves and the envelope are shown in Figure 105 (top).

Example 105. Let $\overline{F}$, $\overline{f}$, $\mathcal{M}$ be as in the previous example 104. This time, let a one parameter system of spheres $\tilde{\mathcal{F}}_t$ be defined as the image of the curve $\tilde{g} : \mathbb{R} \to \mathcal{M}$

$$\begin{aligned} \tilde{g} : \mathbb{R} &\to \mathcal{M} \\ t &\mapsto \tilde{g}_t\overline{F}, \end{aligned}$$

where

$$\tilde{g}(t) = \tilde{g}_t = \begin{pmatrix} 1 & 0 & 0 & 5-t \\ 0 & 1 & 0 & t^2 \\ 0 & 0 & 1 & 5t+3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The sphere $\overline{F}$ is invariant to any rotation around each of the axes x, y, z, hence $\tilde{\mathcal{F}}_t$ is the same one parameter system as in the previous example. However, the computation of the envelope is much easier. Again, every characteristic curve $\tilde{\chi}_t$ of the system is isomorphic to a curve $\tilde{\tau}_t$,

$$\tilde{\tau}_t = \{(x, y, z) \in \mathbb{R}, \overline{f}(x, y, z) = 0 \wedge \tilde{h}_t(x, y, z) = 0\},$$

and $\tilde{h}(x, y, z)$ is a plane passing though the origin, but this time, it has much more simple implicit equation

$$\tilde{h}_t(x, y, z) = -2(-x + 2ty + 5z).$$

Again, $\tau_t(s)$ are again great circles of $\overline{F}$, but the resulting parameterization of τ_t and of characteristic curves is different. Some of the curves $\tilde{\tau}_t$ and some of the characteristic curves and the envelope are shown in Figure 105 (bottom).

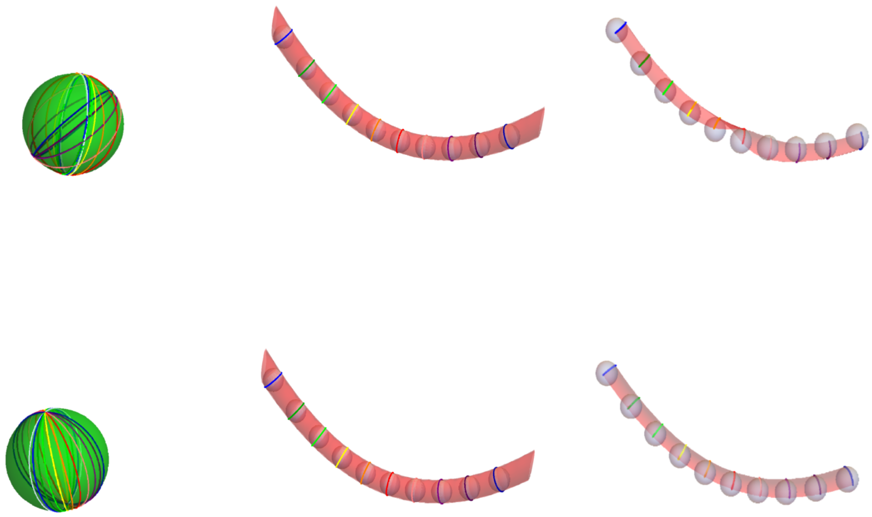

Figure 3.8: Curves $\tau_t(s)$ and $\tilde{\tau}_t(s)$ for several choices of $t \in [-\frac{5}{2}, 2]$ (left), envelope of the one parameter systems $\mathcal{F}_t$ and $\tilde{\mathcal{F}}_t$ and characteristic curves (middle) and a patch of the envelope for $t \in (-\frac{5}{2}, 2)$ and $s \in (-1, 1)$ (right).

The case when characteristic curves are rational is rather rare. For more general one parameter system of quadrics the curve

$$\chi_{\gamma_t} = \{(x, y, z) \in \mathbb{R}^3, \overline{f}(x, y, z) = 0 \wedge d\psi_{\mathbf{1}}(\gamma_t)(x, y, z) = 0\}$$

from Theorem 100 is an intersection of two quadratic surfaces, hence, generally, a curve of degree 4.

To decide whether a curve is rational, we can either find its explicit parameterization or compute its *genus* (see [13, Chapter 3]). Then it can be show that a curve is rational if and only if its genus is zero ([13, Thm. 4.11]).

Example 106. Let $\overline{F} = \{(x, y, z) \in \mathbb{R}^3, \overline{f}(x, y, z) = 0\}$, $\overline{f} = x^2 + y^2 + z^2 - 1$, be a unit sphere. Let g_t be a smooth curve in the affine group represented by a matrix g_t

$$\begin{pmatrix} \frac{t\left(9t^3+8t+36\right)}{(t^2+1)(9t^4+10t^2+36t+72)} & -\frac{6\left(t^3-6t-12\right)}{(t^2+1)(9t^4+10t^2+36t+72)} & \frac{6t(7t+2)}{(t^2+1)(9t^4+10t^2+36t+72)} & \frac{t+3}{t^2+1} \\ -\frac{6\left(t^3+6t+12\right)}{(t^2+1)(9t^4+10t^2+36t+72)} & -\frac{t\left(9t^3-10t-36\right)}{(t^2+1)(9t^4+10t^2+36t+72)} & \frac{6t\left(3t^2+6t-2\right)}{(t^2+1)(9t^4+10t^2+36t+72)} & \frac{t^2}{t^2+1} \\ \frac{6t(5t-2)}{9t^4+10t^2+36t+72} & -\frac{6t\left(3t^2+6t+2\right)}{9t^4+10t^2+36t+72} & -\frac{9t^4-8t^2-36t-72}{9t^4+10t^2+36t+72} & 2t+1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Them $\mathcal{F}_t = \{g_t\overline{F}, t \in \mathbb{R}\}$ is a one parameter system of ellipsoids. For instance for $t = 1$ the affine map g_1 has the form

$$g_1 = \begin{pmatrix} \frac{53}{254} & \frac{51}{127} & \frac{27}{127} & 2 \\ -\frac{57}{127} & \frac{37}{254} & \frac{21}{127} & \frac{1}{2} \\ \frac{18}{127} & -\frac{66}{127} & \frac{107}{127} & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$F_1 = \{(x, y, z) \in \mathbb{R}^3, f_1(x, y, z) = 0\},$$

where

$$f_1(x, y, z) = \overline{f}(g_1^{-1})(x, y, z) = 4x^2 - 16x + 4y^2 - 4y + z^2 - 6z + 25.$$

In Figure 106 there are some elements of the system, the unit sphere $\overline{F}$ (green) and the ellipsoid F_1 (red).

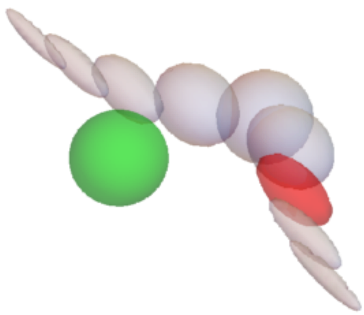

Figure 3.9: The one parameter system $\mathcal{F}_t$ of ellipsoids, the unit sphere $\overline{F}$ (green) and ellipsoid F_1 (red)

.

For each $t \in \mathbb{R}$ the characteristic curve of F_t is the image of a curve τ_t under g_t. In this case, τ_t is the intersection of the unit sphere $\overline{F}$ and a surface $H_t = \{(x, y, z) \in \mathbb{R}^3, h_t(x, y, z) = 0\}$. For $t = 1$ it hods

$$\begin{aligned} h_1(x, y, z) = & 15805x^2 + 2376xy - 3852xz + 30099x + 11773y^2 + 14124yz \\ & + 50927y + 4680z^2 - 11938z. \end{aligned}$$

If we project τ_1 onto a plane parallel to the $xy-$plane, we obtain the curve

$$\begin{aligned}\tau_1'(x,y) =& 138603529x^4 - 55945296x^3y + 761673102x^3 + 377789906x^2y^2 \\ &+ 938931574x^2y + 1137757741x^2 - 75105360xy^3 + 760959870xy^2 \\ &+ 3196754202xy + 189756288x + 249798025y^4 + 385225798y^3 \\ &+ 2602978277y^2 + 813901344y - 120613444.\end{aligned}$$

It is a curve of degree 4. We compute the genus of this curve and we realise it is equal to 1, hence the plane curve τ_1' in not a rational curve. But then also τ is not rational and therefore also the characteristic curve of F_1 is not a rational curve.

In Figure 106 there is the unit sphere $\overline{F}$, the surface H_1, their intersection and the curve τ_1'.

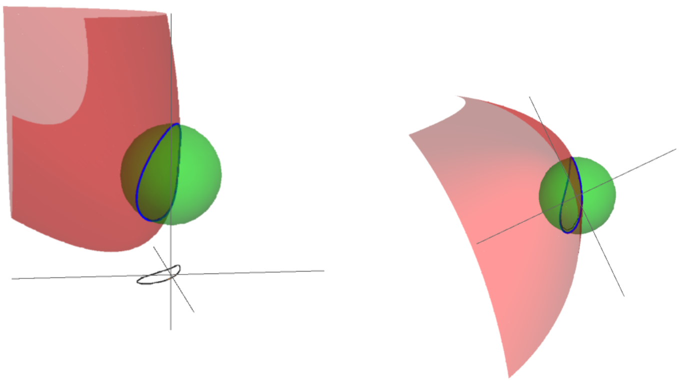

Figure 3.10: Unit sphere $\overline{F}$, a surface H_1, their intersection and its projection onto plane from two different angles.

Conclusion

We clarified the definition of one parameter systems of quadratic surfaces in $\mathbb{R}^3$ and their envelopes and proved a characterisation of envelope surfaces. We discussed algebraic methods for envelope computation. We commented on two proves of rationality of an envelope of a one parameter systems of spheres, cones and cylinders of revolution and presented models of Laguerre geometry formally and in details. Finally, we introduced a new kinematic approach to one parameter systems of quadrics and were able to describe a characteristic curve of the system as an image of an intersection of possible quite simple surfaces. Using this approach we presented a new proof of the fact that envelopes of one parameter systems of cones of revolution are rational and even provided explicit parameterization of characteristic curves and the envelope. We applied the theory to one parameter systems on other surfaces and discussed the rationality of their envelopes by several examples.

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