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Matrix games with random payoff

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Název práce: Maticové hry s náhodnou výplatou

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Abstrakt: V této práci rozebírame teorií maticových her a speciálně případ, kdy je výplatní matice náhodná. Prezentujeme několik modelů řešení pro tento typ her, založených na charakteristikách výplatní funkce, jako i na teorií optimalizace s pravděpodobnostními omezeními. V praktické části řešime příklad motivovaný soupeřením dvou zprostředkovatelů elektrické energie na uzavřetém trhu. Využívame různe přístupy prezentované v teoretické části na to, abychom našli optimální stratégie pro obě společnosti.

Klíčová slova: teorie her, maticové hry, náhodná výplata

Title: Matrix games with random payoff

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Abstract: In this thesis we study matrix games and more specifically the case when the payoff matrix is random. We present several solution models for such games, based both on characteristics of the payoff and on the theory of chance-constrained programming. In the numeric study we formulate and solve a real-world motivated problem of two electricity providers on a closed market. We use different approaches presented in the theoretical part of the thesis to find optimal strategies for both companies.

Keywords: game theory, matrix games, random payoff

Contents

1. Introduction

In the theory of optimization, there is a still-growing field of the game theory. This part of mathematics was first developed as a separate field by John von Neuman in late 1920's and later famously followed by John Forbes Nash Jr. and others throughout the 20th century. This field of mathematics, as the name suggests, focuses on the study of games, but not just games in a common sense, such as chess or poker, but in more general sense as competitive situations. This found many applications from economics and political science to psychology and biology. The modern game theory reshaped how we think about competitions in the real world.

The game theory studies various types of games and in 20th century its subject grew from the original study of zero-sum competitions by von Neuman to a large and diverse number of cases. The most important characteristics of games are as follows.

Games by the number of players		
Single-player games	Two-player games	Multiple-player games

Games by the type of the competition		
Competitive games	Cooperative games	Competitively-cooperative games

Games by the dependency on time		
Games evolving in time	Repeating games	Non-repeating games

Games by the representation of the game	
Normal-form games	Extensive-form games

Games by the payoff	
Constant-sum games	Non-constant sum games

Although, there are many different categories and subcategories for each of those types of games, in this thesis, we will further focus on the theory of constant-sum two-player competitive games and more specifically on the theory of matrix games. We primarily focus on the case when the matrix is random. For such a situation there is no longer single way to define an optimal strategy, as is the case for deterministic matrix games. We therefore have to specify a suitable solution model, which takes on the problem of uncertain payoff with some specific assumptions (axioms). From this we have multiple possible approaches to classify optimal solutions of such games.

Review of used literature

The main basis for our work on the deterministic constant-sum games in the first chapter is [7]. We also briefly mention [9], that shows the connection between dominance and rationality of players. For more theoretical results in the theory of dominance we refer to [11], [6], [8] and [10], where authors provided several approaches and discussed properties of iterated dominance in general games. In [11], [6] and [8] authors defined the iterated elimination of strictly dominated strategies and studied its properties, mainly the existence and uniqueness of the maximal reduction. In [10] authors studied more general version of such procedures in a general choice problem and provided satisfactory conditions for their uniqueness.

The theory of games with random payoff is somewhat less developed than the theory of deterministic games, main works, that we mention and discuss their results in this thesis are [5], [4], [3] and [2]. First of the mentioned papers discusses the theory of matrix games with random payoff with joint probabilistic constrains and its main results for the general case are the relations of the two optimization problems that are solved by the first and the second player. The second paper is the original work that developed the theory of matrix games with random payoff. In their paper authors generalize deterministic optimization problems, that are solved by the two players to the case of random payoff and prove that those problems have deterministic equivalents. In [3], authors give different approach to generalization of the deterministic problem, which is 'less cautious' and no longer defends against the worst payoff, but instead it defends against the least likely payoff, that is, it optimizes the payoff subject to the minimal probability of it happening. Authors use individual constrains and prove general properties of this problem and relations between the payoff maximizing and confidence maximizing problems, from that they are able to formulate a solution algorithm for the payoff maximizing problem in games with random matrix with continuous distribution. The last reviewed paper was [2], which discusses the results for the least likely payoff constrained models with joint constrains under the assumption of a specific distribution of elements of payoff matrix. Main results of this paper are equivalent deterministic problems for payoff and confidence maximizing models. In [1], the general results of theory of stochastic programming, that we use to formulate equivalent deterministic programs in the cases of discrete distributions with finite support are formulated.

Structure of the thesis

In the second chapter we discuss the general theory of constant-sum games and of their optimal strategies. We formulate the basic definitions and provide examples for their special cases. We present results developed in [7] for the zero-sum games and show how they may be used to study general constant-sum games. After that, we define a strict dominance and discuss its basic results. Lastly, we discuss the implications of those results for the case of matrix games and formulate the optimization problem, that the players in a matrix games want to solve together with its equivalent formulation.

In the third chapter we further develop the theory of matrix games by introducing the randomness of payoff. We present and categorize several different models, which are used in the case of random payoff. Mainly, we distinguish between the models, which are based on the characteristics of the payoff and more complex models based on the theory of stochastic optimization. We present the model of expected payoff and as an alternative to it the model of naive quantile payoff. Then we formulate different models based on the theory of chance-constrained programming and using the results of stochastic optimization, we formulate equivalent deterministic mixed-integer programs for the case when the payoff matrix has a discrete distribution with finitely many values.

In the fourth chapter we formulate a real-world motivated problem, which may be interpreted in the sense of a matrix game with random payoff and solve it using different approaches presented in the third chapter. On this example we show the differences between the models and also point out the importance of expert judgement for the choice of a suitable model for the specific case that is studied.

2. Game theory

In this chapter we take a closer look at the case of the deterministic constant-sum games, terminology related to them and their properties. After that, we will discuss implications of those results in the case of matrix games and lastly, we will formulate the optimization problem, that players want to solve in a matrix game.

2.1 Basic terminology for games with constant sum

Definition 1 (The two-player constant-sum game)**.** *Let S_1 and S_2 be non-empty sets of strategies of the first and the second player respectively and $u = (u_1, u_2) : S_1 \times S_2 \to \mathbb{R}^2$ be the game's payoff function. The triple:*

$$G = (S_1, S_2, u),$$

we call a two-player game with a constant sum, if $\forall s_1 \in S_1, \forall s_2 \in S_2 : u_1(s_1, s_2) = -u_2(s_1, s_2) + c$ for some $c \in \mathbb{R}$, where u_i denotes the payoff function of the player i.

An important special case of a constant-sum game is the zero-sum game for $c = 0$. In this case $\forall s_1 \in S_1, \forall s_2 \in S_2 : u_1(s_1, s_2) = -u_2(s_1, s_2)$, thus for the simplicity we explicitly define only the payoff function of the first player

$$u : S_1 \times S_2 \to \mathbb{R}$$

and the payoff function of the second player then implicitly is $-u$.

In general S_1 and S_2 may be abstract sets of strategies that can players play. An important categorization of games is given by the following definition.

Definition 2 (Sets of strategies)**.** *Let $G = (S_1, S_2, u)$ be a two-player game with a constant sum,*

1. *if S_1 and S_2 are sets of different actions that can each player take, we call this a game in pure strategies. In this case, we will use the notation P instead of S_1 and Q instead of S_2, $p \in P$, $q \in Q$ we call a pure strategy of the first and second player respectively.*

2. *If*
$$S_1 = \{x; x = \sum_{p \in P} \lambda_x(p)p, \ \sum_{p \in P} \lambda_x(p) = 1, \ \forall p \in P : \lambda_x(p) \geq 0\}$$
and
$$S_2 = \{y; y = \sum_{q \in Q} \lambda_y(q)q, \ \sum_{q \in Q} \lambda_y(q) = 1, \ \forall q \in Q : \lambda_y(q) \geq 0\},$$
we will call this game a game in mixed strategies and use the notation of X instead of S_1 and Y instead of S_2, $x \in X$, $y \in Y$ we call a mixed strategy of the first and second player respectively.

As we will later see, categorization of games based on Definition 2 is crucial for us, as results for games in pure strategies differ from results for games in mixed strategies.

It is handy to think about mixed strategies as probability distributions over the set of pure strategies, as this interpretation is best applicable in practical results of finding optimal strategies, where player can play only single action at a time, but in many repetitions of the play and focuses on finding the strategy which yields him the highest payoff in the long-term. This intuition may be used to solve repeating games using the methods of non-repeating games. Mixed strategy may also represent an intensity indicating for example the split of resources by each player.

The payoff function for the game in mixed strategies is defined through the payoff function for the game in pure strategies as follows:

$$\forall x \in X, \forall y \in Y : u(x,y) = \sum_{p \in P} \sum_{q \in Q} \lambda_x(p)\lambda_y(q)u(p,q),$$

where P and Q are the sets of pure strategies and $x = \sum_{p \in P} \lambda_x(p)p$ and $y = \sum_{q \in Q} \lambda_y(q)q$ are convex combinations of pure strategies. This may be also regarded as an expected payoff and comes from the idea, that each mixed strategy is just a probability distribution over pure strategies.

Definition 3 (Two-player matrix game)**.** *Let $m, n \in \mathbb{N}$. The two-player zero-sum game is called a two-player matrix game, if $S_1 \subseteq \mathbb{R}^m$, $S_2 \subseteq \mathbb{R}^n$ and $\forall \boldsymbol{s_1} \in S_1, \forall \boldsymbol{s_2} \in S_2 : u(\boldsymbol{s_1}, \boldsymbol{s_2}) = \boldsymbol{s_1}^T A \boldsymbol{s_2}$ for some $m \times n$ matrix A.*

- For simplicity we will refer to the two-player matrix game as a "matrix game". In some literature matrix games as a term may include also bimatrix games, in which the payoffs of the two players are given by different matrices.

- We will denote $I = \{1, \dots, m\}$ the set of row indices and $J = \{1, \dots, n\}$ the set of column indices of A.

- Even if we did not impose any further criteria on the sets of strategies, other than the fact, that they are sets of m and n-dimensional real vectors, it is easy to see, that without loss of generality, from the linearity of the payoff function, we can just think of sets that are subsets of a convex closure of the standard basis of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.

- Sets of pure strategies then in a matrix game are $P = \{\boldsymbol{p} \in \mathbb{R}^m; \sum_{i=1}^m p_i = 1, p_i \in \{0,1\}, i \in I\}$ and $Q = \{\boldsymbol{q} \in \mathbb{R}^n; \sum_{j=1}^n q_j = 1, q_j \in \{0,1\}, j \in J\}$, for the first and the second player respectively. It is from this that the first and second player in a matrix game are also referred to as a row or a column player. For $i \in I$ we will write $\boldsymbol{p_i} \in P$ and mean a pure strategy such that $p_i = 1$. Similarly for $j \in J$ and $\boldsymbol{q_j} \in Q$.

- We will use the following notation for a matrix game

$$G = (S_1, S_2, u_A),$$

where $u_A : S_1 \times S_2 \to \mathbb{R}$ denotes the payoff function of the first player given by the matrix A.

Example. Sets $P = Q = \{(0,1)^T, (1,0)^T\}$ and

$$A = (a_{ij})_{2\times 2} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

define a matrix game in pure strategies with payoff matrix A and player strategy sets P and Q. We see that even though the payoff function was not clearly defined for this game, we can represent it as a bilinear form defined by matrix A and for any two actions $\boldsymbol{p} \in P, \boldsymbol{q} \in Q$ we get $u_A(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{p}^T A \boldsymbol{q}$. We can clearly expand this definition to $X = Conv(P) = Conv(Q) = Y$.

2.2 Optimal strategies

Now we will classify optimal strategies and closely related terminology. We will begin by defining terminology for the constant-sum games and then we will discuss their interpretations in the case of matrix games.

2.2.1 The best response and the Nash equlibrium

Definition 4 (The best response strategy)**.** *Let $G = (S_1, S_2, u)$ be a constant-sum game, we define the set of best response strategies of the first player for a given $s_2 \in S_2$ as*

$$BR(s_2) = \{s_1 \in S_1;\, u_1(s_1, s_2) \geq u_1(s_1', s_2), \forall s_1' \in S_1\},$$

element of $BR(s_2)$ is called the best response strategy for the first player given s_2.

Similarly for the second player and given $s_1 \in S_1$

$$BR(s_1) = \{s_2 \in S_2;\, u_2(s_1, s_2) \geq u_2(s_1, s_2'), \forall s_2' \in S_2\},$$

element of $BR(s_1)$ is called the best response strategy for the second player given s_1.

Best response strategies in a sense of this definition are such strategies, for which no higher payoff may be achieved with any other strategy, given opponent's strategy, but this set may be an empty one, as we will show in the next example.

Example. Let $G = (P, Q, u)$ be a constant sum game with $P = (0, 1)$, $Q = (0, 1)$ and payoff function $u : P \times Q \to \mathbb{R}^2$ such that for $p \in P$, $q \in Q : u_1(p, q) = p - q$ and $u_2(p, q) = q - p + 1$. Clearly this is a well defined constant sum game as $u_1(p, q) = -u_2(p, q) + 1$.

Now let $q \in Q$ be given, we will show that $BR(q) = \emptyset$. Suppose, that $BR(q) \neq \emptyset$, so there is such $p^* \in P$, that for any other $p' \in P : u_1(p^*, q) \geq u_1(p', q)$ but from definition, P is an open set in $\mathbb{R}$, so that $\forall p \in P, \exists \epsilon > 0$ such that $p + \epsilon \in P$. Specially this holds true for p^* and from definition of u_1 we get that $u_1(p^* + \epsilon, q) > u_1(p^*, q)$ which is a contradiction. Similar argument may be made for the second player.

The Definition 4 motivates the definition of a key term in the game theory - the Nash equilibrium of a game.

Definition 5 (The Nash equilibrium). *Let $G = (S_1, S_2, u)$ be a constant sum game then $(s_1^*, s_2^*) \in S_1 \times S_2$ is called a (Nash) equilibrium of the game G, if*

$$\forall s_1 \in S_1 \forall s_2 \in S_2 : u_1(s_1, s_2^*) \leq u_1(s_1^*, s_2^*) \wedge u_2(s_1^*, s_2) \leq u_2(s_1^*, s_2^*).$$

Or in other words

$$s_1^* \in BR(s_2^*) \wedge s_2^* \in BR(s_1^*).$$

If G is a game in pure strategies, we say that (s_1^*, s_2^*) is a Nash equilibrium in pure strategies. Similarly, if G is a game in mixed strategies, we say that (s_1^*, s_2^*) is a Nash equilibrium in mixed strategies.

In previous example we showed, that the Nash equilibrium might not exist in a case when sets of pure strategies are infinite. In the following example we will show that it may not exist in pure strategies, even if those sets are finite.

Example. Let $G = (P, Q, u)$ be a constant sum game in pure strategies such that $P = Q = \{0, 1\}$ and $u : P \times Q \to \mathbb{R}^2$ which satisfies

$$u_1(p, q) = \begin{cases} p - 1 & \text{if } q = 1, \\ -p & \text{if } q = 0 \end{cases}$$

and

$$u_2(p, q) = \begin{cases} -q & \text{if } p = 1, \\ q - 1 & \text{if } p = 0. \end{cases}$$

It is easy to show that G is a well defined constant sum game with constant $c = -1$.

Now suppose the player two plays 1, then the best response for the player one is to play 1 as well, but if the player one plays 1 the best response for the player two is to play 0. Similarly, if the player two plays 0 then the best response for the player one is to play 0, but if the player one plays 0 the best response for the player two is to play 1, therefore the game has no Nash equilibrium in pure strategies.

This example also shows that the Nash equilibrium in pure strategies may not exist, even if sets of best responses are non-empty for both players in every combination of players' strategies.

Now, we will formulate some of the results presented in the Chapter 7 of [7].

Definition 6 ([7]; Definition 7.2). *Let $G = (X, Y, u)$ be a zero-sum game we define*

1. *game's upper valuation as $v_U^* := \inf_{y \in Y} \sup_{x \in X} u(x, y)$,*
2. *game's lower valuation as $v_L^* := \sup_{x \in X} \inf_{y \in Y} u(x, y)$.*
3. *game's upper value as $v_U := \min_{y \in Y} \sup_{x \in X} u(x, y)$,*
4. *game's lower value as $v_L := \max_{x \in X} \inf_{y \in Y} u(x, y)$.*

If $v_L = v_U$ we say that the game has a value $v := v_L = v_U$.

Strategy $x^* \in X$ of the first player is called optimal, if $\forall y \in Y : u(x^*, y) \geq v_L^*$, strategy $y^* \in Y$ of the second player is called optimal, if $\forall x \in X : u(x, y^*) \leq v_U^*$.

Lemma 1 ([7]; Lemma 7.3.)**.** *Every $G = (X, Y, u)$ zero-sum game has an upper and a lower valuation. Furthermore $v_L^* \leq v_U^*$.*

Lemma 2 ([7]; Lemma 7.4.)**.** *Let $G = (X, Y, u)$ be a zero-sum game. It follows that:*

1. *The first player has at least one optimal strategy, iff the lower value of the game exists.*
2. *The second player has at least one optimal strategy, iff the upper value of the game exists.*

For proofs of lemmas 1 and 2 we refer to the [7].

Lemma 3 ([7]; Theorem 7.5.)**.** *Let $G = (X, Y, u)$ be a zero-sum game. If X and Y are compact metric spaces and $u : X \times Y \to \mathbb{R}$ is a continuous function then game's upper and lower values exist. Furthermore,*

$$\forall x \in X \exists y^*(x) \in Y \text{ s.t. } u(x, y^*(x)) = \min_{y \in Y} u(x, y) \in \mathbb{R} \tag{2.1}$$

$$\forall y \in Y \exists x^*(y) \in X \text{ s.t. } u(x^*(y), y) = \max_{x \in X} u(x, y) \in \mathbb{R} \tag{2.2}$$

$$v_U = \min_{y \in Y} \max_{x \in X} u(x, y) \in \mathbb{R} \tag{2.3}$$

$$v_L = \max_{x \in X} \min_{y \in Y} u(x, y) \in \mathbb{R} \tag{2.4}$$

Proof. We propose a following proof of this lemma. For a given $x \in X$, $u(x, .) : Y \to \mathbb{R}$ is a continuous function on a compact set Y. From the Extreme value theorem there exists a $y^*(x) \in Y$ such that $\min_{y \in Y} u(x, y) = u(x, y^*(x)) \in \mathbb{R}$. Similarly for a given $y \in Y$, $u(., y) : X \to \mathbb{R}$ is a continuous function on a compact set X, so there exists a $x^*(y) \in X$ such that $\max_{x \in X} u(x, y) = u(x^*(y), y) \in \mathbb{R}$. From the Definition 6, $v_L = \max_{x \in X} u(x, y^*(x)) = \max_{x \in X} \min_{y \in Y} u(x, y) \in \mathbb{R}$ and $v_U = \min_{y \in Y} u(x^*(y), y) = \min_{y \in Y} \max_{x \in X} u(x, y) \in \mathbb{R}$.

□

Lemma 4 ([7];Theorem 7.6.)**.** *Zero-sum game $G = (X, Y, u)$ has a value, if and only if it has a Nash equilibrium.*

Proof. This proof was formulated in [7].
"$\Rightarrow$": Suppose G has a value $v \in \mathbb{R}$. From Lemma 2 there are $x^* \in X$ and $y^* \in Y$ optimal strategies for the first and the second player respectively, so that

$$\forall x \in X, \forall y \in Y : u(x, y^*) \leq v \leq u(x^*, y). \tag{2.5}$$

This implies that $u(x^*, y^*) \leq v \leq u(x^*, y^*)$, so that $v = u(x^*, y^*)$, because (2.5) specially holds for $x^* \in X$ and $y^* \in Y$. And from the Definition 5 $(x^*, y^*) \in X \times Y$

is a Nash equilibrium.Therefore the first implication holds.
"$\Leftarrow$": Now suppose that G has a Nash equilibrium, that means there is a $(x^*, y^*) \in X \times Y$ such that $\forall x \in X, \forall y \in Y : u(x, y^*) \leq u(x^*, y^*) \leq u(x^*, y)$.

From Lemma 3 it follows that,

$$u(x^*, y^*) = \min_{y \in Y} u(x^*, y) \leq \sup_{x \in X} \inf_{y \in Y} u(x, y) = v_L^*,$$
$$u(x^*, y^*) = \max_{x \in X} u(x, y^*) \geq \inf_{y \in Y} \sup_{x \in X} u(x, y) = v_U^*.$$

From Lemma 1 we know that $v_L^* \leq v_U^*$ so it must be that,

$$u(x^*, y^*) = \max_{x \in X} \inf_{y \in Y} u(x, y) = \min_{y \in Y} \sup_{x \in X} u(x, y) = v_L^* = v_U^* = v_L = v_U = v.$$

That concludes the proof.

□

Lemma 5 ([7];Theorem 7.7., "The minimax theorem")**.** *Let $G = (X, Y, u)$ be a zero-sum game. If $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are compact convex sets and $u : X \times Y \to \mathbb{R}$ be a continuous function, concave in $x \in X$ and convex in $y \in Y$ then G has a value.*

Proof. We propose a following proof of the lemma. Both X and Y are convex compact subsets of metric spaces and u is a continuous function, therefore by Lemma 3, G has a upper and a lower value. Furthermore,

$$v_L = \max_{x \in X} \min_{y \in Y} u(x, y) \in \mathbb{R},$$
$$v_U = \min_{y \in Y} \max_{x \in X} u(x, y) \in \mathbb{R}.$$

$\forall x \in X, u(x, .) : Y \to \mathbb{R}$ is a convex function, therefore $f(y) := \max_{x \in X} u(x, y)$ is also a continuous convex function on Y. Similarly $g(x) := \min_{y \in Y} u(x, y)$ is a continuous concave function on X. By Theorem 2.37 in [7], $f(y)$ and $g(x)$ are global maximum and minimum of u on X and Y for a given $y \in Y$ and $x \in X$ respectively and v_U is a global minimum of f on Y and v_L is a global maximum of g on X. Now let $(x', y^*), (x^*, y') \in X \times Y$ be such that

$$v_U = \min_{y \in Y} \max_{x \in X} u(x, y) = u(x', y^*), \tag{2.6}$$
$$v_L = \max_{x \in X} \min_{y \in Y} u(x, y) = u(x^*, y'). \tag{2.7}$$

It follows from (2.6), (2.7) and previous discussion that

$$\forall y \in Y : u(x', y^*) \leq u(x', y), \tag{2.8}$$
$$\forall x \in X : u(x, y') \leq u(x^*, y'). \tag{2.9}$$

Inequalities (2.8) and (2.9) specially hold true for $(x', y') \in X \times Y$ which implies that

$$v_U = u(x', y^*) \leq u(x', y') \leq u(x^*, y') = v_L,$$

so it must be that $v_L = v_U$ and G has a value.

□

Let us notice that the point $(x', y') \in X \times Y$ from the previous proof is a Nash equilibrium of the game.

Results of the previous discussion may be, by the following theorem, without loss of generality expanded to the class of constant-sum games.

Theorem 6 (Characterization of Nash equilibria in a constant-sum game)**.** *Let $G = (S_1, S_2, u)$ be a constant-sum game such that $u_1(s_1, s_2) = -u_2(s_1, s_2) + c$ for every $s_1 \in S_1, s_2 \in S_2$ and some $c \in \mathbb{R}$. $(s_1^*, s_2^*) \in S_1 \times S_2$ is a Nash equilibrium of G, if and only if it is a Nash equilibrium of a zero-sum game $H = (S_1, S_2, u_1)$.*

Proof.

Suppose $H = (S_1, S_2, u_1)$ has a Nash equilibrium $(s_1^*, s_2^*) \in S_1 \times S_2$ this means that,

$$\forall s_1 \in S_1 : u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \tag{2.10}$$

and

$$\forall s_2 \in S_2 : -u_1(s_1^*, s_2^*) \geq -u_1(s_1^*, s_2). \tag{2.11}$$

Inequality (2.11) holds, if and only if

$$\forall s_2 \in S_2 : u_2(s_1^*, s_2^*) = -u_1(s_1^*, s_2^*) + c \geq -u_1(s_1^*, s_2) + c = u_2(s_1^*, s_2) \tag{2.12}$$

holds. (2.10) and (2.12) hold, if and only if (s_1^*, s_2^*) is a Nash equilibrium of $G = (S_1, S_2, u)$.

□

We call this Theorem the ”Characterization of Nash equilibria in a constant-sum game”, because it shows, that the game’s property of having a Nash equilibrium is independent of the constant c and we can find Nash equilibria by solving the characteristic zero-sum game. Or in other words, how willing to play certain strategy each player is, only depends, on the slope of their payoff function in that point, not on its relative height. In economy, this is known as the fact that rational agents think in margins.

2.2.2 Strictly dominated strategies

In this section we will briefly discuss the relation of strict dominance in a constant-sum game.

Definition 7 (Strict dominance)**.** *Let $G = (S_1, S_2, u)$ be a constant sum game. If $s_1, s_1' \in S_1$, $s_1 \neq s_1'$ and*

$$\forall s_2 \in S_2 : u_1(s_1, s_2) > u_1(s_1', s_2),$$

we write $s_1 \gg_{S_2} s_1'$ and say that strategy s_1 strictly dominates s_1' given S_2. Similarly, if $s_2, s_2' \in S_2$, $s_2 \neq s_2'$ and

$$\forall s_1 \in S_1 : u_2(s_1, s_2) > u_2(s_1, s_2'),$$

we write $s_2 \gg_{S_1} s_2'$ and say that strategy s_2 strictly dominates s_2' given S_1.

We will note that in addition to the strict dominance, there is also a weak or very weak dominance defined similarly but instead of requiring strictly higher payoff, we would require a payoff that is at least as high, as the one of the dominated strategy. This however means, that results for such dominance are not as strong, as for the strict dominance.

Strictly dominated strategies are such, for which exists some other strategy that yields a higher payoff in every situation. This gives us a notion of rationality in the game. Rational players would not play dominated strategies. In reality, players tend to exhibit rationality only to some extend as was shown for example in [9]. Strict dominance is a good way to identify strategies, which are not optimal as we show in the following theorem, of which special case for the matrix games was presented and proved in [7] (Theorem 7.20).

Theorem 7. *Let $G = (X, Y, u)$ be a zero-sum game in mixed strategies and let P, Q be sets of pure strategies for the first and the second player respectively. Let $p \in P$, $q \in Q$ and $P' := P \backslash \{p\}$, $Q' := Q \backslash \{q\}$ then:*

1. *If $\exists x' \in X' = Conv(P') \subseteq X$ such that $x' \gg_Y p$ and $x^* = \sum_{\hat{p} \in P} \lambda_{x^*}(\hat{p})\hat{p} \in X$ is an optimal strategy of the first player, then $\lambda_{x^*}(p) = 0$.*

2. *If $\exists y' \in Y' = Conv(Q') \subseteq Y$ such that $y' \gg_X q$ and $y^* = \sum_{\hat{q} \in Q} \lambda_{y^*}(\hat{q})\hat{q} \in Y$ is an optimal strategy of the first player, then $\lambda_{y^*}(q) = 0$.*

For briefness, we will only show the proof for the case of the first player, the proof for the second player is similar.

Proof. Let $x' \in X'$ be such that $x' = \sum_{p' \in P'} \lambda_{x'}(p')p'$ and $x' \gg_Y p$. Now let $x^* = \sum_{\hat{p} \in P} \lambda_{x^*}(\hat{p})\hat{p} \in X$ be an optimal strategy. It follows from Lemma 2 that G has a lower value $v_L = \max_{x \in X} \inf_{y \in Y} u(x, y) = \inf_{y \in Y} u(x^*, y)$, which implies that $x^* = \arg\max_{x \in X} \inf_{y \in Y} u(x, y)$. Now suppose, that $\lambda_{x^*}(p) > 0$ we define $\hat{x} := \sum_{p' \in P'} \lambda_{x^*}(p')p' + \lambda_{x^*}(p)x' = \sum_{p' \in P'} \lambda_{x^*}(p')p' + \lambda_{x^*}(p) \sum_{p' \in P'} \lambda_{x'}(p')p'$.

Clearly, $\hat{x}$ is a well defined mixed strategy in X, for which we have the following inequality

$$
\begin{aligned}
\forall y \in Y : u(x^*, y) &= \sum_{\hat{p} \in P} \lambda_{x^*}(\hat{p}) u(\hat{p}, y) \\
&< \sum_{p' \in P'} \lambda_{x^*}(p') u(p', y) + \lambda_{x^*}(p) \sum_{p' \in P'} \lambda_{x'}(p') u(p', y) \\
&= \sum_{p' \in P'} \lambda_{x^*}(p') u(p', y) + \lambda_{x^*}(p) u(x', y) = u(\hat{x}, y).
\end{aligned}
\tag{2.13}
$$

Inequality (2.13) is a contradiction with the assumption that x^* is an optimal strategy, therefore it must be that $\lambda_{x^*}(p) = 0$.

□

Strict dominance may be used as a basis of solution algorithms of iterated elimination of dominated strategies. Those algorithms in general games and their properties were discussed by many authors, for example in [11], [6] or [8]. More general approach to the problem was presented in [10], where authors generalized this method to a general choice problem and provided conditions for the algorithm to yield a unique maximal reduction.

2.2.3 Optimality in matrix games

Now we shall discuss results in the case of matrix games. Most important are following theorems. First, shows that every matrix game has a Nash equilibrium in mixed strategies. Second shows that it has a Nash equilibrium in pure strategies, if and only if the game's matrix has a saddle point.

Theorem 8. *Let $G = (X, Y, u_A)$ be a matrix game in mixed strategies, then exists $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in X \times Y$ such that*

$$\boldsymbol{x}^* \in BR(\boldsymbol{y}^*) \wedge \boldsymbol{y}^* \in BR(\boldsymbol{x}^*).$$

Proof.
Let $G = (X, Y, u_A)$ be a matrix game in mixed strategies. It follows that X and Y are convex polyhedra, which means that both sets are compact and convex. $u_A(\boldsymbol{x}, \boldsymbol{y}) := \boldsymbol{x}^{\boldsymbol{T}} A \boldsymbol{y}$ is a bilinear form, so it is a concave function in $\boldsymbol{x} \in X$ and convex function in $\boldsymbol{y} \in Y$. Therefore, the satisfactory conditions for the existence of value of G by Lemma 5 are fulfilled and by the Lemma 4, G has a Nash equilibrium.

□

Theorem 9 ([7],Theorem 7.15.)**.** *Let $G = (P, Q, u_A)$ be a matrix game in pure strategies with matrix $A = (a_{ij})_{m \times n}$. G has a Nash equilibrium, if and only if exists $k \in I$ and $l \in J$ such that*

$$\max\{a_{il}; i \in I\} = a_{kl} = \min\{a_{kj}; j \in J\}. \tag{2.14}$$

Proof.

"$\Leftarrow$": Suppose that exists $k \in I$ and $l \in J$, such that (2.14) holds, since

$$a_{kl} = (\boldsymbol{p_k})^T A \boldsymbol{q_l}, \tag{2.15}$$

from the Definition 4, $\boldsymbol{p_k} \in BR(\boldsymbol{q_l})$ and $\boldsymbol{q_l} \in BR(\boldsymbol{p_k})$ and from the Definition 5, $(\boldsymbol{p_k}, \boldsymbol{q_l}) \in P \times Q$ is a Nash equilibrium.

"$\Rightarrow$": Now suppose that there exists $(\boldsymbol{p_k}, \boldsymbol{q_l}) \in P \times Q$ a Nash equilibrium of G. This means that $\boldsymbol{p_k} \in BR(\boldsymbol{q_l}) = \{\boldsymbol{p} \in P; \forall \boldsymbol{p'} \in P : \boldsymbol{p}^T A \boldsymbol{q_l} \geq (\boldsymbol{p'})^T A \boldsymbol{q_l}\}$ and $\boldsymbol{q_l} \in BR(\boldsymbol{p_k}) = \{\boldsymbol{q} \in Q; \forall \boldsymbol{q'} \in Q : (\boldsymbol{p_k})^T A \boldsymbol{q} \leq (\boldsymbol{p_k})^T A \boldsymbol{q'}\}$ as both P and Q are sets of pure strategies inequalities

$$\forall \boldsymbol{p'} \in P : (\boldsymbol{p_k})^T A \boldsymbol{q_l} \geq (\boldsymbol{p'})^T A \boldsymbol{q_l},$$

$$\forall \boldsymbol{q'} \in Q : (\boldsymbol{p_k})^T A \boldsymbol{q_l} \leq (\boldsymbol{p_k})^T A \boldsymbol{q'},$$

may be rewritten as

$$\forall i \in I : a_{kl} = (\boldsymbol{p_k})^T A \boldsymbol{q_l} \geq (\boldsymbol{p_i})^T A \boldsymbol{q_l} = a_{il},$$

$$\forall j \in J : a_{kl} = (\boldsymbol{p_k})^T A \boldsymbol{q_l} \leq (\boldsymbol{p_k})^T A \boldsymbol{q_j} = a_{kj}.$$

Or in other words a_{kl} satisfies (2.14).

$\square$

From Theorem 8 and Lemma 3 we have that a matrix game with matrix A always has a unique value (in mixed strategies) and it is equal to

$$\min_{\boldsymbol{y} \in Y} \max_{\boldsymbol{x} \in X} \boldsymbol{x}^T A \boldsymbol{y} = \max_{\boldsymbol{x} \in X} \min_{\boldsymbol{y} \in Y} \boldsymbol{x}^T A \boldsymbol{y}.$$

We will consider the following function $v : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, that for a given $m \times n$ matrix A gives the value of a matrix game in mixed strategies with A. Formally,

$$v : A \mapsto \min_{\boldsymbol{y} \in Y} \max_{\boldsymbol{x} \in X} \boldsymbol{x}^T A \boldsymbol{y}.$$

We would say that the player 'is solving' or that 'we want to find a solution of a matrix game' with matrix A and we would mean, finding an strategy, that yields $v(A)$. As we will see from the following section, this may be attained "independently" of the specific optimal strategy of the other player.

2.2.4 Formulation of the linear program

To find optimal strategies we use the mathematical programming. In the case of matrix games such mixed strategies are solutions of a linear program (LP). Standard methods on how to solve such programs may be found in [7] and other optimization literature. We will formulate the LP for the first player, the LP for the second player is equivalently derived from Definition 6 and Lemma 3. For a given matrix game in mixed strategies $G = (X, Y, u_A)$ we want to

$$\max_{\boldsymbol{x} \in X} \min_{\boldsymbol{y} \in Y} \boldsymbol{x^T} A \boldsymbol{y}. \tag{2.16}$$

That is

$$\max_{\boldsymbol{x} \in X} \min_{\boldsymbol{y} \in Y} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j. \tag{2.17}$$

ρ is a feasible solution of (2.16) in $\boldsymbol{x} \in X$, if and only if $\forall \boldsymbol{y} \in Y : \boldsymbol{x}^T A \boldsymbol{y} \geq \rho$ and specially it is true for all pure strategies $\boldsymbol{q_j}$, $j \in J$. As both sets of mixed strategies are convex and $u(\boldsymbol{x}, .) : Y \to \mathbb{R}$ is linear for a given $\boldsymbol{x} \in X$, it must be that $\sum_{i=1}^{m} x_i a_{ij} \geq \rho$, $\forall j \in J$ is both satisfactory and necessary for (2.17), from which we have the equivalent formulation of (2.16) in the form

$$\max_{\boldsymbol{x} \in X, \rho \in \mathbb{R}} \rho \quad \text{subject to: } \sum_{i=1}^{m} x_i a_{ij} \geq \rho,\ j \in J. \tag{2.18}$$

This equivalent formulation is more convenient as it excludes the minimization problem of the second player.

3. Matrix games with random payoff

In this chapter we will focus on defining a matrix game with random payoff and providing instruments to classify and find optimal strategies in the case when the players' payoff depends, not only on their strategies, but on a random element as well.

3.1 Random payoff

First we will define a random vector, random matrix and a closely related terminology, which will be used in this chapter. We will use $(\Omega, \mathcal{A}, \mathsf{P})$ to denote the probability space.

Definition 8. *(Basic terminology).*

1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathbb{R}, \mathcal{B}, \mu)$ be a measurable space. Measurable map $X : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ we call a (real-valued) random variable.*
2. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathbb{R}^n, \mathcal{B}^n, \mu)$ be a measurable space. Vector of random variables, $\boldsymbol{X} = (X_1, X_2, \dots, X_n)^T : (\Omega, \mathcal{A}) \to (\mathbb{R}^n, \mathcal{B}^n)$, we call a random vector.*
3. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(\mathbb{R}^{mn}, \mathcal{B}^{mn}, \mu)$ be a measurable space. $A = (a_{ij})_{m \times n}$ we call a random matrix, if*

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{21} & \dots & a_{2n} & \dots & a_{mn} \end{pmatrix}^T : (\Omega, \mathcal{A}) \to (\mathbb{R}^{mn}, \mathcal{B}^{mn})$$

   is a random vector.

For a random variable X we will write $X \in \mathcal{L}^n$ and mean that $\mathsf{E}\,|X|^n < \infty$.

Without loss of generality, if $\boldsymbol{A}$ has a discrete distribution, we will denote the different realizations of A as A_i and their probabilities $p_i = \mathsf{P}[A = A_i]$, where $i \in \{1, \dots, N\}$ for some $N \in \mathbb{N} \cup \{\infty\}$ and we have that either $A_i \leq A_{i+1}$ or $A_i \nleq A_{i+1}$ & $A_{i+1} \nleq A_i$ for all $i \in \{1, \dots, N-1\}$. Furthermore, we will use the convention that, if $N = \infty$, then $\{1, \dots, N\} = \mathbb{N} = \{1, \dots, N-1\}$.

We denote the cumulative distribution function (CDF) of a vector $\boldsymbol{X}$ as $F_{\boldsymbol{X}}$ and by the CDF F_A of a random matrix A we will mean the CDF of the random vector $\boldsymbol{A}$.

We formally defined a random matrix as a rearranged random vector. We could have defined it separately, but this definition is more convenient for us, as we can use the standard notation and operations as with random vectors, such as the quantile function and "$\leq$" vector relation.

Definition 9 (Matrix game with random payoff)**.** *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space and* $A = (a_{ij})_{m\times n}$ *be a random matrix. We define* $u_A : \Omega \times S_1 \times S_2 \to \mathbb{R}$ *such that* $\forall \boldsymbol{s_1} \in S_1, \forall \boldsymbol{s_2} \in S_2, \forall\omega \in \Omega : u_A(\omega)(\boldsymbol{s_1}, \boldsymbol{s_2}) = \boldsymbol{s_1}^T A(\omega)\boldsymbol{s_2}$*. The triple*

$$G = (S_1, S_2, u_A),$$

we call a matrix game with random payoff. By a realization of G *in the case of* $\omega \in \Omega$ *we mean the deterministic matrix game* $G(\omega)$ *with the payoff function* $u_A(\omega)$*.*

This definition expands the Definition 3 used in the previous chapter by introducing the concept of randomness into the definition of the matrix game. Thus, deterministic matrix games may be thought of as matrix games with random payoff, where the payoff matrix is constant almost surely. For more convenience when we will speak of the realization of the game we will refer to it trough the realization of the game's matrix A.

3.2 Solution models for matrix games with random payoff

The case of matrix games with random payoff differs from the case of deterministic matrix games, as in contrary to the deterministic case, there may be several different reasonable approaches on how to classify an optimal solution of such a game. One may use a different model in every different situation, as it may be interpreted as more suitable for the specific real-world problem that is studied.

In this section we will present standard approaches, how to classify and find optimal strategies in matrix games with random payoff. We will model situation when players have to choose their strategies before they know the evaluation of the game. Such decisions are referred to as zero-order decisions or in some literature as the first stage decisions, this terminology comes from the two (or possibly multiple) stage formulation of the stochastic program. From characteristic of such decisions, it follows that optimal strategies of both players are not functions of ω.

3.2.1 Payoff's distribution characteristics based models

Those are the most simple and presumably, the most intuitive models, which players tend to use in a situation when their payoff is a random variable. In those models players optimize their strategies with respect to some distribution characteristic of the payoff matrix (or an approximation of it). As this characteristic is a deterministic matrix, this transforms the problem to our original form of deterministic matrix game, that was discussed in the second chapter.

Expected payoff model

This is the most simple model used to solve random matrix games. It is intuitive and easy approach to deal with the randomness of the payoff. However, as we will see, its results are not precise and the payoff, given optimal strategy, may be actually very bad in a sense, that the losses of the player may be very high, with a high probability.

Let $G = (X, Y, u_A)$ be a matrix game in mixed strategies with random payoff matrix $A = (a_{ij})_{m\times n}$ such that $a_{ij} \in \mathcal{L}^1$, $i \in I$, $j \in J$. Let us denote the expected payoff matrix $\mathbf{E} := \mathsf{E}\ A = (\mathsf{E}\ a_{ij})_{m\times n} = (\epsilon_{ij})_{m\times n}$. We say that strategy satisfies (or is optimal in) the expected payoff model if it solves the deterministic matrix game with matrix $\mathbf{E}$.

Example. Suppose a game with random matrix $A_n = (a_{ij})_{2\times 2}$ with mutually independent a_{ij} defined as $a_{11} \sim U(0,1)$, $a_{12} \sim U(-1,0)$, $a_{21} \sim U(-n, n-1)$ and $a_{22} \sim U(-n+1, n)$, where $n \in \mathbb{N}$ and $U(a,b)$ denotes a discrete uniform distribution over $\{a, a+1, \ldots, b-1, b\}$ for a, $b \in \mathbb{Z} : a < b$. Then

$$\mathbf{E} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

and so the optimal strategy of the first player in the expected payoff model would be $x^* = (0.5, 0.5)^T$ and of the second player $y^* = (0.5, 0.5)^T$, but while using this strategy, row player's loss would be as high as $-n/2$ with non-zero probability and for $n \gg 1$, the payoff while using this strategy would be smaller than 0 with probability close to 1/2. If the player would use a $(1,0)^T$ as a strategy instead, his payoff would be less than 0 with just a probability of 1/4 and it would not exceed the loss of -1. This difference would be fatal in a situation when there is a maximal limit $L \in \mathbb{N}$, that the player can afford to lose and $n > 2L$ (the utility[1] of the first player in a case when $u(\boldsymbol{x}, \boldsymbol{y}) < L$ would be $-\infty$). In such a case for a high enough n, the strategy which is optimal in the expected payoff model would be unplayable for a rational player.

Naive quantile payoff model

First, we need to formulate what we mean by a "naive quantile matrix".

Definition 10 (Naive quantile matrix)**.** *Let $A = (a_{ij})_{m\times n}$ be a random matrix, such that every component of A has a marginal distribution function $F_{a_{ij}}$. Let us denote $\forall i \in I, \forall j \in J, \forall h \in [0,1] : F^{-1}_{a_{ij}}(h) := \inf\{b \in \mathbb{R}; F_{a_{ij}}(b) \geq h\}$, that is, $F^{-1}_{a_{ij}}$ is a quantile function of a random variable a_{ij}. If $h \in [0,1]$ and $\forall i \in I, \forall j \in J : F^{-1}_{a_{ij}}(\sqrt[mn]{h}) \in \mathbb{R}$, we define a naive h-quantile matrix as $A^{(h)} := (F^{-1}_{a_{ij}}(\sqrt[mn]{h}))_{m\times n}$.*

In some cases the naive quantile matrix is easier to compute than the actual quantile matrix, but generally it is not quantile matrix and it does not satisfy the quantile condition $\mathsf{P}[A \leq A^{(h)}] \geq h$. The following lemma shows, that it is a quantile matrix in a case, when the elements of the matrix are mutually independent.

Lemma 10. *Let $m, n \in \mathbb{N}$, $h \in [0,1]$ and $A = (a_{ij})_{m\times n}$ be a random matrix such that a_{ij} are mutually independent random variables for all $i \in I, j \in J$. If $A^{(h)}$ exists, then*

[1]Utility is a theoretical function used in the decision theory, which measures the value, that the agent gives to certain choices. In the game theory, it may or may not be equal to the payoff function (depending on a situation and the interpretation of the payoff).

$$\mathsf{P}[A \leq A^{(h)}] := \mathsf{P}[\boldsymbol{A} \leq \boldsymbol{A^{(h)}}] \geq h. \tag{3.1}$$

Furthermore, if a_{ij} are all continuous random variables, then (3.1) *is satisfied as an equation.*

Proof. We have

$$\mathsf{P}[A \leq A^{(h)}] := \mathsf{P}[\boldsymbol{A} \leq \boldsymbol{A^{(h)}}] = \mathsf{P}(\bigcap_{i=1}^{m} \bigcap_{j=1}^{n} \{a_{ij} \leq F_{a_{ij}}^{-1}(\sqrt[mn]{h})\}),$$

since a_{ij} are independent this is equal to

$$\prod_{i=1}^{m} \prod_{j=1}^{n} \mathsf{P}[a_{ij} \leq F_{a_{ij}}^{-1}(\sqrt[mn]{h})] = \prod_{i=1}^{m} \prod_{j=1}^{n} F_{a_{ij}}(F_{a_{ij}}^{-1}(\sqrt[mn]{h})) \geq \prod_{i=1}^{m} \prod_{j=1}^{n} \sqrt[mn]{h} = h.$$

In the case, when a_{ij} are continuous $F_{a_{ij}}^{-1}$ is a compositional inverse of $F_{a_{ij}}$, therefore the inequality becomes equality.

□

Let $G = (X, Y, u_A)$ be a matrix game with $m \times n$ random matrix A for which exists $A^{(1-\alpha)}$, where $\alpha \in (0, 1]$ is a desired confidence level. We say, that a strategy of the first player satisfies (is optimal in) the naive quantile payoff model on a confidence level α, if it solves the deterministic matrix game with $A^{(1-\alpha)}$. Similarly, we say that strategy of the second player satisfies (is optimal in) the naive quantile payoff model on a confidence level α, if it solves the deterministic matrix game with $A^{(\alpha)}$.

This model may be interpreted as the model that players would intuitively use, if they knew only the marginal distributions of matrix A. If they knew the distribution of A, they could use mn-dimensional quantiles in a similar manner, however, as the topic of multidimensional quantiles is an extensive one, we will not address it further in this thesis.

3.2.2 Chance-constrained programming based models

More precise models are based on the theory of chance-constrained programming. Those models generalize the linear program (2.16) and (2.18) by introducing probability into the constrains of the problem.

Now we will propose a categorization of those models based on the structure of problem they solve from the point of view of the game theory.

Based on the type of the objective function of problems, that players are solving we will refer to as

1. the payoff maximizing model, if it maximizes the payoff ρ of the player, given some confidence level $\alpha \in (0, 1]$,

2. the confidence maximizing model, if it maximizes the confidence level $\alpha \in (0, 1]$, given some player's desired payoff $\rho \in \mathbb{R}$.

With respect to the bounds of the problem players are solving, we will refer to

1. the worst payoff constrained model, if it is constrained by the probability of the minimum payoff for the player,
2. the least likely payoff constrained model, if it is constrained by the minimum probability of the payoff.

This categorization gives us four basic categories of models based on the chance-constrained programming from the game theory's point of view. From the chance-constrained programming, we have two other standard categories, models with joint or individual constrains, which reflects the position of the probability operator in the problem that players want to solve. However, those two are not always disjoint categories and may depend on the further properties of the model. For example, if the random matrix has mutually independent elements, the joint and individual constrains are equivalent in the worst payoff constrained models.

Payoff maximizing model with joint worst payoff constrains

This model is based on a chance-constrained programming problems for the first and the second player defined as

$$\max_{\boldsymbol{x}\in X,\, \rho\in\mathbb{R}} \rho \quad \text{subject to} \quad \mathsf{P}[A^T\boldsymbol{x} \geq \rho\mathbf{1}] \geq \alpha \tag{3.2}$$

and

$$\min_{\boldsymbol{y}\in Y,\, \tau\in\mathbb{R}} \tau \quad \text{subject to} \quad \mathsf{P}[A\boldsymbol{y} \leq \tau\mathbf{1}] \geq \beta, \tag{3.3}$$

for given confidence levels $\alpha,\, \beta \in (0,1]$ of the first and the second player respectively. Where $\mathbf{1} = \begin{pmatrix}1, & \ldots, & 1\end{pmatrix}^T$ and we use a convention that $\mathbf{1}$ has always a corresponding dimension.

Using results of the Chapter 3 of [1], under the assumption that A has a discrete distribution with finitely many values, problems (3.2) and (3.3) may be rewritten using our conventions in the form of following mixed integer programs

$$\begin{aligned} \max_{\boldsymbol{x}\in X,\, \rho\in\mathbb{R}, \boldsymbol{z}\in\{0,1\}^N} \rho \quad \text{subject to: } & \rho\mathbf{1} - A_k^T\boldsymbol{x} \leq \boldsymbol{u}_k z_k, && k = 1,\ldots,N, \\ & \sum_{k=1}^N p_k z_k \leq 1-\alpha, \\ & z_k \in \{0,1\}, && k = 1,\ldots,N \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \min_{\boldsymbol{y}\in Y,\, \tau\in\mathbb{R}, \boldsymbol{z}\in\{0,1\}^N} \tau \quad \text{subject to: } & A_k\boldsymbol{y} - \tau\mathbf{1} \leq \boldsymbol{w}_k z_k, && k = 1,\ldots,N, \\ & \sum_{k=1}^N p_k z_k \leq 1-\beta, \\ & z_k \in \{0,1\}, && k = 1,\ldots,N. \end{aligned} \tag{3.5}$$

Where $\boldsymbol{u}_k$ and $\boldsymbol{w}_k$ are vectors of constants, such that the corresponding inequality is feasible whenever $z_k = 1$. In some literature $\boldsymbol{u}_k$ and $\boldsymbol{w}_k$ are referred to as BigMs, the problem does not depend on their exact value, as long as the corresponding bound is always feasible. This formulation of the problem may be interpreted, as that the player "chooses" in which of the realizations of A, he gets a lower payoff than the optimal and they want, that the combined probability of such realizations is less than or equal to the complement probability of their confidence level.

This formulation is useful also in the case when the payoff matrix has large, possibly countably infinite, support or in the case when it has a continuous distribution. In those cases we can use standard statistical sampling methods to find an approximation of its distribution with finite support and use it as a basis for our mixed-integer programs to find approximate solutions of this model.

In further discussion we will denote ρ^* the optimal value of (3.2) and τ^* the optimal value of (3.3). In [5] authors gave conditions for "weak duality" and "strong duality" of players' optimal values in this model to hold.

Theorem 11 ([5]; Theorem 3.0.1., "Weak duality")**.** *If $\alpha > 0.5$ and $\beta > 0.5$ then $\rho^* \leq \tau^*$.*

Theorem 12 ([5]; Theorem 4.0.2., "Strong duality" for a discrete distribution)**.** *For any two confidence levels $\alpha,\ \beta$, if there exists $N_0 \in \{1, \dots, N-1\}$ such that $\sum_{i=N_0+1}^{N} p_i < \alpha \leq \sum_{i=N_0}^{N} p_i$ and $\sum_{i=N_0+1}^{N} p_i < \beta \leq \sum_{i=N_0}^{N} p_i$ then $\rho^* = \tau^*$.*

"Duality" in the case when A has a discrete distribution is somewhat stronger as in the case when it has a continuous distribution. In Theorem 12 we did not have to impose any explicit criteria on the confidence levels at which players play. However, the existence of index N_0 depends on them. In the case when A has a continuous distribution we would explicitly require confidence levels to satisfy $\alpha + \beta = 1$.

Theorem 13 ([5]; Theorem 4.0.3., "Strong duality" for a continuous distribution)**.** *If A has a continuous distribution which satisfies*

1. *for any two $\omega_1,\ \omega_2 \in \Omega : A(\omega_1) \leq A(\omega_2)$ or $A(\omega_1) \geq A(\omega_2)$,*
2. *cumulative distribution function of A is strictly increasing*

and $\alpha + \beta = 1$, then $\rho^* = \tau^*$.

For the proofs of Theorems 11, 12 and 13 we refer to [5].

From the previous discussion we see, that in fact, the "strong duality" is, in most cases, too strong requirement for matrix games with random payoff. As we can suppose that rational players would require their confidence levels to be above 0.5 and generally do not know the confidence level of the other player. In some cases it would actually be reasonable to assume that their confidence levels are close to 1. Fact, that in the most relevant cases, strong duality does not hold, may be interpreted in a way, that players in reality play "different games". As we will see in the following model, this interpretation is not as abstract, as it may first seem.

Payoff maximizing model with individual worst payoff constrains

Let $G = (X, Y, u_A)$ be a matrix game with random payoff matrix $A = (a_{ij})_{m\times n}$ and sets of mixed strategies X and Y. Both players choose a confidence level $\alpha_j, \beta_i \in [0,1]$ for every pure strategy $j \in J$ or $i \in I$ of the other player and then solve

$$\max_{\boldsymbol{x}\in X,\, \rho\in\mathbb{R}} \rho \quad \text{subject to } \mathsf{P}[\sum_{i=1}^{m} x_i a_{ij} \geq \rho] \geq \alpha_j,\ j \in J \tag{3.6}$$

and

$$\min_{\boldsymbol{y}\in Y,\, \tau\in\mathbb{R}} \tau \quad \text{subject to } \mathsf{P}[\sum_{j=1}^{n} a_{ij} y_j \leq \tau] \geq \beta_i,\ i \in I. \tag{3.7}$$

Let A_i - the i-th row of A - and A^j - the j-th column of A - have discrete distributions with N_i and N_j different realizations respectively for each $i \in I$ and $j \in J$ and $N_1 = \sum_{j\in J} N_j$, $N_2 = \sum_{i\in I} N_i$. We denote $p_i(k)$, $k = 1,\dots,N_i$ and $p_j(k)$, $k = 1,\dots,N_j$ the probabilities of the k-th realization of each row or column respectively, then by the Chapter 3 of [1] the previous problems are equivalent to

$$\begin{aligned}
&\max_{\boldsymbol{x}\in X,\, \rho\in\mathbb{R}, \boldsymbol{z}\in\{0,1\}^{N_1}} \rho && \qquad (3.8)\\
&\text{subject to: } \rho - \sum_{i=1}^{m} x_i a_{ij}(k) \leq u_j(k) z_j(k), && j \in J,\ k = 1,\dots,N_j,\\
&\qquad \sum_{k=1}^{N_j} p_j(k) z_j(k) \leq 1 - \alpha_j, && j \in J\\
&\qquad z_j(k) \in \{0,1\}, && j \in J,\ k = 1,\dots,N_j
\end{aligned}$$

and

$$\begin{aligned}
&\min_{\boldsymbol{y}\in Y,\, \tau\in\mathbb{R}, \boldsymbol{z}\in\{0,1\}^{N_2}} \tau && \qquad (3.9)\\
&\text{subject to: } \sum_{j=1}^{n} y_j a_{ij}(k) - \tau \leq w_i(k) z_i(k), && i \in I,\ k = 1,\dots,N_i,\\
&\qquad \sum_{k=1}^{N_i} p_i(k) z_i(k) \leq 1 - \beta_i, && i \in I\\
&\qquad z_i(k) \in \{0,1\}, && i \in I,\ k = 1,\dots,N_i,
\end{aligned}$$

where $u_j(k)$ and $w_i(k)$ are constants, such that the corresponding inequalities are feasible whenever $z_j(k) = 1$ and $z_i(k) = 1$.

Generally, this is a different model than the previously discussed, as players choose a confidence level for each specific pure strategy of the other player and as a_{ij} do not have to be independent, those two models may yield different solutions. This model is better in a situation, when it is reasonable to assume, that the other player is less likely to play certain strategies than others.

In [4] authors further developed this model with the assumption that a_{ij} are mutually independent. In that case it is also equivalent to the model with joint constrains.

As shown in their paper, with this assumption, players want to find an optimal strategy for an optimal deterministic game, from a specific set of feasible games.

To show this, authors proved several lemmas and theorems. We will only mention the most important of them, for more detailed discussion we will refer to [4].

In [4] (Lemma 1), authors proved the following equivalence for the first player. Given this model, for any $j \in J$, $\boldsymbol{x} \in X$, $\rho \in \mathbb{R}$ and a_{ij}

$$\mathsf{P}[\sum_{i=1}^{m} x_i a_{ij} \geq \rho] \geq \alpha_j, \tag{3.10}$$

if and only if

$$\mathsf{P}(\bigcup_{G_j} \{a_{ij} \geq \gamma_{ij}, \forall i \in I\}) \geq \alpha_j,$$

where

$$G_j := \{\gamma_{ij}; i \in I, \sum_{i=1}^{m} x_i \gamma_{ij} \geq \rho\}.$$

In Lemma 2 and Lemma 3 they showed sufficient and necessary conditions for above equivalence to hold. First, if there exist $\gamma_{ij} \in \mathbb{R}$, $i \in I$, $j \in J$ such that

$$\prod_{i=1}^{m} \mathsf{P}[a_{ij} \geq \gamma_{ij}] \geq \alpha_j \text{ and } \sum_{i=1}^{m} x_i \gamma_{ij} \geq \rho \tag{3.11}$$

then ρ and $\boldsymbol{x}$ satisfy (3.10).

If (3.10) holds, then there exists $\gamma_{ij} \in \mathbb{R}$, $i \in I$, $j \in J$ such that

$$1 - \prod_{i=1}^{m} \mathsf{P}[a_{ij} < \gamma_{ij}] \geq \alpha_j, \ \sum_{i=1}^{m} x_i \gamma_{ij} \geq \rho. \tag{3.12}$$

Analogous conditions may be derived for the second player. The only difference is the change of corresponding inequalities.

It follows from [4], Lemmas 1, 2 and 3, that the studied problem (3.6) is equivalent to a deterministic optimization problem, where the first player wants to

$$\begin{gathered} \max_{\boldsymbol{x} \in X, \, \rho \in \mathbb{R}, \, \Gamma \in \mathbb{R}^{m \times n}} \rho \quad \text{subject to} \\ \sum_{i=1}^{m} x_i \gamma_{ij} \geq \rho, \ \prod_{i=1}^{m} (1 - F_{a_{ij}}(\gamma_{ij}) + \delta_{ij}) \geq \alpha_j, \ j \in J. \end{gathered} \tag{3.13}$$

Where $\Gamma = (\gamma_{ij})_{m \times n}$ and $\delta_{ij} = \mathsf{P}[a_{ij} = \gamma_{ij}]$[2].

We see that this transformed our former problem to the form of finding an optimal deterministic matrix game with matrix Γ which satisfies

$$\prod_{i=1}^{m} (1 - F_{a_{ij}}(\gamma_{ij}) + \delta_{ij}) \geq \alpha_j, \ j \in J.$$

[2]In their original paper authors worked with left continuous CDF, so they did not have to include the δ_{ij} term. Also note that in the case when a_{ij} are continuous random variables $\delta_{ij} = 0$.

Similarly, the problem of for the second player may be rewritten as the problem of finding an optimal deterministic game with matrix $\Phi = (\phi_{ij})_{m\times n}$ that satisfies

$$\prod_{j=1}^{n} F_{a_{ij}}(\phi_{ij}) \geq \beta_i,\ i \in I.$$

We denote

$$\begin{aligned}
T_1 &:= \{\Gamma \in \mathbb{R}^{m\times n}; \prod_{i=1}^{m}(1 - F_{a_{ij}}(\gamma_{ij}) + \delta_{ij}) \geq \alpha_j,\ j \in J\},\\
T_2 &:= \{\Phi \in \mathbb{R}^{m\times n}; \prod_{j=1}^{n} F_{a_{ij}}(\phi_{ij}) \geq \beta_i,\ i \in I\},\\
T &:= T_1 \cap T_2.
\end{aligned}$$

T_1 and T_2 are sets of feasible games for the first and the second player respectively. In their Theorem 1, authors showed that inequality payoff constrains of (3.11) and (3.12) may be replaced with equality constrains.

We will denote Γ^* and Φ^* optimal games and x^* and y^* their optimal strategies for each player respectively. It is easy to show that $v(.)$, as defined in the end of our Chapter 1, is a non-decreasing function. In [4], authors showed this in Theorem 2 and its corollary. In their Theorem 3 and its corollary they also showed that, if in each column or row there is at least one continuous random variable, all inequalities in constrains of (3.11) and (3.12) may be replaced with equalities.

Further discussion was on the relations of deterministic games that will the individual players "play". In Theorem 4 and its Corollary authors showed that in cases, which are in reality the most interesting, that is, when the confidence levels are above 0.5, set T is empty, that is, both players have different sets of feasible games, which is a result similar to the one from previous model. In [4], Theorem 5 shows that in fact, in this case $\gamma_{ij} < \phi_{ij}$ for all $i \in I$; $j \in J$ and any two $\Gamma \in T_1$, $\Phi \in T_2$. Which means that $v(\Gamma) < v(\Phi)$ for any two $\Gamma \in T_1$, $\Phi \in T_2$.

If we denote $v_1^* := v(\Gamma^*)$ and $v_2^* := v(\Phi^*)$ the optimal values of the optimal games for first and second player respectively, Theorem 8 in [4] shows that for any two $\Gamma \in T_1$ and $\Phi \in T_2$, $v_2^* - v_1^* \leq \max(\phi_{ij}) - \max(\gamma_{ij})$ or $v_2^* - v_1^* \leq \max_i(\frac{1}{n}\sum_{j=1}^{n}\phi_{ij}) - \max_j(\frac{1}{m}\sum_{i=1}^{m}\gamma_{ij})$. Those inequalities may be interpreted as approximations of gaps in optimal values of game for the first and the second player.

Solutions in pure strategies. Previous discussion gives us a better understanding of properties of feasible games for each player in this model, from them we can now get conditions for pure strategy solutions of this model.

We denote $\hat{\gamma}_{ij} = \max\{\gamma_{ij}; 1 - F_{a_{ij}}(\gamma_{ij}) + \delta_{ij} \geq \alpha_j\}$, $\hat{\phi}_{ij} = \min\{\phi_{ij}; F_{a_{ij}}(\phi_{ij}) \geq 1 - \beta_i\}$ and $\hat{\Gamma} = (\hat{\gamma}_{ij})_{m\times n}$, $\hat{\Phi} = (\hat{\phi}_{ij})_{m\times n}$. It is easy to show that generally $\hat{\Gamma} \notin T_1$ and $\hat{\Phi} \notin T_2$ because if $\hat{\Gamma} \in T_1$ then it would have to be true that for all $j \in J$

$$\prod_{i=1}^{m}(1 - F_{a_{ij}}(\hat{\gamma}_{ij}) + \hat{\delta}_{ij}) \geq (\alpha_j)^m \geq \alpha_j,$$

which implies that $\alpha_j = 1$. Similarly for the $\hat{\Phi}$.

For this reason if $\hat{\Gamma}$ has a saddle at $\hat{\gamma}_{kr}$ we define $\tilde{\gamma}_{ij} := \hat{\gamma}_{ij}$ if $j \in J$, $i = k$ and $\tilde{\gamma}_{ij} := -\infty$ else. Now let $\tilde{\Gamma} := (\tilde{\gamma}_{ij})_{m \times n}$, then $\tilde{\Gamma} \in T_1$ as

$$\prod_{i=1}^{m}(1 - F_{a_{ij}}(\tilde{\gamma}_{ij}) + \tilde{\delta}_{ij}) = \alpha_j$$

Theorem 14 ([4],Theorem 10)**.** *Let $\hat{\Gamma}$ and $\tilde{\Gamma}$ be defined as above. If $v(\hat{\Gamma}) = v(\tilde{\Gamma})$ then $\boldsymbol{p_k} \in P$ is a solution of* (3.13)*.*

Proof. Proof similar to this was presented in [4].

From the definition of $\hat{\Gamma}$, we have $\forall \Gamma \in T_1 : \hat{\Gamma} \geq \Gamma$ and from the fact that v is a non-decreasing function of Γ we have $v(\hat{\Gamma}) \geq v(\Gamma)$.Therefore, if $v(\tilde{\Gamma}) = v(\hat{\Gamma})$ we also have that $v(\tilde{\Gamma}) \geq v(\Gamma)$ for all $\Gamma \in T_1$ and as $\tilde{\Gamma} \in T_1$, it is the optimal game for the first player with only one optimal strategy $\boldsymbol{p_k}$ therefore $\boldsymbol{p_k}$ is the optimal solution of (3.13).

□

From this theorem we have that if $\hat{\Gamma}$ has a sadle point in a row k then the $\boldsymbol{p_k}$ is an optimal strategy of the first player in our original problem (3.6). Similar argument may be made for the second player.

Payoff maximizing model with individual least likely payoff constrains

The last of models we will discuss is the payoff maximizing model with individual least likely payoff constrains. This model is less cautious and based on a axiom, that the payoff a_{ij} happens with probability of $x_i y_j$, this results in a following formulation of the problems players want to solve. Let $G = (X, Y, u_A)$ be a matrix game with random matrix A. Player one wants to solve

$$\max_{\boldsymbol{x} \in X,\, \rho \in \mathbb{R}} \rho \;\text{ subject to }\; \sum_{i=1}^{m} x_i \,\mathsf{P}[a_{ij} \geq \rho] \geq \alpha_j,\, j \in J, \tag{3.14}$$

for given confidence levels $\alpha_j \in (0, 1]$, $j \in J$ and player two wants to solve

$$\min_{\boldsymbol{y} \in Y,\, \rho \in \mathbb{R}} \tau \;\text{ subject to }\; \sum_{j=1}^{n} y_j \,\mathsf{P}[a_{ij} \leq \tau] \geq \beta_i,\, i \in I, \tag{3.15}$$

for given confidence levels $\beta_i \in (0, 1]$, $i \in I$.

Assuming that each a_{ij} has a finitely many values, $a_{ij}(1), \ldots, a_{ij}(N_{ij})$ if we denote $p_{ij}(n) = \mathsf{P}[a_{ij} = a_{ij}(n)]$ and $N = \sum_{i \in I, j \in J} N_{ij}$, then using methods presented in the Chapter 3 of [1], the problem (3.14) is equivalent to

$$\begin{aligned}
&\max_{\boldsymbol{x} \in X,\, \rho \in \mathbb{R}, \boldsymbol{z} \in \{0,1\}^N} \rho && \qquad (3.16)\\
&\text{subject to: } \rho - a_{ij}(k) \leq u_{ij}(k) z_{ij}(k), && i \in I,\, j \in J,\, k = 1, \ldots, N_{ij}\\
&\sum_{i=1}^{m} \sum_{k=1}^{N_{ij}} x_i p_{ij}(k) z_{ij}(k) \leq 1 - \alpha_j, && j \in J,\\
&z_{ij}(k) \in \{0, 1\}, && i \in I,\, j \in J,\, k = 1, \ldots, N_{ij},
\end{aligned}$$

where $u_{ij}(k)$ is a constant such that the corresponding inequality is satisfied whenever $z_{ij}(k) = 1$.

Similarly, in this case, the problem (3.15) is equivalent to

$$\begin{aligned}
&\min_{\boldsymbol{y}\in Y,\,\rho\in\mathbb{R},\boldsymbol{z}\in\{0,1\}^N} \tau && (3.17)\\
&\text{subject to: } a_{ij}(k) - \tau \leq w_{ij}(k) z_{ij}(k), && i \in I,\ j \in J,\ k = 1, \ldots, N_{ij}\\
&\sum_{j=1}^{n} \sum_{k=1}^{N_{ij}} y_j p_{ij}(k) z_{ij}(k) \leq 1 - \beta_i, && i \in I,\\
&z_{ij}(k) \in \{0, 1\}, && i \in I,\ j \in J,\ k = 1, \ldots, N_{ij},
\end{aligned}$$

where $w_{ij}(k)$ is a constant such that the corresponding inequality is satisfied whenever $z_{ij}(k) = 1$.

This model was first presented in [3] where authors studied its connection with the corresponding confidence maximizing model. Model with similar idea but with joint constraints and the assumption of a specific distributions of a_{ij} was studied in [2].

4. Numeric study

In this chapter we will focus on applications of previously developed theory. We will formulate a real-world motivated problem and use different techniques presented in the third chapter of this thesis to solve it.

4.1 Formulating the problem

We will consider a case of duopoly of two electricity providers. We will assume, that each day both companies decide how much power to generate and that energy consumption each day is given by the same random demand.

Let us denote the two companies A and B. We assume, that due to the government regulations, non of the companies has a power to change the price of one GWh of energy on the market. Both A and B have different power plants which they can use to generate power, each of those with different production efficiency resulting in different power costs and each day they can either produce energy or not.

Each of the two is currently making some portion of the whole market and to sustain it, the company wants to form at least that portion of the overall profit on the market. Therefore, if we denote P_A, P_B profits of the two companies at a specific day and $r \in (0,1)$ the portion of the market currently held by company A, their payoffs will be $u_A = P_A - r(P_A + P_B)$ and $u_B = P_B - (1-r)(P_A + P_B) = -P_A + r(P_A + P_B) = -u_A$ respectively. As $u_A + u_B = 0$ for any pair (P_A, P_B) this is a well defined zero-sum game. In fact, as we will show, we can represent this competition as a matrix game with random payoff.

If running, the amount of electricity produced by each plant is given by the following table.

Power plant	Watter	Coal	Nuclear
Energy produced daily [GWh]	30	20	50

Now let $s \in \mathbb{R}$ be the selling price of 1 GWh of electricity on the market in millions of EUR. We denote $\boldsymbol{p} = (p_W, p_C, p_N)^T \in \{0,1\}^3$ the vector of indicators of which plants are being used by the company A and $\boldsymbol{c}_A \in \mathbb{R}^3$ the vector of daily costs of energy production per plant, expressed in millions of EUR, for the company A. Similarly, for the company B, $\boldsymbol{q} \in \{0,1\}^3$ and $\boldsymbol{c}_B \in \mathbb{R}^3$. Furthermore, let $\boldsymbol{o} = (o_W, o_C, o_N)^T \in \mathbb{R}^3$ be the daily output of each power plant given by the previous table, expressed in GWh.

If we assume that every day at least one plant must be running, both companies have 7 different pure strategies, we will denote them as $i, j \in \{1, \ldots, 7\}$, where the indicator vector $\boldsymbol{p}_i = (p_W^i, p_C^i, p_N^i)^T$ corresponding to the given pure strategy i is determined by the binary form of i with formula $i = p_W^i 2^2 + p_C^i 2^1 + p_N^i 2^0$ and similarly for the company B. Both companies know that the elements of the payoff matrix are in the form

$$
\begin{aligned}
a_{ij} =& (1-r)(s(\boldsymbol{p_i}^T\boldsymbol{o} + \frac{\boldsymbol{p_i}^T\boldsymbol{o}}{(\boldsymbol{p_i}+\boldsymbol{q_i})^T\boldsymbol{o}}\min(0, C-\boldsymbol{p_i}^T\boldsymbol{o}-\boldsymbol{q_j}^T\boldsymbol{o})) - \boldsymbol{p_i}^T\boldsymbol{c}_A) \\
&- r(s(\boldsymbol{q_j}^T\boldsymbol{o} + \frac{\boldsymbol{q_i}^T\boldsymbol{o}}{(\boldsymbol{p_i}+\boldsymbol{q_i})^T\boldsymbol{o}}\min(0, C-\boldsymbol{p_i}^T\boldsymbol{o}-\boldsymbol{q_j}^T\boldsymbol{o})) - \boldsymbol{q_i}^T\boldsymbol{c}_B),
\end{aligned}
\tag{4.1}
$$

where C denotes the random demand for electricity and d_A, d_B are vectors of daily costs per power plant for the company A and B respectively. We will assume that C has a discrete distribution with finitely many values from $\{C_1, \ldots, C_k\}$, where $\forall j \in \{1, \ldots, k\} : C_j \in \mathbb{N}$ and $\forall j \in \{1, \ldots, k-1\} : C_j < C_{j+1}$.

As all a_{ij} depend on the same distribution of C, A may be represented as a measurable function of C with finitely many realizations $A(C_1), ..., A(C_k)$ with their respective probabilities. We will assume that $\forall j \in \{1, \ldots, k\} : \mathsf{P}[C = C_j] = \mathsf{P}[D = j]$, where $D \sim Bi(k, p)$ for some probability $p \in [0, 1]$.

4.2 Solutions of the problem

In our study we will solve this problem with following parameters.

Specific parameters	
Parameter	Value
Portion $[r]$	1/3
Costs for A $[\boldsymbol{c}_A]$	$(10,\ 10,\ 40/3)^T$
Costs for B $[\boldsymbol{c}_B]$	$(20/3,\ 10,\ 40/3)^T$
Selling price $[s]$	1

We will consider C with values in $\{100, 101, \ldots, 120\}$ and their respective probabilities are given by $D \sim Bi(21, 0.7)$. Therefore, C may be expressed as a simple linear transformation of D with $C = 99 + D$.

From this we have expected value and variance of C given as

$$\mathsf{E}\ C = 99 + \mathsf{E}\ D = 99 + 0.7 \cdot 21 = 113.7$$

and

$$\mathrm{var}(C) = \mathrm{var}(99 + D) = \mathrm{var}(D) = 21 \cdot 0.7 \cdot 0.3 = 4.41$$

respectively.

With those set parameters we can now compute the exact pure strategy solutions in different models we studied in the Chapter 3.

4.2.1 Solution in the expected payoff model

From the above parameters and the linearity of expectation operator, we can compute each element of the expected payoff matrix ϵ_{ij} as

$$
\begin{aligned}
\mathsf{E}\ a_{ij} =& (1-r)(s(\boldsymbol{p}_i^T\boldsymbol{o} + \frac{\boldsymbol{p}_i^T\boldsymbol{o}}{(\boldsymbol{p}_i+\boldsymbol{q}_i)^T\boldsymbol{o}}\,\mathsf{E}\ \min(0, C - \boldsymbol{p}_i^T\boldsymbol{o} - \boldsymbol{q}_j^T\boldsymbol{o})) - \boldsymbol{p}_i^T\boldsymbol{c}_A) \qquad (4.2)\\
&- r(s(\boldsymbol{q}_j^T\boldsymbol{o} + \frac{\boldsymbol{q}_i^T\boldsymbol{o}}{(\boldsymbol{p}_i+\boldsymbol{q}_i)^T\boldsymbol{o}}\,\mathsf{E}\ \min(0, C - \boldsymbol{p}_i^T\boldsymbol{o} - \boldsymbol{q}_j^T\boldsymbol{o})) - \boldsymbol{q}_i^T\boldsymbol{c}_B),
\end{aligned}
$$

where

$$
\mathsf{E}\ \min(0, C - \boldsymbol{p}_i^T\boldsymbol{o} - \boldsymbol{q}_j^T\boldsymbol{o}) = \sum_{k=1}^{21} \min(0, 99 + k - \boldsymbol{p}_i^T\boldsymbol{o} - \boldsymbol{q}_j^T\boldsymbol{o}) \cdot \binom{21}{k} \cdot 0.7^k \cdot 0.3^{21-k}.
$$

From this we have the expected payoff matrix $\mathbf{E}$ with the precision of two decimal places in the form

$$
\begin{pmatrix}
12.22 & 21.11 & 8.36 & 17.78 & 4.72 & 14.44 & 2.22\\
-5.56 & 3.33 & -8.89 & 0 & -12.22 & -3.33 & -14.51\\
17.31 & 27.78 & 11.17 & 24.44 & 7.38 & 19.54 & 4.47\\
1.11 & 10 & -2.22 & 6.67 & -5.55 & 3.33 & -7.22\\
20.96 & 34.44 & 14.96 & 31.10 & 11.17 & 23.18 & 8.19\\
7.78 & 16.67 & 3.92 & 13.33 & 0.28 & 10 & -2.22\\
20.12 & 37.96 & 14.54 & 30.67 & 10.82 & 22.34 & 7.84
\end{pmatrix}.
$$

$\mathbf{E}$ has a saddle point in ϵ_{57}, therefore, by the Theorem 9, the optimal pure strategy of the first player in this model is $i^* = 5$ and of the second player $j^* = 7$, which corresponds with vectors $\boldsymbol{p}_{i^*} = (1,0,1)^T$ and $\boldsymbol{q}_{i^*} = (1,1,1)^T$. So the optimal strategy in the expected payoff model for the first player is to run water and nuclear plants and for the second player to run all three plants. In this case non of the players has an incentive to diverge from their strategy, as both would earn less (lose more in the case of the second player). The optimal payoff for the company A in this case is approximately 8.19 and for the company B -8.19. This number may be interpreted as the payoff, that each company is expected to earn/lose using this strategy.

4.2.2 Solution in the naive quantile payoff model

In this model, we want to find a deterministic matrices $A^{(1-\alpha)}$ and $A^{(\beta)}$. We will consider $\alpha = 0.95$ for the first company and $\beta = 0.95$ for the second company. To find those matrices we need to find individual quantiles of elements of A. We will denote $C(h)$ the h-quantile of C. As a_{ij} are functions of C to find their quantiles we distinguish three cases.

1. a_{ij} is a non-decreasing function of C in this case h-quantile of a_{ij} is equal to $a_{ij}(C(h))$,
2. a_{ij} is a constant function of C, in this case the h-quantile of a_{ij} is eaqual to the value a_{ij},

3. a_{ij} is a non-increasing function of C, in this case the h-quantile of a_{ij} is equal to $a_{ij}(C(1-h))$.

From the Formula (4.1) we have that this depends on the sign of

$$k(i,j) := (1-r)\boldsymbol{p_i}^T\boldsymbol{o} - r\boldsymbol{q_i}^T\boldsymbol{o}$$

and a_{ij} is non-decreasing, if $k(i,j) > 0$, a_{ij} is constant, if $k(i,j) = 0$ and a_{ij} is non-increasing, if $k(i,j) < 0$. This is due to the fact, that k gives the sign in front of the minimum term in (4.1). The minimum itself is a non-decreasing function of C for all $i \in I$ and $j \in J$, so if k is negative (4.1) is non-increasing function of C, if $k = 0$, then a_{ij} does not depend on C.

With parameters as were set in the previous section we have that $a_{17}(C) = 20/9$ and $a_{67}(C) = -20/9$ are constant, $a_{21}, a_{23}, a_{25}, a_{26}, a_{27}, a_{43}, a_{45}$ and a_{47} are non-increasing and rest are non-decreasing functions of C.

Solution from the first player's perspective

The first player in this model wants to find a deterministic matrix $A^{(1-\alpha)}$ and solve deterministic matrix game defined by it. From previous discussion we can compute naive quantile matrix

$$A^{(1-\alpha)} = \begin{pmatrix} 12.22 & 21.11 & 8.64 & 17.78 & 4.89 & 14.44 & 2.22 \\ -5.56 & 3.33 & -8.89 & 0 & -12.22 & -3.33 & -13.89 \\ 18.14 & 27.78 & 11.72 & 24.44 & 7.82 & 20.36 & 4.73 \\ 1.11 & 10.00 & -2.22 & 6.67 & -5.56 & 3.33 & -6.84 \\ 21.89 & 34.44 & 15.62 & 31.11 & 11.72 & 24.11 & 8.56 \\ 7.78 & 16.67 & 4.19 & 13.33 & 0.44 & 10.00 & -2.22 \\ 21.22 & 39.61 & 15.38 & 32.11 & 11.56 & 23.44 & 8.39 \end{pmatrix}.$$

This matrix has the saddle point in 5-th row and 7-th column, therefore the optimal strategy for the first player is the same as the optimal strategy in the expected payoff model $i^* = 5$. The company A has a optimal payoff of approximately 8.56 in this model.

Solution from the second player's perspective

In this case we find a matrix $A^{(\beta)}$ and follow the same procedure than for the first player.

$$A^{(\beta)} = \begin{pmatrix} 12.22 & 21.11 & 8.81 & 17.78 & 4.99 & 14.44 & 2.22 \\ -5.56 & 3.33 & -8.89 & 0 & -12.22 & -3.33 & -13.39 \\ 18.64 & 27.78 & 12.06 & 24.44 & 8.09 & 20.86 & 4.89 \\ 1.11 & 10.00 & -2.22 & 6.67 & -5.37 & 3.33 & -6.53 \\ 22.45 & 34.44 & 16.02 & 31.11 & 12.06 & 24.68 & 8.78 \\ 7.78 & 16.67 & 4.36 & 13.33 & 0.55 & 10.00 & -2.22 \\ 21.89 & 40.61 & 15.89 & 32.98 & 12.00 & 24.11 & 8.72 \end{pmatrix}.$$

$A^{(\beta)}$ also has a saddle point in 5-th row and 7-th column, therefore the optimal strategy of the company B in this model is $j^* = 7$ and the corresponding optimal value is -8.78.

It should not surprise us that $A^{(1-\alpha)} \leq A^{(\beta)}$. This is due to the fact that in the "ideal" situation, when a_{ij} are mutually indepenedent, they would attain values higher than $a_{ij}^{(1-\alpha)}$ with probability of α and lower than $a_{ij}^{(\beta)}$ with probability of β. So those two matrices in that case form approximate bounds of values of A.

4.2.3 Solutions in chance-constrained programming based models

Now we will use methods of chance-constrained programming to find optimal solutions for this game. In this case we want to find and optimal payoff value for each company, therefore we will solve the payoff maximization problems with three different types of constrains

1. joint worst payoff constrains,
2. individual worst payoff constrains,
3. individual least likely payoff constrains.

As our matrix A is a discrete random variable with finitely many values, we can transform the problem to the forms of mixed integer programs (3.4) and (3.5) in the first case, (3.8) and (3.9) in the second case and (3.16) and (3.17) in the last case.

In reality, it would be very unreasonable to use the least likely payoff constrained model with individual constrains in this case, as elements of the matrix are strongly correlated. In fact, they all are functions of the same random variable. This model is also computationally extensive and as we will see, in this case, it gives us the least precise results, that are only numerical approximations of pure strategies.

Joint worst payoff constrains

In this case we will use confidence levels $\alpha = \beta = 0.95$ and optimize with respect to realizations of the matrix A. We would require the constraints to be violated in those realizations of A that will occur with maximum probability of 0.05.

With those parameters we get a solution $i^* = 5$ for the first and $j^* = 7$ for the second player, which corresponds with the solutions of the expected payoff and naive quantile models. In this model the optimal payoff for the company A would be approximately 7.78 and for the company B -8.56. Those are the minimal/maximal payoffs companies would earn/lose in 95 out of 100 plays of this game. The company A would in 95 out of 100 cases earn more and the company B would lose less than those numbers.

Individual worst payoff constrains

In this model we want to maximize the payoff with respect to realizations of each column or row of the payoff matrix A. We will use $j \in J,\ i \in I : \alpha_j = \beta_i = \sqrt[7]{0.95} \sim 0.993$ as the confidence levels for each player and each row or column.

With those set parameters we attain an optimal solution for the first player $i^* = 2$ and for the second player $j^* = 2$. This would correspond with both companies running just the coal plants. The optimal payoff in this case would be approximately 3.33 for the company A and approximately -3.33 for the company B. In this case both companies optimize with respect to each strategy of the other company, this results in a lower optimal values of the game. This is however not very efficient model to use as the columns and rows of A are strongly correlated.

Individual least likely payoff constrains

We solved this model with confidence levels $i \in I,\ j \in J : \alpha_j = \beta_i = \sqrt[7]{0.95} \sim 0.993$. As we mentioned before, this model is not suitable for this exact problem. We want to maximize the payoff with respect to the every realization of each element of the matrix and require the convex combination of probabilities of realizations, for which the constrains are violated to not exceed the corresponding complementary probability. This is unreasonably computationally extensive and as axioms of the model are strongly violated in this case, results are not exact.

With such parameters we get an approximate solution of $i^* = 1$ for the company A and of $j^* = 2$ for the company B. This would correspond with the company A running just the nuclear plant and the company B running just the coal plant. The optimal payoff values in this model would be approximately 0 for the company A and approximately -3.33 for the company B. This is however not exact at all, as in every realization of A elements in the first row are higher than 0.

Discussion of results

From the three chance-constrained programming based models, we saw the best performance of the model with joint constrains. This is due to the fact that joint constraints in this case are reflecting that A is a function of a single random demand C. We considered, that there is no reason to believe, that the other player would not play certain strategies. If it was reasonable to assume, that the other company would tend not to use specific pure strategies, in models with individual constrains, the company could set low confidence levels to the corresponding pure strategies. In such a case the individual constrains may perform better overall than the joint constrains. From this, it is apparent, that when we want to find the most suitable model for a specific problem, we would also require expert judgement and non-mathematical analysis of the problem.

5. Conclusion

In this thesis we studied a special case of zero-sum games in the form of matrix games. We discussed the general theory of constant-sum, zero-sum and matrix games. We showed that zero-sum games are an important class of constant-sum games and we can use the broadly developed theory of zero-sum games to study constant-sum games as well. We presented several possible approaches to form a suitable solution model for a matrix game with random payoff, discussed their properties and applied them to solve a real-world motivated problem.

5.1 Main results of our thesis

In the Chapter 2, we formulated the terminology of constant-sum games and discussed their special cases. We provided proofs for Lemmas 4 and 9 and formulated and proved key properties of constant-sum and matrix games in Theorems 6, 7, 8 and 9. In this Chapter, we also formulated the linear program of the first player (2.16) in a matrix game and its equivalent form (2.18).

In the Chapter 3, we defined a matrix game with random payoff and presented several solution models for this category of games. First we discussed the properties of the Expected payoff model and showed why it may not be a suitable solution model, then we proposed the ”Naive quantile” payoff model as a simple alternative to this model. Later we introduced more sophisticated solution models based on the theory of chance-constrained programming. We provided the basic categorization of those models and then we examined three of payoff maximizing models in more detail. For each of examined models we formulated equivalent mixed-integer programs that players want to solve in the case when the payoff matrix is a discrete random variable with finite support of reasonable size.

In the Chapter 4, we formulated a model example of two energy providers on a closed market with random demand. We ”zero-summed” this problem and used methods developed in theoretical chapters of this thesis to analyze it. In our problem, three of used models yielded same solutions, this was due the fact that the payoff matrix in this case has a strong saddle point, which is not much affected by its randomness. That makes this solution a strong one, as it does not fully depend on the selected solution model. We also showed that models with individual constrains are not suitable for this problem as their basic axioms are violated by the strong correlation of elements of the matrix and the fact that there is no clear incentive to believe that the other player would prioritize some strategies over others. We also managed to show, that the Naive quantile payoff model, in this case, yields a reasonable approximation of optimal values that each player receives by the model with joint worst payoff constrains despite the fact, that we cannot assume independence of elements of the matrix.

5.2 Potential for further development

There are several possible ways to further develop this thesis.

In the theoretical part of the thesis there is a big potential in the study of the multidimensional quantiles and their application in solution models. Other paths to develop solution models to consider, were also presented in [3] and [2]. In the first case, authors provided specific solution algorithm for the models with least likely individual constrains in the case when the payoff matrix has a continuous distribution. In [2], author studied models with least likely joint constrains, he was able to provide some further results in the case when the payoff matrix has a specific type of marginal distributions. We see a potential in developing this model in the case when it has a discrete distribution with finite support. Another possible development of this theory would be the use of a stochastic dominance. This would require the generalization of results mentioned in the Section 2.2. of the Chapter 2 to the case when the payoff is a random variable. One possible approach to this would be to use a definition of dominance on some prescribed confidence level and try to asses its implications for matrix games with random payoff. Biggest goal to achieve would be to either proof or disproof the existence of a Nash equilibrium in the general matrix game with random payoff, this was, to the best of our knowledge, not done yet. Lastly there is a potential in developing the theory of matrix games with random payoff for matrices of infinite dimensions.

In the numeric example we see the main possibility of advancement in studying a case with larger payoff matrix and in comparison of this formulation of the problem and its predictions with real-world data on the strategies, that are used by major electricity providers in Czech or Slovak republic or other similar sized markets.

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