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Ultrafilters and their applications

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Title: Ultrafilters and their applications

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Abstract: This thesis studies ultrafilters and their various applications in topology, social choice theory and construction of a nonstandard universe.

First of all, we introduce basic properties of ultrafilters and show how to use them to construct nonstandard framework.

Next, we prove Arrow's impossibility theorem which states that every electoral system with a finite set of voters satisfying certain natural conditions necessarily admits at least one dictator who determines the society's preferences. However, if the set of voters is infinite, this is not true anymore and ultrafilters play a key role in the proof. We present two counterexamples in the infinite case using nonstandard framework.

A similar theorem holds in the case where the preferences are real functions. Again, we show two examples of electoral systems that are not dictatorial — one using Banach limits and the other using hyperfinite sums.

Finally, we use the ultrafilters to construct the Čech-Stone compactification of natural numbers. We show that the nonstandard enlargement of natural numbers equipped with suitable topology is the Čech-Stone compactification of the set of natural numbers.

Keywords: ultrafilter, nonstandard universe, social choice theory, Čech-Stone compactification

Contents

Introduction

This thesis studies ultrafilters and their various applications. The first application that we show and that reappears throughout this work is the construction of nonstandard framework. Nonstandard framework is a mathematical approach that embeds a so-called universe that contains common mathematical objects (such as sets, relations, functions, ...) into another (enlarged) universe. This enlarged universe contains special elements which can be used to describe classical definitions and results more intuitively. Some are of the opinion that this approach was already used many centuries ago for example by Newton or Leibniz. However, their efforts struggled with the lack of precise definitions and rigour. Abraham Robinson is considered to be a founder of modern nonstandard analysis. He summarized his contribution in the book *Non-standard Analysis* (Robinson [1966]).

Another application of ultrafilters may be found in the social choice theory. The most famous theorem of this field is known as Arrow's impossibility theorem and states that every electoral system with a finite set of voters satisfying certain natural conditions necessarily admits at least one dictator who determines the society's preferences. However, if the set of voters is infinite, this is not true anymore and ultrafilters play a key role in the proof.

In the first part of this thesis, the basics of ultrafilters and nonstandard framework are presented. The definitions and theorems about ultrafilters are studied in the topology and may be found in many handbooks on topology, for example in *General Topology* (Willard [2004]). The introduction of nonstandard analysis relies mostly on the book *Lectures on the Hyperreals* (Goldblatt [2012]) with several proofs that were not given in detail in the book.

The second and third part focus on the social choice theory. In the second chapter, we consider the case where individual preferences are binary relations. The proof of Arrow's impossibility theorem from Hansson [1976] is given followed by two nonstandard counterexamples in the case where the set of voters is infinite.

Similarly, in the third chapter two methods of construction of a non-dictatorial system are given in the case where the preferences are real functions determining not only which preference is better than the other, but also quantifying how much it is preferred. In this case, one method is standard and uses Banach limits, whereas the other one is nonstandard making use of hyperfinite sums.

The final chapter studies the Čech-Stone compactification of the set of natural numbers. First, necessary definitions and statements are given. Next, we use the ultrafilters to construct the Čech-Stone compactification of natural numbers. Finally, we show that the nonstandard enlargement of natural numbers equipped with suitable topology is the Čech-Stone compactification. This theorem comes from Di Nasso [2015] and we fill in some additional details that were missing in the original paper.

1. Preliminaries

1.1 Ultrafilters

Ultrafilters are special collections of sets. They may be seen as "large sets" since they contain all supersets of their elements. As we will see later, they are used in definitions where certain condition should hold for "almost all" elements, or, in other words, for elements of some "large set". First, we introduce the basic properties of ultrafilters.

Definition 1.1.1. *Let X be a nonempty set. A **filter** on X is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that:*

(i) $\mathcal{F} \neq \emptyset$,

(ii) $\emptyset \notin \mathcal{F}$,

(iii) $K, L \in \mathcal{F} \Rightarrow K \cap L \in \mathcal{F}$,

(iv) $((K \in \mathcal{F}) \wedge (K \subseteq L)) \Rightarrow L \in \mathcal{F}$.

*It is an **ultrafilter** if moreover*

(v) $\forall K \subseteq X : (K \in \mathcal{F}) \vee (X \setminus K \in \mathcal{F})$.

Remark. It follows from conditions (v) and (ii) that for every subset K of X exactly one of the sets K and $X \setminus K$ is in the ultrafilter $\mathcal{F}$.

Lemma 1.1.2. *A filter $\mathcal{F}$ is an ultrafilter if and only if it is maximal, i.e. there is no filter $\mathcal{G}$ such that $\mathcal{F} \subsetneq \mathcal{G}$.*

Proof. First, suppose that the condition (v) of Definition 1.1.1 holds and for contradiction suppose that there exists a filter $\mathcal{G}$ such that $\mathcal{F} \subsetneq \mathcal{G}$. Take $S \in \mathcal{G} \setminus \mathcal{F}$. Then $S \notin \mathcal{F}$, so necessarily $X \setminus S \in \mathcal{F}$. Hence, $X \setminus S \in \mathcal{G}$. From condition (iii) we get $\emptyset = S \cap (X \setminus S) \in \mathcal{G}$ which contradicts condition (ii).

Conversely, suppose that $\mathcal{F}$ is maximal and for $K \subseteq X$, $K \notin \mathcal{F}$ and $X \setminus K \notin \mathcal{F}$. It holds that $K \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, because otherwise $F \subseteq (X \setminus K)$ and $X \setminus K$ would be in $\mathcal{F}$. Hence, we can define

$$\mathcal{U}' = \{A \subseteq X : \exists A_1, \ldots, A_k \in (\mathcal{F} \cup \{K\}),\ A_1 \cap \cdots A_k \subseteq A\}.$$

Then $\mathcal{U}'$ is an ultrafilter and $\mathcal{F}$ is its proper subcollection. That contradicts the maximality of $\mathcal{F}$. □

Lemma 1.1.3. *Let $\mathcal{U}$ be an ultrafilter and A, B sets such that $A \cup B \in \mathcal{U}$. Then $A \in \mathcal{U}$, or $B \in \mathcal{U}$.*

Proof. Suppose that $A \notin \mathcal{U}$. From Definition 1.1.1(v) and (iii) we get $A^C \in \mathcal{U}$ and $A^C \cap (A \cup B) \in \mathcal{U}$. We can write the set B as

$$B = (A \cap B) \cup \left(A^C \cap (A \cup B)\right),$$

so from Definition 1.1.1(iv) it follows that $B \in \mathcal{U}$ since it is a superset of $A^C \cap (A \cup B)$. □

Definition 1.1.4. *We say that an ultrafilter $\mathcal{F}$ is **principal** or **fixed** if*

$$\bigcap_{U \in \mathcal{F}} U \neq \emptyset.$$

*A filter that is not fixed is called **non-principal** or **free**.*

Remark. From Lemma 1.1.3 we get that if an ultrafilter contains a finite set, then it also contains a one-element set and therefore is principal. Such an ultrafilter is of the form $\{U \subseteq X : a \in U\}$ for some $a \in X$. This holds because it contains the singleton $\{a\}$, so by Definition 1.1.1 (iv) the ultrafilter comprises all its supersets. On the other hand, if a set does not contain a, then its complement does, hence it cannot lie in the ultrafilter.

Moreover, all principal ultrafilters are of the form $\{U \subseteq X : a \in U\}$ for some $a \in X$. If a principal ultrafilter $\mathcal{F}$ contains a finite set, then the claim follows from the previous paragraph. Suppose that $\mathcal{F}$ is a principal ultrafiter and $I = \bigcap_{U \in \mathcal{F}} U$ is an infinite set. Let $a \in I$. Then either the set $I \setminus \{a\}$ or $(X \setminus I) \cup \{a\}$ is in $\mathcal{F}$. However, both options contradict the definition of I.

Theorem 1.1.5. *If X is a nonempty finite set, then every ultrafilter on X is of the form $\{U \subseteq X : a \in U\}$, and so is fixed.*

Proof. This follows from the previous remark – the set X belongs to $\mathcal{F}$ and is finite, therefore the ultrafilter is fixed. □

Definition 1.1.6. *Let $\mathcal{F}$ be a family of nonempty sets. We say that $\mathcal{F}$ has the **finite intersection property (FIP)** if whenever I is a finite index set and $\forall i \in I : A_i \in \mathcal{F}$ we have*

$$\bigcap_{i \in I} A_i \neq \emptyset.$$

The Zorn's lemma, as stated below, is equivalent to the axiom of choice and is needed in the theory of ultrafilters. The definitions of terms used in the lemma can be found in appendix A.1.

Theorem 1.1.7 (Zorn's lemma)**.** *If a partially ordered set $(P, \leq)$ has the property that every chain in P has an upper bound in P, then the set P contains a $\leq$-maximal element.*

Theorem 1.1.8. *Let X be a set and let $A \subseteq \mathcal{P}(X)$ have the finite intersection property. Then there is an ultrafilter $\mathcal{U}$ on X such that $A \subseteq \mathcal{U}$.*

Proof. Since A has the finite intersection property, the collection

$$\mathcal{A} = \{B \subseteq X : \exists A_1, \dots, A_k \in A,\ A_1 \cap \dots \cap A_k \subseteq B\}$$

is a filter generated by A. Consider all filters on X that contain $\mathcal{A}$ and order them by set inclusion. To use Zorn's lemma, we need to show that every chain has an upper bound, i.e. that for every linearly ordered set of filters $\{\mathcal{F}_x : x \in I\}$ the union

$$\bigcup_{x \in I} \mathcal{F}_x$$

is a filter. The first two properties of Definition 1.1.1 are trivial. For the third one, take $K, L \in \bigcup_{x \in I} \mathcal{F}_x$. Then there exist $x, y \in I$ such that $K \in \mathcal{F}_x$ and

$L \in \mathcal{F}_y$. From the linear order, either $\mathcal{F}_x \subseteq \mathcal{F}_y$ or $\mathcal{F}_y \subseteq \mathcal{F}_x$. Without loss of generality, suppose that it is the first case. Then $K, L \in \mathcal{F}_y$ and also $K \cap L \in \mathcal{F}_y$ because $\mathcal{F}_y$ is a filter.

Similarly, if $K \in \bigcup_{x \in I} \mathcal{F}_x$ and $K \subseteq L$, then there exist $x \in I$ such that $K \in \mathcal{F}_x$, therefore $L \in \mathcal{F}_x \subseteq \bigcup_{x \in I} \mathcal{F}_x$.

Hence, by Theorem 1.1.7, there exists a maximal filter containing $\mathcal{A}$ and by Lemma 1.1.2 it is an ultrafilter. □

Theorem 1.1.9. *If X is infinite, then there exists a free ultrafilter on X.*

Proof. Take a collection of all cofinite subsets of X, i.e.

$$\mathcal{A} = \{A \subseteq X : X \setminus A \text{ is finite}\}.$$

Then $\mathcal{A}$ has the finite intersection property, so by Theorem 1.1.8, there is an ultrafilter $\mathcal{U}$ on X such that $\mathcal{A} \subseteq \mathcal{U}$. If $\mathcal{U}$ was fixed, then it would be of the form $\{U \subseteq X : a \in U\}$ for some $a \in X$. But the set $X \setminus \{a\}$ is cofinite, so it is contained in $\mathcal{U}$. Hence, the set $\{a\}$ is not in $\mathcal{U}$ for any $a \in X$. Therefore, $\mathcal{U}$ is free. □

1.2 Nonstandard Framework

In this section, we will establish the framework for a nonstandard approach. First, we present the axiomatic approach. Then we introduce the ultrapower construction which may be more intuitive. Both constructions were taken from Goldblatt [2012], Chapters 13 and 14.

1.2.1 General Construction

Definition 1.2.1. *A set X is called **transitive** if $a \in A \in X$ implies $a \in X$.*

Definition 1.2.2. *A **universe** is a set $\mathbb{U}$ satisfying:*

*(i) $\mathbb{U}$ is **strongly transitive**, i.e. for any set $A \in \mathbb{U}$ there exists a transitive set $B \in \mathbb{U}$ such that $A \subseteq B \subseteq \mathbb{U}$;*

(ii) if $a, b \in \mathbb{U}$, then $\{a, b\} \in \mathbb{U}$;

(iii) if $A, B \in \mathbb{U}$, then $A \cup B \in \mathbb{U}$;

(iv) if A is a set in $\mathbb{U}$, then $\mathcal{P}(A) \in \mathbb{U}$.

*If a set $\mathbb{X}$ belongs to $\mathbb{U}$, we then say that $\mathbb{U}$ is a **universe over** $\mathbb{X}$.*

Example. Let $\mathbb{X}$ be a set containing real numbers. For any $n \in \mathbb{N}$ we define:

$$\mathbb{U}_0(\mathbb{X}) = \mathbb{X}$$
$$\mathbb{U}_{n+1}(\mathbb{X}) = \mathbb{U}_n(\mathbb{X}) \cup \mathcal{P}\left(\mathbb{U}_n(\mathbb{X})\right).$$

The **superstructure over** $\mathbb{X}$ is defined as

$$\mathbb{U}(\mathbb{X}) = \bigcup_{n=0}^{\infty} \mathbb{U}_n(\mathbb{X}).$$

An equivalent definition of $\mathbb{U}_{n+1}(\mathbb{X})$ is

$$\mathbb{U}_{n+1}(\mathbb{X}) = \mathbb{X} \cup \mathcal{P}(\mathbb{U}_n(\mathbb{X})).$$

The members of $\mathbb{X}$ are called **individuals** since they have no elements:

$$(\forall x \in \mathbb{X}) : (x \neq \emptyset \wedge (\forall y \in \mathbb{U}(\mathbb{X})) : y \notin x) .$$

All other members of $\mathbb{U}(\mathbb{X})$ are sets. We define a **rank** of an entity $a \in \mathbb{U}(\mathbb{X})$ as the least n such that $a \in \mathbb{U}_n(\mathbb{X})$

We claim that the superstructure $\mathbb{U}(\mathbb{X})$ is a universe over $\mathbb{X}$. From the inductive definition, we have $\mathbb{U}_n(\mathbb{X}) \in \mathbb{U}_{n+1}(\mathbb{X}) \subseteq \mathbb{U}(\mathbb{X})$. In particular, $\mathbb{X} \in \mathbb{U}(\mathbb{X})$.

To prove the strong transitivity, we observe that for every $n \in \mathbb{N}$, $\mathbb{U}_{n+1}(\mathbb{X})$ is transitive: if $a \in A \in \mathbb{U}_{n+1}(\mathbb{X}) = \mathbb{X} \cup \mathcal{P}(\mathbb{U}_n(\mathbb{X}))$, then either $A \in \mathbb{X}$ or $A \in \mathcal{P}(\mathbb{U}_n(\mathbb{X}))$. In the latter case, we get $a \in \mathbb{U}_n(\mathbb{X}) \subseteq \mathbb{U}_{n+1}(\mathbb{X})$. The case $A \in \mathbb{X}$ is not possible since members of $\mathbb{X}$ are individuals but $a \in A$.

Now, if $A \in \mathbb{U}(\mathbb{X})$, then $A \in \mathbb{U}_n(\mathbb{X})$ for some $n \in \mathbb{N}$, so we can define $B = \mathbb{U}_{n+1}(\mathbb{X})$ to be a transitive set such that $A \subseteq B \subseteq \mathbb{U}(\mathbb{X})$.

The other conditions hold due to the following properties:

(i) If $a, b \in \mathbb{U}_n(\mathbb{X})$, then $\{a, b\}$ is a subset of $\mathbb{U}_n(\mathbb{X})$, so it belongs to $\mathbb{U}_{n+1}(\mathbb{X})$.

(ii) If $A, B \in \mathbb{U}_n(\mathbb{X})$, then $A \cup B \in \mathbb{U}_{n+1}(\mathbb{X})$.

(iii) If $A \in \mathbb{U}_n(\mathbb{X})$, then subsets of A belong to $\mathbb{U}_{n+1}(\mathbb{X})$ and the power set $\mathcal{P}(A)$ itself is a member of $\mathbb{U}_{n+2}(\mathbb{X})$.

$\triangle$

All members of $\mathbb{U}$ are individuals or sets. However, the sets are enough to describe many other mathematical structures. For example, an ordered pair $\langle a, b\rangle$ can be expressed as the set $\{\{a\}, \{a, b\}\}$. The m-tuples can be defined inductively as $\langle a_1, \ldots, a_{n+1}\rangle = \langle\langle a_1, \ldots, a_m\rangle, a_{m+1}\rangle$. An m-ary relation is a set of m-tuples and a function can be expressed as a set of ordered pairs $\langle a, f(a)\rangle$.

Further, we need to define the language associated with the universe $\mathbb{U}$, denoted $\mathcal{L}_{\mathbb{U}}$.

The primary component of $\mathcal{L}_{\mathbb{U}}$ are **terms**. All variables and members of $\mathbb{U}$ (called **constants**) are terms. Moreover, given terms $\tau_1, \ldots, \tau_m, \tau, \sigma$, the tuple $\langle \tau_1, , \ldots, \tau_m\rangle$ and function-value $\tau(\sigma)$ are also $\mathcal{L}_{\mathbb{U}}$-terms.

Atomic formulae are of the form $\tau = \sigma$ or $\tau \in \sigma$ for $\mathcal{L}_{\mathbb{U}}$-terms τ, σ.

Each atomic formula is an $\mathcal{L}_{\mathbb{U}}$-**formula**. If ψ and ϕ are $\mathcal{L}_{\mathbb{U}}$-formulae, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$. In addition, if τ is an $\mathcal{L}_{\mathbb{U}}$-term and x is a variable not occurring in τ, then $(\forall x \in \tau)\phi$ and $(\exists x \in \tau)\phi$ are also $\mathcal{L}_{\mathbb{U}}$-formulae. A **sentence** is a formula whose variables are bound by existential or universal quantifiers.

If $\mathbb{U} \xrightarrow{*} \mathbb{U}'$ is a mapping between two universes, then we can replace all occurrences of constants $a \in \mathbb{U}$ in $\mathcal{L}_{\mathbb{U}}$-formula ϕ by ${}^*a$ and get an $\mathcal{L}_{\mathbb{U}'}$-formula ${}^*\phi$.

Definition 1.2.3. *A* ***nonstandard framework*** *for a set $\mathbb{X}$ is a universe $\mathbb{U}$ over $\mathbb{X}$ together with a* ***transfer map*** *$\mathbb{U} \xrightarrow{*} \mathbb{U}'$ such that:*

(i) ${}^*a = a$ *for all* $a \in \mathbb{X}$,

(ii) ${}^*\emptyset = \emptyset$,

(iii) *a sentence* φ *is true if and only if* ${}^*\varphi$ *is true.*

The last property is called the **transfer principle**.

Definition 1.2.4. *An element* $a \in \mathbb{U}'$ *is called:*

(i) ***standard****, if it is of the form* ${}^*b$ *for some* $b \in \mathbb{U}$*;*

(ii) ***nonstandard****, if it is not standard;*

(iii) ***internal****, if* $a \in {}^*A$ *for some* $A \in \mathbb{U}$*;*

(iv) ***external****, if it is not internal.*

As we will see later, internal sets are very important because if we have a true sentence describing the certain property for all subsets of a set A, then it transfers to a true sentence describing the property of all internal subsets of ${}^*A$. Similarly, a sentence that holds for all functions in the standard universe transfers to a sentence that holds for all internal functions in the nonstandard universe.

Definition 1.2.5. *We say that a universe* $\mathbb{U}'$ *is an* ***enlargement*** *of a universe* $\mathbb{U}$ *relative to* $\xrightarrow{*}$ *if whenever* $A \in \mathbb{U}$ *is a collection of sets with the finite intersection property, then*

$$\bigcap_{B \in A} {}^*B \neq \emptyset.$$

The enlargements will play an important role in Section 2.4 and Chapter 4. Not every nonstandard framework is an enlargement. However, the following theorem holds and will be proved once the ultrapower construction of a nonstandard framework is introduced.

Theorem 1.2.6 (Enlargement theorem, Goldblatt [2012])**.** *For an infinite set* $\mathbb{X}$ *there exists an enlargement of* $\mathbb{U}(\mathbb{X})$ *that is of the form* $\mathbb{U}({}^*\mathbb{X})$*.*

1.2.2 Ultrapower Construction

Let I be an infinite set. In most of this work, $I = \mathbb{N}$ would be sufficient. However, in Section 2.4 and Chapter 4, the enlargement is needed, hence more complicated set is used – this is shown in the proof of Theorem 1.2.6.

Let $\mathbb{U}(\mathbb{X})^I$ be a set of all functions from I to $\mathbb{U}(\mathbb{X})$. For constants $a \in \mathbb{U}(\mathbb{X})$ we denote a_I the constant function with value a. For two functions $f, g \in \mathbb{U}(\mathbb{X})^I$ we denote

$$[[f = g]] := \{i \in I : f(i) = g(i)\}.$$

Let $\mathcal{F}$ be a non-principal ultrafilter on I. This ultrafilter serves as an indication of "large sets". Two functions from $\mathbb{U}(\mathbb{X})^I$ can be identified if they agree on a "large set", i.e.

$$[[f = g]] \in \mathcal{F}.$$

In other words, two functions are identified if $f(i) = g(i)$ for "almost all" $i \in I$. For this, we need the ultrafilter $\mathcal{F}$ to be non-principal, because otherwise a one-element set would belong to $\mathcal{F}$ and we would identify functions that agree only on one element, which does not correspond to the intuition about "large sets".

In fact, $[[f = g]]$ is an equivalence relation. The only non-trivial condition is transitivity and it holds due to Definition 1.1.1(iii).

In the same way, we define:

$$\begin{aligned}
[[f \in g]] &= \{i \in I : f(i) \in g(i)\}, \\
[[f \in a]] &= [[f \in a_I]] = \{i \in I : f(i) \in a\}, \\
Z_n &= \{f \in \mathbb{U}(\mathbb{X})^I : [[f \in \mathbb{U}_n(\mathbb{X})]] \in \mathcal{F}\}, \\
Z &= \bigcup \{Z_n : n \in \mathbb{N}\}.
\end{aligned}$$

For $f \in Z_0$ we denote

$$[f] = \{g \in Z_0 : [[f = g]] \in \mathcal{F}\}$$

and

$$\mathbb{Y} = \{[f] : f \in Z_0\}.$$

We define inductively $\mathbb{U}_n(\mathbb{Y})$: For $f \in Z_0$, $[f] \in \mathbb{U}_0(\mathbb{Y}) = \mathbb{Y}$ was already defined. Suppose that for all $f \in Z_n$ $[f]$ is defined and $[f] \in \mathbb{U}_n(\mathbb{Y})$. Then for $g \in Z_{n+1} \setminus Z_n$ we define

$$[g] = \{[f] : f \in Z_n \text{ and } [[f \in g]] \in \mathcal{F}\} \in \mathbb{U}_{n+1}(\mathbb{Y}).$$

Then the superstructure over $\mathbb{Y}$ takes the form

$$\mathbb{U}(\mathbb{Y}) = \bigcup_{n \in \mathbb{N}} \mathbb{U}_n(\mathbb{Y}).$$

We have two natural maps:

$$a \mapsto a_I \text{ from } \mathbb{U}(\mathbb{X}) \text{ to } Z.$$

and

$$f \mapsto [f] \text{ from } Z \text{ to } \mathbb{U}(\mathbb{Y}).$$

Their composition is a map $* : a \mapsto [a_I]$ from $\mathbb{U}(\mathbb{X})$ to $\mathbb{U}(\mathbb{Y})$.
If $a \in \mathbb{X}$, then ${}^*a \in \mathbb{Y}$. If a has a rank $n + 1$, then

$${}^*a = \{[f] : f \in Z_n \text{ and } [[f \in a]] \in \mathcal{F}\}.$$

Accordingly,

$${}^*\mathbb{U}_n(\mathbb{X}) = \{[f] : [[f \in \mathbb{U}_n(\mathbb{X})]] \in \mathcal{F}\} = \{[f] : f \text{ has rank } \leq n\}.$$

In particular,

$${}^*\mathbb{X} = \{[f] : [[f \in \mathbb{X}]] \in \mathcal{F}\} = \mathbb{Y}.$$

Hence,

$$\mathbb{U}(\mathbb{Y}) = \mathbb{U}({}^*\mathbb{X}),$$

so the map $\xrightarrow{*}$ goes from $\mathbb{U}(\mathbb{X})$ to $\mathbb{U}({}^*\mathbb{X})$. In order to show that $\mathbb{U}(\mathbb{X})$ together with the map $\mathbb{U}(\mathbb{X}) \xrightarrow{*} \mathbb{U}({}^*\mathbb{X})$ is a nonstandard framework, it remains to verify the transfer principle and to check that ${}^*\emptyset = \emptyset$. The latter holds since

$${}^*\emptyset = \{[f] : [[f \in \emptyset]] \in \mathcal{F}\} = \emptyset$$

The transfer principle holds due to Łoś's Theorem:

Theorem 1.2.7 (Łoś, Goldblatt [2012]). *For any $\mathcal{L}_{\mathbb{U}(\mathbb{X})}$-formula $\varphi(x_1, \dots, x_p)$ and any $f_1, \dots, f_p \in Z$ the sentence*

$$^*\varphi([f_1], \dots, [f_p])$$

is true if and only if

$$\{i \in I : \varphi(f_1(i), \dots, f_p(i)) \text{ is true}\} \in \mathcal{F}.$$

Altogether, $\mathbb{U}(\mathbb{X}) \xrightarrow{*} \mathbb{U}(^*\mathbb{X})$ is a transfer map.

We can now construct a special ultrafilter such that if we follow the ultrapower construction with this ultrafilter, we get an enlargement.

Proof of Theorem 1.2.6. Let $I = \mathcal{P}_F(\mathbb{U}(\mathbb{X}))$ be a set of all finite subsets of $\mathbb{U}(\mathbb{X})$. For $A \in I$ we define

$$I_A = \{B \in I : A \subseteq B\}.$$

Then the collection $\{I_A : A \in I\}$ has the finite intersection property because for any $A_1, \dots, A_n \in I$ we can define $B = A_1 \cup \dots \cup A_n$. For this B we have

$$B \in \bigcap_{i \in \{1, \dots, n\}} I_{A_i}.$$

By Theorem 1.1.8, there exists an ultrafilter $\mathcal{U}$ containing the collection $\{I_A : A \in I\}$. This ultrafilter is non-principal: for contradiction suppose that there exists

$$V \in \bigcap_{U \in \mathcal{U}} U.$$

From the definition of $\mathcal{U}$, this means that for every $A \in I$, a finite subset of $\mathbb{U}(\mathbb{X})$, the set V is in I_A, i.e. $A \subseteq V$ and $V \in I$. On the one hand, we have that V is a finite subset of $\mathbb{U}(\mathbb{X})$, on the other hand

$$\bigcup_{A \in I} A \subseteq V$$

and $\bigcup_{A \in I} A$ is infinite. That is the contradiction. Hence, $\mathcal{U}$ is a non-principal ultrafilter.

We want to prove that the superstructure $\mathbb{U}(^*\mathbb{X})$ associated with $\mathcal{U}$ is an enlargement. Let $A \in \mathbb{U}(\mathbb{X})$ be a collection of sets with the finite intersection property. For any $i \in I$, the intersection

$$i \cap A := \{i \cap a : a \in A\}$$

may be empty or finite. If it is finite, then by the FIP the intersection

$$A_i = \bigcap_{a \in A} (i \cap a)$$

is not empty, hence it contains an element x. We define a function f:

$$f(i) = \begin{cases} \emptyset & \text{if } A_i = \emptyset \\ x & \text{otherwise} \end{cases}.$$

Then f has bounded rank since

$$f(i) \in \bigcup_{a \in A} a \text{ and } A \in \mathbb{U}(\mathbb{X}),$$

so $[f] \in \mathbb{U}({}^*\mathbb{X})$. We claim that

$$[f] \in \bigcap_{a \in A} {}^*a.$$

Take $a \in A$. By the ultrafilter construction,

$$[f] \in {}^*a \iff \{i \in I : f(i) \in a\} \in \mathcal{U}.$$

The set $\{a\}$ is finite, so $\{a\} \in I$. Thus, from the construction of the ultrafilter $\mathcal{U}$, we have

$$I_{\{a\}} = \{b \in I : a \subseteq b\} \in \mathcal{U}.$$

If $a \subseteq b$, then $a = b \cap a$ and $b \cap a$ is one of the sets whose intersection is A_b, therefore $f(b) \in (b \cap a) = a$. Hence,

$$\{i \in I : f(i) \in a\} \supseteq \{b \in I : a \subseteq b\} \in \mathcal{U}$$

and from Definition 1.1.1(iv),

$$\{i \in I : f(i) \in a\} \in \mathcal{U}$$

as well. Thus, the proof is completed. □

Example. Take $I = \mathbb{N}$ and $\mathcal{U}$ a non-principal ultrafilter over $\mathbb{N}$. The result of the ultrapower construction is a universe $\mathbb{U}({}^*\mathbb{X})$ whose elements are sequences. For easier understanding, consider the set of real numbers. First, we build the set of sequences of real numbers

$$\mathbb{R}^{\mathbb{N}} = \{(r_n)_{n \in \mathbb{N}} : \forall n \in \mathbb{N} : r_n \in \mathbb{R}\}$$

We identify two sequences of real numbers, $r = (r_n)_n$ and $s = (s_n)_n$, if

$$[[s = r]] = \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{U}.$$

This defines the equivalence classes

$$[r] = \{s \in \mathbb{R}^{\mathbb{N}} : [[r = s]] \in \mathcal{U}\}.$$

We use the notation $[(r_n)_n]$, $[r_n]$ or $[r]$ to denote the equivalence class of the sequence $r = (r_n)_{n \in \mathbb{N}}$. Then we can write

$${}^*\mathbb{R} = \{[r] : r \in \mathbb{R}^{\mathbb{N}}\}.$$

The elements of $\mathbb{R}$ are embedded into ${}^*\mathbb{R}$ using constant sequences, i.e. if $r \in \mathbb{R}$, then ${}^*r = [(r)_n]$. An example of a nonstandard hyperreal number is an element of ${}^*\mathbb{R}$ defined by the sequence $\omega = (1, 2, 3, 4, \dots)$. For every $r \in \mathbb{R}$ the set $[[(r)_n = \omega]]$ has at most one element, so it cannot belong to $\mathcal{U}$.

The order on ${}^*\mathbb{R}$ is defined in the following way:

$$[r_n] < [s_n] \Leftrightarrow [[r < s]] = \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{U}$$
$$[r_n] \geq [s_n] \Leftrightarrow [[r \geq s]] = \{n \in \mathbb{N} : r_n \geq s_n\} \in \mathcal{U}$$

and similarly for other inequalities. We have, $[(r)_n] < [\omega]$ because the set $[[(r)_n < \omega]]$ is finite for every $r \in \mathbb{R}$, hence also not in $\mathcal{U}$.

We define addition and multiplication in a natural way:

$$[r_n] + [s_n] = [(r_n + s_n)_n]$$
$$[r_n] \cdot [s_n] = [(r_n \cdot s_n)_n].$$

The subsets of ${}^*\mathbb{R}$ are sequences of subsets of $\mathbb{R}$. If $[(A_n)_n] \subseteq {}^*\mathbb{R}$ and $[r_n] \in {}^*\mathbb{R}$, then

$$[r_n] \in [A_n] \Leftrightarrow [[r \in A]] = \{n \in \mathbb{N} : r_n \in A_n\} \in \mathcal{U}.$$

So, if $A \subseteq \mathbb{R}$, then

$${}^*A = \{[r] \in {}^*\mathbb{R} : [[r \in A]] \in \mathcal{U}].$$

A function $f : \mathbb{R} \to \mathbb{R}$ extends to ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ as

$${}^*f([r_n]) = [(f(r_n))_n] = [(f(r_1), f(r_2), f(r_3), \dots)].$$

Then

$${}^*f([r_n]) = [s_n] \Leftrightarrow \{n \in \mathbb{N} : f(r_n) = s_n\} \in \mathcal{U}.$$

In general, if we have a sequence of functions $f_n : A_n \to \mathbb{R}$, then we can define the function $[f_n] : [A_n] \to {}^*\mathbb{R}$ by the formula

$$[f_n]([r_n]) = [f_n(r_n)],$$

so

$$[f_n]([r_n]) = [s_n] \Leftrightarrow \{n \in \mathbb{N} : f_n(r_n) = s_n\} \in \mathcal{U}.$$

All elements obtained by the construction mentioned above are internal. An example of an external subset of ${}^*\mathbb{R}$ is the set $\mathbb{N}$.

1.2.3 Properties of the Transfer Principle

Lemma 1.2.8. *Let K and L be subsets of $\mathbb{X}$. Then the following properties hold:*

(i) ${}^*(K \cap L) = {}^*K \cap {}^*L$,

(ii) ${}^*(K \cup L) = {}^*K \cup {}^*L$,

(iii) ${}^*(K \setminus L) = {}^*K \setminus {}^*L$,

(iv) $K \subseteq L \Leftrightarrow {}^*K \subseteq {}^*L$,

(v) ${}^*K = \{{}^*k : k \in K\}$ *if K is finite,*

(vi) ${}^*\mathcal{P}(K) \subseteq \mathcal{P}({}^*K)$.

Proof. We will use the transfer principle applied on appropriately chosen formulae.

(i) The formula

$$(\forall x \in \mathbb{X}) : [x \in K \cap L \Leftrightarrow (x \in K \wedge x \in L)]$$

transfers to

$$(\forall x \in {}^*\mathbb{X}) : [x \in {}^*(K \cap L) \Leftrightarrow (x \in {}^*K \wedge x \in {}^*L)] \, .$$

(ii) The transferred version of the formula

$$(\forall x \in \mathbb{X}) : [x \in K \cup L \Leftrightarrow (x \in K \vee x \in L)]$$

is

$$(\forall x \in {}^*\mathbb{X}) : [x \in {}^*(K \cup L) \Leftrightarrow (x \in {}^*K \vee x \in {}^*L)] \, .$$

(iii) The formula describing the set difference is

$$(\forall x \in \mathbb{X}) : [x \in K \setminus L \Leftrightarrow (x \in K \wedge x \notin L)] \, .$$

Its $*$-transform is

$$(\forall x \in {}^*\mathbb{X}) : [x \in {}^*(K \setminus L) \Leftrightarrow (x \in {}^*K \wedge x \notin {}^*L)] \, .$$

(iv) The set inclusion is described by the formula

$$(\forall x \in \mathbb{X}) : [x \in K \Rightarrow x \in L]$$

By transfer, the formula

$$(\forall x \in {}^*\mathbb{X}) : [x \in {}^*K \Rightarrow x \in {}^*L]$$

holds.

(v) If $K = \{k_1, k_2, \dots, k_n\}$ is finite, then we can write the formula

$$(\forall x \in \mathbb{X}) : [x \in K \Leftrightarrow (x = k_1 \vee x = k_2 \vee \cdots \vee x = k_n)] \, .$$

It transfers to

$$(\forall x \in {}^*\mathbb{X}) : [x \in {}^*K \Leftrightarrow (x = {}^*k_1 \vee x = {}^*k_2 \vee \cdots \vee x = {}^*k_n)] \, .$$

(vi) For every set $K \subseteq \mathbb{X}$, the following formula defines its power set:

$$(\forall x \in \mathcal{P}(K))(\forall y \in x) : y \in K.$$

By transfer, the set ${}^*\mathcal{P}(K)$ is defined by

$$(\forall x \in {}^*\mathcal{P}(K))(\forall y \in x) : y \in {}^*K,$$

so ${}^*\mathcal{P}(K) \subseteq \mathcal{P}({}^*K)$. From the previous part, if K is finite then $\mathcal{P}({}^*K) = {}^*\mathcal{P}(K)$ because ${}^*K$ can be identified with K.

□

Lemma 1.2.9.

$$^*\mathcal{P}(K) = \{A \subsetneq {}^*K : A \text{ is internal}\}.$$

Proof. If $A \in {}^*\mathcal{P}(K) \in \mathbb{U}'$, then it is obviously internal. Conversely, take A an internal subset of $^*K$. Because it is internal, there exists $B \in \mathbb{U}$ such that $A \in {}^*B$. The following sentence describes subsets of K:

$$(\forall y \in B) : [(\forall x \in y : \ x \in K) \Rightarrow y \in \mathcal{P}(K)] .$$

If transfers to

$$(\forall y \in {}^*B) : [(\forall x \in y : \ x \in {}^*K) \Rightarrow y \in {}^*\mathcal{P}(K)] .$$

Since $A \in {}^*B$ and $A \subseteq {}^*K$, we get that $A \in {}^*\mathcal{P}(K)$. □

If the set K is infinite, then $\mathcal{P}({}^*K)$ contains external entities, hence

$$^*\mathcal{P}(K) \subsetneq \mathcal{P}({}^*K).$$

1.2.4 Arithmetic of Hyperreals

Definition 1.2.10. *A hyperreal number a is called:*

(i) ***infinitesimal*** *if for all $\varepsilon \in \mathbb{R}^+$ we have $|a| < \varepsilon$. The set of all infinitesimal numbers is denoted by $\mathbb{I}$;*

(ii) ***limited*** *if there exist $r, s \in \mathbb{R}$ such that $r < a < s$;*

(iii) ***unlimited*** *if it is not limited, i. e. either*

$$\forall r \in \mathbb{R} : a > r \ (\textbf{\textit{positive unlimited}})$$

or

$$\forall r \in \mathbb{R} : a < r \ (\textbf{\textit{negative unlimited}}).$$

Example. If we consider the ultrapower construction with the non-principal ultrafilter $\mathcal{U}$ over $\mathbb{N}$, then the hyperreal numbers are equivalence classes of sequences of real numbers. The sequence

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$$

is one example of an infinitesimal number. Indeed, for every $\varepsilon \in \mathbb{R}^+$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. But the set $\{i \in \mathbb{N} : i < n\}$ is finite, hence cannot be in $\mathcal{U}$. So its complement, the set $\{i \in \mathbb{N} : i \geq n\}$ is in $\mathcal{U}$ and for all its elements we have $\frac{1}{i} < \varepsilon$. This exactly means that the sequence defines an infinitesimal number.

Similarly, the sequence

$$(1, 2, 3, 4, \dots)$$

is a positive unlimited hyperreal number.

Definition 1.2.11. *We say that a hyperreal number a is* ***infinitely close*** *to a hyperreal number b (and denote* $a \simeq b$*) if* $a - b \in \mathbb{I}$.

Remark. $\simeq$ is an equivalence relation on ${}^*\mathbb{R}$.

Lemma 1.2.12. *Every limited hyperreal number r is infinitely close to exactly one real number, called* ***standard part*** *of r and denoted st(r).*

Proof. Define a set $R = \{x \in \mathbb{R} : x < r\}$. Since r is limited, there exist real numbers $s, t \in R$ such that $s < r < t$, so $s \in R$ and t is an upper bound of R. Therefore, there exists the supremum of R, $r_0 = \sup R$.

To prove that $r_0 = \mathrm{st}(r)$, let $\varepsilon \in \mathbb{R}^+$. Then $r \leq r_0 + \varepsilon$ and $r > r_0 - \varepsilon$ because otherwise $r_0 - \varepsilon$ would be an upper bound of R lower than r_0. Altogether, $|r - r_0| \leq \varepsilon$. Since ε was arbitrary, we can conclude that $r_0 = \mathrm{st}(r)$.

If there are two real numbers $s, t \in \mathbb{R}$, both infinitely close to r, then, by transitivity of $\simeq$, also $s \simeq t$. This by definition means that

$$\forall \varepsilon \in \mathbb{R}^+ : |s - t| < \varepsilon.$$

Hence, $s = t$ and the standard part is unique. □

Lemma 1.2.13. *Let x, y be limited hyperreal numbers. Then the following hold:*

(i) $st(x + y) = st(x) + st(y)$,

(ii) $st(x \cdot y) = st(x) \cdot st(y)$,

(iii) $x \leq y \Longrightarrow st(x) \leq st(y)$.

Proof.

(i) Let $\varepsilon \in \mathbb{R}^+$. Then

$$|x - \mathrm{st}(x)| < \frac{\varepsilon}{2} \quad \text{and} \quad |y - \mathrm{st}(y)| < \frac{\varepsilon}{2}.$$

We have

$$|(x + y) - (\mathrm{st}(x) + \mathrm{st}(y))| \leq |x - \mathrm{st}(x)| + |y - \mathrm{st}(y)| < \varepsilon,$$

so the sum of standard parts of x and y is infinitely close to the standard part of $x + y$. Since $\mathrm{st}(x) + \mathrm{st}(y)$ is real and the standard part is unique, we can conclude that $\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y)$.

(ii) Let $\varepsilon \in \mathbb{R}^+$. The hyperreal numbers x and y are limited, so there exist $K, L \in \mathbb{R}^+$ such that $|x| < K$ and $|\mathrm{st}(y)| < L$. From the definition of the standard part, we know that

$$|x - \mathrm{st}(x)| < \frac{\varepsilon}{2 \cdot K} \quad \text{and} \quad |y - \mathrm{st}(y)| < \frac{\varepsilon}{2 \cdot L}.$$

We compute:

$$\begin{aligned} |(x \cdot y) - (\mathrm{st}(x) \cdot \mathrm{st}(y))| &\leq |x \cdot y - x \cdot \mathrm{st}(y)| + |x \cdot \mathrm{st}(y) - \mathrm{st}(x) \cdot \mathrm{st}(y)| \\ &= |x| \cdot |y - \mathrm{st}(y)| + |\mathrm{st}(y)| \cdot |x - \mathrm{st}(x)| \\ &\leq K \cdot |y - \mathrm{st}(y)| + L \cdot |x - \mathrm{st}(x)| < \varepsilon \end{aligned}$$

Hence, $\mathrm{st}(x \cdot y) = \mathrm{st}(x) \cdot \mathrm{st}(y)$.

(iii) If $\mathrm{st}(x) > \mathrm{st}(y)$, then there exists $\varepsilon \in \mathbb{R}^+$ such that $\mathrm{st}(x) - \mathrm{st}(y) > \varepsilon$. As before, we know form the definition that

$$|x - \mathrm{st}(x)| < \frac{\varepsilon}{2} \text{ and } |y - \mathrm{st}(y)| < \frac{\varepsilon}{2}.$$

We estimate

$$x - y = (x - \mathrm{st}(x)) + (\mathrm{st}(x) - \mathrm{st}(y)) + (\mathrm{st}(y) - y) > -\frac{\varepsilon}{2} + \varepsilon - \frac{\varepsilon}{2} = 0.$$

On the other hand, we know that $x - y \leq 0$ since $x \leq y$, so we have $0 < x - y \leq 0$ and that is a contradiction.

□

2. Ordinal Preference Relations

In this chapter, we deal with a part of the social choice theory. We suppose that every member of a society (a voter) has an individual preference on the set of alternatives. In this chapter, the preferences are special binary relations. So, for example, the first voter prefers an alternative A over B but also prefers C over A. The goal is to establish one preference relation (election result) that represents all the individual relations and satisfies certain properties that a reasonable electoral system should satisfy.

2.1 Definitions

Let A be the **set of alternatives**, $|A| \geq 3$. Let V be the **set of voters**. Without loss of generality suppose throughout this thesis that $V \subseteq \mathbb{N}$.

Definition 2.1.1. *We say that a binary relation $P \subseteq A \times A$ is a **weak order on** A if it is:*

*(i) **asymmetric**, i.e.* $\forall a, b \in A : (a, b) \in P \Rightarrow (b, a) \notin P$,

*(ii) **negatively transitive**, i.e.*

$$\forall a, b, c \in A : [(a, b) \notin P \wedge (b, c) \notin P] \Rightarrow (a, c) \notin P.$$

Remark. We will denote the fact that a is related to b by P by $(a, b) \in P$ or aPb.

Lemma 2.1.2. *If P is a weak order on A, then it holds:*

(i) $\forall a, b, c \in A : aPb \Rightarrow (aPc \vee cPb)$,

(ii) P is transitive, i.e. $\forall a, b, c \in A : (aPb \wedge bPc) \Rightarrow aPc$,

(iii) $\forall a, b, c \in A : [(\neg bPa \wedge bPc) \vee (aPb \wedge \neg cPb)] \Rightarrow aPc$.

Proof.

(i) This is just the inversion of negative transitivity.

(ii) Suppose that aPb and bPc. Applying (i) on aPb and c we get aPc or cPb. But from asymmetry of P and bPc we have $\neg cPb$, so aPc must hold.

(iii) Suppose that $\neg bPa \wedge bPc$ (the other part can be proven in the same way). From (i) applied on bPc and a we have bPa or aPc. Since we already know $\neg bPa$, aPc is valid.

□

Let $\mathscr{P}$ denote the set of all weak orders on A and be called the **set of preferences**. Then $\mathscr{P} \subseteq \mathcal{P}(A \times A)$ and $F = \{f : V \to \mathscr{P}\}$ denotes the **set of all situations**.
A set of **social welfare functions** is the set $\Sigma = \{\sigma : F \to \mathscr{P}\}$.

Definition 2.1.3. *Let $f, g \in F$ be situations, $a, b \in A$ alternatives, $v \in V$ and let $U \subseteq V$ be a set. We say that:*

(i) $(a, b) \in f(U)$ *if* $\forall v \in U : (a, b) \in f(v)$,

(ii) $f(v) = g(v)$ *on* $\{a, b\}$ *if* $(a, b) \in f(v) \Leftrightarrow (a, b) \in g(v)$,

(iii) $f = g$ *on* $\{a, b\}$ *if* $\forall v \in V : f(v) = g(v)$ *on* $\{a, b\}$.

Definition 2.1.4. *Let σ be a social welfare function. A subset $U \subseteq V$ is **σ-decisive** if*

$$\forall a, b \in A \, \forall f \in F : (a, b) \in f(U) \Rightarrow (a, b) \in \sigma(f).$$

Definition 2.1.5 (Arrow's conditions)**.**

(i) ***Unanimity***: $\forall a, b \in A \, \forall f \in F : (a, b) \in f(V) \Rightarrow (a, b) \in \sigma(f)$,

(ii) ***Independence of irrelevant alternatives***:

$$\forall a, b \in A \, \forall f, g \in F : f = g \text{ on } \{a, b\} \Rightarrow \sigma(f) = \sigma(g) \text{ on } \{a, b\}.$$

(iii) ***Non-dictatorship***:

$$\nexists v \in V \, \forall f \in F \, \forall a, b \in A : (a, b) \in f(v) \Rightarrow (a, b) \in \sigma(f).$$

Remark. If the non-dictatorship condition is not satisfied, then any such $v \in V$ is called a **dictator**. In the view of decisive sets, a dictator is a singleton that is decisive.

The unanimity condition states that if everyone prefers one alternative to another, also society's aggregation should prefer it. The independence of irrelevant alternatives condition says that the relation between two alternatives should depend only on individual relations between these two alternatives and nothing else. So, for example, if we remove or add one alternative, this should not change society's preference between the other alternatives. The non-dictatorship condition states that no individual dictates his preferences, no matter what the rest of the voters prefer.

2.2 Arrow's Impossibility Theorem

Theorem 2.2.1 (Arrow's theorem)**.** *If the set of voters is finite and $|A| \geq 3$, then there is no social welfare function satisfying Arrow's conditions (Unanimity, Independence of irrelevant alternatives and Non-dictatorship).*

This theorem first appeared in Arrow [1950]. It is a consequence of the next statement and the properties of ultrafilters stated in Section 1.1.

Theorem 2.2.2. *Let σ be a social welfare function that satisfies Unanimity and Independence of irrelevant alternatives. Then the σ-decisive subsets of V form an ultrafilter.*

Proof. In this proof we will proceed as in Hansson [1976]. Let $\mathcal{F}$ denote the set of all σ-decisive sets. We need to check the conditions from Definition 1.1.1:

(i) Due to the Unanimity condition, $V \in \mathcal{F}$ so $\mathcal{F} \neq \emptyset$.

(ii) If the empty set was decisive then for any alternatives $a, b \in A$ and for any situation $f \in F$ it would hold simultaneously $(a, b) \in \sigma(f)$ and $(b, a) \in \sigma(f)$. That's in contradiction to the asymmetry of $\sigma(f)$.

(iii) Take $K, L \in \mathcal{F}$ and fix $a, b \in A$ and $f \in F$ such that

$$\forall v \in K \cap L : (a, b) \in f(v).$$

To prove that $(a, b) \in \sigma(f)$, we define other situation $g \in F$ such that the relation between a and b is the same as in f. Moreover, take another alternative $z \in A$ and define its relations with a and b in g in the following way:

- $(a, c) \in g(v), (b, c) \in g(v)$ if $v \in K \setminus L$,
- $(a, c) \in g(v), (c, b) \in g(v)$ if $v \in K \cap L$,
- $(c, a) \in g(v), (c, b) \in g(v)$ if $v \in L \setminus K$.

Now we have $(a, c) \in \sigma(g)$ because $\forall v \in K : (a, c) \in g(v)$ and $K \in \mathcal{F}$ as well as $(c, b) \in \sigma(g)$ due to the fact that $\forall v \in L : (c, b) \in g(v)$ and $L \in \mathcal{F}$. By transitivity, $(a, b) \in \sigma(g)$. Finally, because of the Independence of irrelevant alternatives condition, $(a, b) \in \sigma(f)$.

(iv) This condition is trivial by the following sequence of implications:

$$(a, b) \in f(L) \xRightarrow{K \subseteq L} (a, b) \in f(K) \xRightarrow{K \in \mathcal{F}} (a, b) \in \sigma(f).$$

(v) We denote the fact that X is decisive for x over y by $D(X, x, y)$, i.e.

$$D(X, x, y) \equiv \forall f \in F : [(x, y) \in f(X) \Rightarrow (x, y) \in \sigma(f)].$$

The set X being decisive means that $D(X, x, y)$ holds for every $x, y \in A$. Note that

$$D(X, x, y) \Rightarrow \neg D(X^C, y, x). \tag{2.1}$$

First, we need the following lemma.

Lemma 2.2.3.

$$\neg D(X^C, x, y) \Rightarrow [(\forall z \neq y : D(X, z, x)) \wedge (\forall z \neq x : D(X, y, z))]$$

Proof. We prove $\neg D(X^C, x, y) \Rightarrow (\forall z \neq y : D(X, z, x))$, the other part is similar. If $D(X^C, x, y)$ is false then it means that there exists $f \in F$ such that

$$\Big(\forall v \in X^C : (x, y) \in f(v)\Big) \wedge (x, y) \notin \sigma(f).$$

Fix an alternative $z \in A$, $z \neq y$ and a situation $g \in F$. We need to prove the following implication:

$$(\forall v \in X : (z, x) \in g(v)) \Rightarrow (z, x) \in \sigma(g).$$

We construct the situation $h \in F$ such that:

(a) the relation between x and y is the same as in f,

(b) the relation between x and z is the same as in g,

(c) $(z, y) \in h(V)$

It is possible because the only conflict would appear if $(y, x) \in h(v)$ and $(x, z) \in h(v)$ for some $v \in V$. If $v \in X^C$ then $(x, y) \in f(v)$, so the first part cannot be satisfyied due to asymmetry and the definition of h. Similarly, if $v \in X$ then $(z, x) \in g(v)$ and the second part is not possible.
We know that for $v \in X^C : (x, y) \notin \sigma(f)$. Due to Independence of irrelevant alternatives, we have $(x, y) \notin \sigma(h)$.
Because of Unanimity, $(z, y) \in \sigma(h)$, hence, using the property (iii) of Lemma 2.1.2, $(z, x) \in \sigma(h)$. We can conclude that $(z, x) \in \sigma(g)$, again due to the Independence of irrelevant alternatives condition and that finishes the proof. □

Let the set $K \subseteq V$ be given. Either the complement K^C is in $\mathcal{F}$ and we are done, or it isn't and then there exist alternatives $x, y \in A$ such that $\neg D(K^C, x, y)$. By the first part of Lemma 2.2.3, we get that

$$(\forall z \neq y : D(K, z, x)).$$

It follows from (2.1) that

$$(\forall z \neq y : \neg D(K^C, x, z)),$$

so altogether

$$(\forall z : \neg D(K^C, x, z)).$$

We may see that this is similar to our assumption apart from z being arbitrary. If we follow the same procedure but apply the second part of Lemma 2.2.3 we get

$$(\forall w, z : \neg D(K^C, w, z)).$$

By using Lemma 2.2.3 once again we get

$$(\forall w, z : D(K, z, w))$$

since there are at least three alternatives. We can conclude that $D(K, z, w)$ holds for every $w, z \in A$, therefore K is decisive.

Hence, $\mathcal{F}$ is an ultrafilter. □

Proof of Theorem 2.2.1. By Theorem 2.2.2 we know that the set of all decisive subsets of V forms an ultrafilter. Since V is finite, Theorem 1.1.5 implies that it is of the form $\{U \subseteq V : a \in U\}$ for some $a \in V$. Hence, a is the dictator. □

2.3 Non-dictatorship Via Ultrafilters

Theorem 2.3.1. *If the set of voters is countably infinite and $|A| \geq 3$, then there exists a social welfare function satisfying Unanimity and Independence of irrelevant alternatives such that no finite set of voters is decisive.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on the set V. Let $\mathbb{X}$ be a set containing $\mathbb{R}$, the set of voters V and the set of alternatives A, let $\mathbb{U}(\mathbb{X})$ be a universe and $\mathbb{U}(\mathbb{X}) \xrightarrow{*} \mathbb{U}(\mathbb{X})'$ be the embedding into the nonstandard universe obtained by the ultrapower construction described in Section 1.2.2. Similarly as in Example on page 10, we can describe the involved structures as "sequences" of elements of $\mathbb{U}(\mathbb{X})$ or, more properly, elements of $\mathbb{U}(\mathbb{X})^V$. In this section, $[[a = b]]$ denotes the set $\{v \in V : a_v = b_v\}$ and similarly for other relation symbols surrounded by $[[\cdot]]$. We will work with the following entities:

- The set of voters: ${}^*V = \{[v_n] \in {}^*\mathbb{N} : [[v \in V]] \in \mathcal{U}\}$;
- The set of alternatives: ${}^*A = \{[a_n] \in {}^*\mathbb{R} : [[a \in A]] \in \mathcal{U}\}$;
- A weak order $P \subseteq A \times A$ transfers to ${}^*P \subseteq {}^*A \times {}^*A$:

$$([a_v], [b_v]) \in {}^*P \Leftrightarrow \{v \in V : (a_v, b_v) \in P\} \in \mathcal{U};$$

- The set of weak orders (preferences): ${}^*\mathscr{P} \in {}^*\mathcal{P}(A \times A)$ is a set of internal subsets of ${}^*A \times {}^*A$ that satisfy transferred asymmetry and negative transitivity, i.e.

$$\forall a, b \in {}^*A : (a, b) \in {}^*P \Rightarrow (b, a) \notin {}^*P$$

  and

$$\forall a, b, c \in {}^*A : [(a, b) \notin {}^*P \wedge (b, c) \notin {}^*P] \Rightarrow (a, c) \notin {}^*P;$$

- The situation is a function from the set of voters V to the set of preferences $\mathscr{P}$, so the set of all situations (denoted ${}^*F$) is a set of all internal functions from ${}^*V$ to ${}^*\mathscr{P}$. For $[f_n] \in {}^*F$ we have

$$\forall([a_n], [b_n]) \in {}^*A \times {}^*A \; \forall [v_n] \in {}^*V :$$

$$(([a_n], [b_n]) \in [f_n]([v_n]) \Leftrightarrow \{n \in V : (a_n, b_n) \in f_n(v_n)\} \in \mathcal{U});$$

- The set of all social welfare functions $\Sigma = \{\sigma : F \to \mathscr{P}\}$ transfers to ${}^*\Sigma = \{\sigma : {}^*F \to {}^*\mathscr{P} : \sigma \text{ is internal}\}$.

We need to transfer also the assumptions:
The Unanimity condition

$$\forall a, b \in A \; \forall f \in F : (a, b) \in f(V) \Rightarrow (a, b) \in \sigma(f)$$

transfers to *-Unanimity:

$$\forall a, b \in {}^*A \; \forall f \in {}^*F \; [\forall v \in {}^*V : (a, b) \in f(v)] \Rightarrow (a, b) \in {}^*\sigma(f). \tag{2.2}$$

The Independence of irrelevant alternatives

$$\forall a, b \in A\, \forall f, g \in F : f = g \text{ on } \{a, b\} \Rightarrow \sigma(f) = \sigma(g) \text{ on } \{a, b\}$$

transfers to *-Independence of irrelevant alternatives:

$$\forall a, b \in {}^*A\, \forall f, g \in {}^*F : f = g \text{ on } \{a, b\} \Rightarrow {}^*\sigma(f) = {}^*\sigma(g) \text{ on } \{a, b\}. \tag{2.3}$$

The σ-decisivness of $U \subseteq V$

$$\forall a, b \in A\, \forall f \in F : (\forall u \in U : (a, b) \in f(u)) \Rightarrow (a, b) \in \sigma(f)$$

transfers to the definition of a $^*\sigma$-decisive subset $^*U \subseteq {}^*V$:

$$\forall a, b \in {}^*A\, \forall f \in {}^*F : (\forall u \in {}^*U : (a, b) \in f(u)) \Rightarrow (a, b) \in {}^*\sigma(f). \tag{2.4}$$

Next, we can define a special social welfare function $\overline{\sigma}$ for which we check the assumptions and show that no finite $U \subseteq V$ can be $\overline{\sigma}$-decisive. The non-existence of a dictator will directly follow.

Given $[f_n] \in {}^*F$, we define $\overline{\sigma} : {}^*F \longmapsto {}^*\mathscr{P}$:

$$\overline{\sigma}([f_n]) = [(f_n(n))_{n \in V}].$$

Then it holds that

$$\forall [a_n], [b_n] \in {}^*A : ([a_n], [b_n]) \in \overline{\sigma}([f_n]) \Leftrightarrow \{n \in V : (a_n, b_n) \in f_n(n)\} \in \mathcal{U}.$$

We check the assumptions:

- Asymmetry: If $([a_n], [b_n]) \in \overline{\sigma}([f_n])$, then

  $$\{n \in V : (a_n, b_n) \in f_n(n)\} \in \mathcal{U}.$$

  But every $f_n(n)$ is asymmetric, so

  $$\{n \in V : (a_n, b_n) \in f_n(n)\} \subseteq \{n \in V : (b_n, a_n) \notin f_n(n)\},$$

  hence

  $$\{n \in V : (b_n, a_n) \notin f_n(n)\} \in \mathcal{U}.$$

  The complement of an element of the ultrafilter cannot belong to this ultrafilter, so

  $$\{n \in V : (b_n, a_n) \in f_n(n)\} = \{n \in V : (b_n, a_n) \notin f_n(n)\}^C \notin \mathcal{U}.$$

  Hence, $([b_n], [a_n]) \notin \overline{\sigma}([f_n])$.

- Negative transitivity: If

  $$\{n \in V : (a_n, b_n) \notin f_n(n)\} \in \mathcal{U}$$

  and

  $$\{n \in V : (b_n, c_n) \notin f_n(n)\} \in \mathcal{U},$$

then

$$\{n \in V : (a_n, b_n) \notin f_n(n)\} \cap \{n \in V : (b_n, c_n) \notin f_n(n)\} \in \mathcal{U}.$$

For elements of this intersection we have $(a_n, c_n) \notin f_n(n)$, so

$$\{n \in V : (a_n, c_n) \notin f_n(n)\} \supseteq$$

$$\supseteq \{n \in V : (a_n, b_n) \notin f_n(n)\} \cap \{n \in V : (b_n, c_n) \notin f_n(n)\},$$

hence $([a_n], [c_n]) \notin \overline{\sigma}([f_n])$. Together with the previous part, this implies that $\overline{\sigma}$ is a weak order.

- *-Unanimity: Suppose that

  $$\forall [a_n], [b_n] \in {}^*A \; \forall [f_n] \in {}^*F \; \forall [v_n] \in {}^*V : ([a_n], [b_n]) \in [f_n]([v_n]).$$

  By the definition, it is equivalent to

  $$\forall [v_n] \in {}^*V : \{n \in V : (a_n, b_n) \in f_n(v_n)\} \in \mathcal{U}.$$

  In particular, $[v_n] = (1, 2, 3, \dots) \in {}^*V$, hence

  $$\{n \in V : (a_n, b_n) \in f_n(n)\} \in \mathcal{U},$$

  which is by the definition exactly $([a_n], [b_n]) \in \overline{\sigma}([f_n])$, so (2.2) holds.

- *-Independence of irrelevant alternatives: Suppose that

  $$\forall [a_n], [b_n] \in {}^*A \; \forall [f_n], [g_n] \in {}^*F : [f_n] = [g_n] \text{ on } \{[a_n], [b_n]\},$$

  i.e. $\forall [v_n] \in {}^*V$:

  $$([a_n], [b_n]) \in [f_n]([v_n]) \Leftrightarrow ([a_n], [b_n]) \in [g_n]([v_n])$$

  and

  $$([b_n], [a_n]) \in [f_n]([v_n]) \Leftrightarrow ([b_n], [a_n]) \in [g_n]([v_n]),$$

  i. e. $\forall [v_n] \in {}^*V$:

  $$\{n \in V : (a_n, b_n) \in f_n(v_n)\} \in \mathcal{U} \Leftrightarrow \{n \in V : (a_n, b_n) \in g_n(v_n)\} \in \mathcal{U}$$

  and

  $$\{n \in V : (b_n, a_n) \in f_n(v_n)\} \in \mathcal{U} \Leftrightarrow \{n \in V : (b_n, a_n) \in g_n(v_n)\} \in \mathcal{U}.$$

  It holds in particular for $[v_n] = (1, 2, 3, \dots) \in {}^*V$, hence

  $$([a_n], [b_n]) \in \overline{\sigma}([f_n]) \Leftrightarrow ([a_n], [b_n]) \in \overline{\sigma}([g_n])$$

  and

  $$([b_n], [a_n]) \in \overline{\sigma}([f_n]) \Leftrightarrow ([b_n], [a_n]) \in \overline{\sigma}([g_n]),$$

  so (2.3) is satisfied.

Now let us consider a finite set of voters, $U \subseteq V$, $|U| = n$ for some $n \in \mathbb{N}$, and check whether it can be decisive. Since U is finite, we have ${}^*U = U$. As stated above, U is $\overline{\sigma}$-decisive if

$$\forall [a_n], [b_n] \in {}^*A \; \forall [f_n] \in {}^*F: \; (\forall u \in U : ([a_n], [b_n]) \in [f_n](u)) \Rightarrow ([a_n], [b_n]) \in \overline{\sigma}([f_n]).$$

We want to prove that U is not decisive, it means proving the negated statement

$$\exists [a_n], [b_n] \in {}^*A \; \exists [f_n] \in {}^*F \; \forall u \in {}^*U : ([a_n], [b_n]) \in [f_n](u) \wedge ([a_n], [b_n]) \notin \overline{\sigma}([f_n]). \tag{2.5}$$

Let us define $[a_n], [b_n] \in {}^*A$ and $[f_n] \in {}^*F$ in the following way:

1. If A is finite, we denote $A = \{a_1, a_2, a_3, \ldots, a_N\}$ for some $N \in \mathbb{N}$. We define $f \in F$:

$$f(v) = \begin{cases} \{(a_{i+1}, a_i), i \in \{1, 2, \ldots, N\}\} \cup \{(a_1, a_N)\} & \text{if } v \in U \\ \{(a_i, a_{i+1}), i \in \{1, 2, \ldots, N\}\} \cup \{(a_N, a_1)\} & \text{if } v \in V \setminus U \end{cases}$$

   Take

$$\begin{aligned} f &= [(f)_n] \\ a &= [(a_2, \ldots, a_N, a_1, a_2, \ldots, a_N, a_1, \ldots)] \\ b &= [(a_1, a_2, a_3, \ldots, a_N, a_1, a_2, \ldots, a_N, a_1, \ldots)]. \end{aligned}$$

2. If A is infinite, then, by the axiom of choice, there exists a linear order on the set A. We denote elements of A according to this order $A = \{a_1, a_2, a_3, \ldots\}$ where $a_1 < a_2 < a_3 < \ldots$ We define $f \in F$:

$$f(v) = \begin{cases} \{(a_{i+1}, a_i), i \in \{1, 2, \ldots\}\} & \text{if } v \in U \\ \{(a_i, a_{i+1}), i \in \{1, 2, \ldots\}\} & \text{if } v \in V \setminus U \end{cases}$$

   Take $f = [(f)_n]$, $a = [(a_{n+1})_n]$, $b = [(a_n)_n]$.

Then $a, b \in {}^*A$ and $f \in {}^*F$. Moreover, $(a, b) \in f(U)$ because

$$\forall v \in U : \{n \in V : (a_n, b_n) \in f(v)\} = \{n \in V : (a_{n+1}, a_n) \in f(v)\} = V.$$

In addition,

$$\{n \in V : (b_n, a_n) \in f(n)\} = \{n \in V : (a_n, a_{n+1}) \in f(n)\} = V \setminus U.$$

Because U is finite, it holds that $V \setminus U \in \mathcal{U}$, hence $(b, a) \in \overline{\sigma}(f)$. By asymmetry, $(a, b) \notin \overline{\sigma}(f)$ which completes the proof of nondecisiveness of any finite $U \subseteq V$.

To finish the proof in the standard universe, we define $\sigma \in \Sigma$ in the following way:

$$\forall a, b \in A \; \forall f \in F : (a, b) \in \sigma(f) \Leftrightarrow (a, b) \in \overline{\sigma}(f).$$

It follows from this definition, the transfer principle and properties of $\overline{\sigma}$ that σ is the sought-for social welfare function. $\square$

Remark. Note that $v = (1, 2, 3, \dots) \in {}^*V \setminus V$ is "the dictator" for the social welfare function $\overline{\sigma}$. However, it is not in V, so it not a dictator in the sense of Definition 2.1.5(iii).

Moreover, by applying the transfer principle on the original Arrow's theorem (formulation from Kirman and Sondermann [1972]) we get the existence of a nonstandard dictator for every internal social welfare function. Without loss of generality suppose that $V \subseteq \mathbb{N}$. Then every finite set in V is bounded by some $n \in \mathbb{N}$.

$$
\begin{aligned}
&\Big(\exists n \in \mathbb{N}\, \forall v \in V : v \leq n\Big) \Rightarrow && V \text{ is finite} \\
\neg\Big(&\exists \sigma \in \Sigma : \ [\exists a, b, c \in A : \neg(a = b) \wedge \neg(a = c) \wedge \neg(b = c)] && |A| \geq 3 \\
&\quad \wedge [\forall f \in F \exists! p \in \mathscr{P} : \sigma(f) = p] && \sigma \text{ is a function} \\
&\quad \wedge [\forall a, b \in A \forall f \in F : (\forall v \in V : (a, b) \in f(v)) \Rightarrow (a, b) \in \sigma(f)] && \text{unanimity} \\
&\quad \wedge [\forall a, b \in A \forall f, g \in F : f = g \text{ on } \{a, b\} \Rightarrow \sigma(f) = \sigma(g) \text{ on } \{a, b\}] && \text{IIA} \\
&\quad \wedge \neg [\exists v_0 \in V \forall a, b \in A \forall f \in F : (a, b) \in f(v_0) \Rightarrow (a, b) \in \sigma(f)]\Big) && \text{non-dictatorship}
\end{aligned}
$$

The transferred formula is:

$$
\begin{aligned}
&\Big(\exists n \in {}^*\mathbb{N}\, \forall v \in {}^*V : v \leq n\Big) \Rightarrow \\
\neg\Big(&\exists \sigma \in {}^*\Sigma : [\exists a, b, c \in {}^*A : \neg(a = b) \wedge \neg(a = c) \wedge \neg(b = c)] \\
&\quad \wedge [\forall f \in {}^*F \exists! p \in {}^*\mathscr{P} : \sigma(f) = p] \\
&\quad \wedge [\forall a, b \in {}^*A \forall f \in {}^*F : (\forall v \in {}^*V : (a, b) \in f(v)) \Rightarrow (a, b) \in \sigma(f)] \\
&\quad \wedge [\forall a, b \in {}^*A \forall f, g \in {}^*F : f = g \text{ on } \{a, b\} \Rightarrow \sigma(f) = \sigma(g) \text{ on } \{a, b\}] \\
&\quad \wedge \neg [\exists v_0 \in {}^*V \forall a, b \in {}^*A \forall f \in {}^*F : (a, b) \in f(v_0) \Rightarrow (a, b) \in \sigma(f)]\Big)
\end{aligned}
$$

The first part means that ${}^*V$ is a hyperfinite set. This transferred formula implies:

$$
\begin{aligned}
&\forall \sigma \in {}^*\Sigma : \\
&\qquad \Big[[\exists n \in {}^*\mathbb{N}\, \forall v \in {}^*V : v \leq n] \\
&\qquad \wedge [\exists a, b, c \in {}^*A : \neg(a = b) \wedge \neg(a = c) \wedge \neg(b = c)] \\
&\qquad \wedge [\forall f \in {}^*F \exists! p \in {}^*\mathscr{P} : \sigma(f) = p] \\
&\qquad \wedge [\forall a, b \in {}^*A \forall f \in {}^*F : (\forall v \in {}^*V : (a, b) \in f(v)) \Rightarrow (a, b) \in \sigma(f)] \\
&\qquad \wedge [\forall a, b \in {}^*A \forall f, g \in {}^*F : f = g \text{ on } \{a, b\} \Rightarrow \sigma(f) = \sigma(g) \text{ on } \{a, b\}] \Big] \\
&\qquad \Rightarrow [\exists v_0 \in {}^*V \forall a, b \in {}^*A \forall f \in {}^*F : (a, b) \in f(v_0) \Rightarrow (a, b) \in \sigma(f)],
\end{aligned}
$$

i.e. a nonstandard dictator exists. If V is finite, then $V = {}^*V$, thus the dictator is one of the voters. If V is infinite, then ${}^*V \setminus V \neq \emptyset$ and the dictator may not be one of the voters. This result is similar to the "invisible dictator" in Kirman and Sondermann [1972]. They find the "invisible dictator" in the Čech-Stone compactification of V. He may not be one of the voters but he dictates the results. As we will see in Section 4.3, nonstandard and invisible dictators are in fact the same.

Furthermore, in Kirman and Sondermann [1972], they show a stronger claim: they show that to every member v of the Čech-Stone compactification of V there exists a social welfare function such that v is its (invisible) dictator.

2.4 Non-dictatorship Via Concurrence

In this part, another method of nonstandard analysis is used to prove Theorem 2.3.1.

Definition 2.4.1 (Goldblatt [2012]). *A binary relation R is called **concurrent** or **finitely satisfiable** if for any finite subset $\{x_1, \cdots, x_n\} \subseteq \mathrm{dom}(R)$ there exists an element $y \in \mathrm{range}(R)$ with $x_i R y$ for every $i \in \{1, \cdots, n\}$.*

Theorem 2.4.2 (Goldblatt [2012], Theorem 14.2.1). *If $\mathbb{U} \xrightarrow{*} \mathbb{U}'$ is a universe embedding, then the following are equivalent.*

(i) $\mathbb{U}'$ is an enlargement of $\mathbb{U}$ relative to $\xrightarrow{*}$.

(ii) For any concurrent relation $R \in \mathbb{U}$ there exists an entity $b \in \mathbb{U}'$ such that ${}^*x({}^*R)b$ for all $x \in \mathrm{dom}(R)$.

Proof of Theorem 2.3.1. Suppose that $U \in \mathcal{P}_F(V)$ (U is a finite subset of V) and $\sigma \in \Sigma$. We define a relation $R \subseteq \mathcal{P}_F(V) \times \Sigma : (U, \sigma) \in R \Leftrightarrow \sigma$ satisfies unanimity and independence of irrelevant alternatives and U is not σ-decisive.

We want to prove that for every finite $M \subseteq \mathcal{P}_F(V)$:

$$\exists \sigma \in \Sigma \; \forall U \in M : \; (U, \sigma) \in R.$$

If $M \subseteq \mathcal{P}_F(V)$, then $N = \bigcup_{U \in M} \{v \in U\}$ is a finite set. We define σ to be a social welfare function satisfying unanimity and independence of irrelevant alternatives such that it doesn't take elements of N into account, i.e $\sigma(f) = \sigma(g)$ whenever $f \restriction_{V \setminus N} = g \restriction_{V \setminus N}$. One example of such a function may be the dictatorship of some $v \in V \setminus N$ (i.e. $\forall f \in F : \sigma(f) = f(v)$).

A set $U \in M$ is not σ-decisive because we can redefine some fixed $f \in F$:

$$\overline{f}(v) = \begin{cases} f(v) & \text{if } v \in V \setminus U \\ \{(a, b) : (b, a) \in \sigma(f)\} & \text{if } v \in U \end{cases}$$

Clearly $f \restriction_{V \setminus N} = \overline{f} \restriction_{V \setminus N}$, therefore $\sigma(f) = \sigma(\overline{f})$. It holds that

$$\exists a, b \in A : (a, b) \in \overline{f}(U) \wedge (b, a) \in \overline{f}(U^C) \wedge (a, b) \notin \sigma(\overline{f}).$$

so $U \in M$ is not σ-decisive.

By Theorem 2.4.2, $\exists\overline{\sigma} \in {}^*\Sigma\ \forall U \in \mathcal{P}_F(V)$: ${}^*U = U$ is not $\overline{\sigma}$-decisive. Therefore, no finite subset of V is $\overline{\sigma}$-decisive. We can complete the proof in the standard universe in the same way as in the first proof of Theorem 2.3.1 by defining $\sigma \in \Sigma$ as follows:

$$\forall a, b \in A\ \forall f \in F : (a, b) \in \sigma(f) \Leftrightarrow (a, b) \in \overline{\sigma}(f).$$

□

3. Cardinal Preference Relations

In this chapter, the preference relations are real functions. It is an extension of the ordinal preferences because with functions we can express how much we prefer one alternative over another, so the aggregation procedure can take this into account and chose the alternative that is overall well-rated even if it is not always the most preferred.

3.1 Definitions

Let A be a (finite nonempty) set of alternatives, $4 \leq |A| < \infty$. Let $\mathbb{R}^A$ denote a set of functions from A to $\mathbb{R}$. Let $V = \{1, 2, \dots\}$ be a finite or infinite set of society members (voters).

Definition 3.1.1. *A set $X \subset \mathbb{R}^A$ is called a* ***cardinal preference relation over*** *A if the following conditions are satisfied:*

(i) $X \neq \emptyset$,

(ii) $\forall x, y \in X \exists \alpha \in \mathbb{R}^+ \exists \beta \in \mathbb{R} \forall a \in A : x(a) = \alpha y(a) + \beta$,

(iii) $\forall x \in \mathbb{R}^A \forall \alpha \in \mathbb{R}^+ \forall \beta \in \mathbb{R} : x \in X \Rightarrow y(a) = \alpha x(a) + \beta \in X, a \in A$.

An element $x \in X$ is called a ***(cardinal) utility over*** *A.*

Let Ξ (or Ξ_A) denote a set of all cardinal preference relations over A. Then Ξ is a partition of nonempty subsets of $\mathbb{R}^A$ (i.e. $\Xi \subseteq \mathcal{P}(\mathbb{R}^A)$, $\Xi = \cup_{i \in I} X_i$, $X_i \cap X_j = \emptyset$ if $i \neq j$, $X_i \neq \emptyset$).

An element $(X_1, X_2, \dots) \in \Xi^{|V|}$ will be called a ***cardinal profile*** *and denoted by $\underline{X}$.*

A ***cardinal welfare function****, or a* ***cardinal collective choice rule****, or* ***procedure for aggregation of cardinal preferences*** *is a function $f : \Xi^{|V|} \to \Xi$.*

Remark. A cardinal preference relation defined in the previous definition is sometimes referred to as "cardinal in the sense of von Neumann-Morgenstern" as there are multiple ways how to define cardinal preference relations.

Definition 3.1.2. *Let X be a cardinal preference relation. We say that an element $x \in X$ is a* ***zero-one normalized representative of*** *X, if*

(i) for each least preferred alternative $a \in A$ we have $x(a) = 0$,

(ii) for each most preferred alternative $a \in A$ strictly preferred to the least one we have $x(a) = 1$.

Definition 3.1.3. *Let f be a cardinal welfare function. We say that:*

- *f satisfies* ***cardinal independence of irrelevant alternatives (CIAA)*** *if:*

$$\forall B \subset A, |B| = 3 \; \forall \underline{X}, \underline{Y} \in \Xi^{|V|} : \underline{X}|_B = \underline{Y}|_B \Rightarrow f(\underline{X})|_B = f(\underline{Y})|_B,$$

- *f satisfies **unanimity (U)** if:*

$$\forall \{a,b\} \subset A, \forall \underline{X} \in \Xi^{|V|}:$$

$$(\forall i \in V \; \forall x_i \in X_i: \; x_i(a) > x_i(b) \Rightarrow y(a) > y(b) \text{ for some (or all) } y \in f(\underline{X}))\,,$$

- *f is **cardinally dictatorial** if there is $j \in V$ such that for all $\underline{X} \in \Xi^{|V|}$ we have $f(\underline{X}) = X_j$.*

Remark. If $|B| = 2$ then there is a bijection from ordinal to cardinal profiles restricted to B.

In the following definition, we constrain the set of voters to be finite since this definition is only required for Theorem 3.2.1 which assumes that the set of voters is finite.

Definition 3.1.4. *Suppose that $|V| = n < \infty$. A sequence in Ξ **converges** if a sequence of its zero-one normalized representatives converges pointwise. A convergence in Ξ^n is defined as a convergence in coordinates. We say that a cardinal welfare function $f\colon \Xi^n \to \Xi$ is **continuous** if whenever a sequence of cardinal profiles $((X_1^k, X_2^k, \ldots, X_n^k))_{k\in\mathbb{N}}$ converges to a cardinal profile $(Y_1, Y_2, \ldots, Y_n)$, we have*

$$f((X_1^k, X_2^k, \ldots, X_n^k)) \overset{k\to\infty}{\longrightarrow} f((Y_1, Y_2, \ldots, Y_n)).$$

3.2 (Non-)dictatorship

The following theorem is an analogue of the Arrow's Impossibility Theorem (Theorem 2.2.1) in the case where the preferences are real functions. The proof can be found in Kalai and Schmeidler [1977].

Theorem 3.2.1 (Cardinal impossibility theorem, Kalai and Schmeidler [1977])**.** *If $|A| \geq 4$ and $|V| < \infty$ then a procedure for aggregation of cardinal preferences is continuous and satisfies cardinal independence of irrelevant alternatives and unanimity if and only if it is cardinally dictatorial.*

Definition 3.2.2 (Väth [2006])**.** *A linear functional f on ℓ_∞ (the space of bounded sequences) is called a **Banach limit** or a **Banach-Mazur limit**, if it satisfies the following properties:*

(i) If $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty$ is a constant sequence, i.e. $\forall n \in \mathbb{N} : x_n = c$, then $f(x) = c$,

(ii) f is positive, i.e. $f(x) \geq 0$ whenever $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty : \forall n \in \mathbb{N}\; x_n \geq 0$,

(iii) f is shift invariant, i.e. $f(x) = f(y)$ for $x = (x_n)_{n\in\mathbb{N}}$ and $y = (x_{n+1})_{n\in\mathbb{N}}$.

Notation. If $x = (x_n)_{n\in\mathbb{N}} \in \ell_\infty$ and f is a linear functional on ℓ_∞, the following types of notation will be used: $f(x)$, $f((x_n)_{n\in\mathbb{N}})$ or $f((x_n)_n)$.

Theorem 3.2.3. *Let $|A| \geq 4$ and V be countably infinite. Then there exists a cardinal welfare function satisfying CIIA and U which is not cardinally dictatorial.*

Proof. For each cardinal profile $\underline{X} = (X_1, X_2, \dots)$ take its zero-one normalized representatives $(x_1, x_2, \dots)$ and fix it throughout the proof.

Method 1. **Banach limit**: We define a cardinal welfare function f at each point $a \in A$ as a Banach limit of the sequence $(x_i(a))_i$. To prove that f is not cardinally dictatorial we need to prove that for every $j \in V$ there exists a cardinal profile $\underline{X}$ such that $f(\underline{X}) \neq X_j$, i. e.

$$\forall x \in f(\underline{X})\ \forall y \in X_j\ \forall \alpha \in \mathbb{R}^+\ \forall \beta \in \mathbb{R}\ \exists a \in A : y(a) \neq \alpha x(a) + \beta.$$

Fix $i \in V$ and $a \in A$. We define $\underline{X}$ in the following way:

$$x_i(b) = \begin{cases} 0 & \text{if } b = a \\ 1 & \text{if } b \neq a \end{cases},$$

and for $j \in V, j \neq i$:

$$x_j(b) = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{if } b \neq a \end{cases}.$$

Due to properties (iii) and (i) of Definition 3.2.2, we derive

$$f(\underline{X})(a) = f((1, 1, \cdots, 1, \underset{i\text{-th position}}{0}, 1, \cdots)) \overset{(iii)\text{applied } i \text{ times}}{=} f((1, 1, \cdots)) \overset{(i)}{=} 1$$

and for $b \neq a$

$$f(\underline{X})(b) = f((0, 0, \cdots, 0, \underset{i\text{-th position}}{1}, 0, \cdots)) \overset{(iii)\text{applied } i \text{ times}}{=} f((0, 0, \cdots)) \overset{(i)}{=} 0.$$

Altogether,

$$f(\underline{X})(b) = \begin{cases} 1 & \text{if } b = a \\ 0 & \text{if } b \neq a \end{cases}.$$

We need to prove that $f(\underline{X})$ is not a positive linear transformation of x_i for any $i \in V$. Suppose that there exist $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$ such that $f(\underline{X}) = \alpha x_i + \beta$. If we evaluate both functions at point a and at any other point b, we get the system of equations

$$\begin{aligned} 1 &= \alpha \cdot 0 + \beta \\ 0 &= \alpha \cdot 1 + \beta. \end{aligned}$$

Its only solution is $\alpha = -1$, $\beta = 1$, which is a contradiction to $\alpha > 0$. So $f(\underline{X}) \notin X_i$ and f is not cardinally dictatorial.

Method 2. **Nonstandard proof**: Let $\mathcal{U}$ be a non-principal ultrafilter on the set V. Let $\mathbb{U} \overset{*}{\to} \mathbb{U}'$ be the embedding into the nonstandard universe as in the proof of the Theorem 2.3.1. Take arbitrary infinite $N \in {}^*\mathbb{N}$. We define a cardinal welfare function f at each point $a \in A$ as the standard part of a sum of the zero-one normalized representatives at ${}^*a$ divided by the number of voters:

$$f(\underline{X})(a) = \operatorname{st}\left(\frac{1}{N}\sum_{i=1}^{N} {}^*x_i({}^*a)\right).$$

We can get the nonexistence of a cardinal dictator by the same cardinal profile as in the Method 1. Fix $i \in V$ and $a \in A$. Indeed,

$$f(\underline{X})(a) = \mathrm{st}\left(\frac{1}{N}\sum_{i=1}^{N} {}^*x_i({}^*a)\right) = \mathrm{st}\left(\frac{N-1}{N}\right) = 1,$$

and for $b \in A$, $b \neq a$:

$$f(\underline{X})(b) = \mathrm{st}\left(\frac{1}{N}\sum_{i=1}^{N} {}^*x_i({}^*b)\right) = \mathrm{st}\left(\frac{1}{N}\right) = 0.$$

By the same reasoning as in Method 1. we get $f(\underline{X}) \notin X_i$.

□

Remark. In fact, Method 2. is a special case of Method 1. because the function f defined as

$$f(x) = \mathrm{st}\left(\frac{1}{N}\sum_{i=1}^{N} {}^*x_i\right) \quad \text{for } x = (x_i)_i \in \ell_\infty$$

is a Banach limit. The linearity follows from the following properties from Lemma 1.2.13:

1. $\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y)$,
2. $\mathrm{st}(x \cdot y) = \mathrm{st}(x) \cdot \mathrm{st}(y)$.

We may easily check the conditions from Definition 3.2.2:

(i) if $\forall n \in \mathbb{N} : x_n = c$, then

$$f(x) = \mathrm{st}\left(\frac{1}{N}\sum_{i=1}^{N} {}^*c\right) = \mathrm{st}\left(\frac{1}{N} \cdot N \cdot {}^*c\right) = c,$$

(ii) if $\forall n \in \mathbb{N} :\ x_n \geq 0$ then $\left(\frac{1}{N}\sum_{i=1}^{N} {}^*x_i\right) \geq 0$, so again by Lemma 1.2.13, $f(x) \geq 0$,

(iii)

$$\begin{aligned} f((x_i)_i) &= \mathrm{st}\left(\frac{1}{N}\sum_{i=1}^{N} {}^*x_i\right) = \mathrm{st}\left(\frac{1}{N}\left({}^*x_1 + \sum_{i=1}^{N} {}^*x_{i+1} - {}^*x_{N+1}\right)\right) \\ &= \mathrm{st}\left(\frac{{}^*x_1 - {}^*x_{N+1}}{N}\right) + \mathrm{st}\left(\frac{1}{N}\sum_{i=1}^{N} {}^*x_{i+1}\right) \\ &= \mathrm{st}\left(\frac{1}{N}\sum_{i=1}^{N} {}^*x_{i+1}\right) = f((x_{i+1})_i) \end{aligned}$$

since $\frac{{}^*x_1 - x_{N+1}}{N} \simeq 0$.

More information on the relation between Banach limits and hyperfinite sums can be found in Väth [2006].

Example. Another example of a Banach limit (and consequently of a non-dictatorial social welfare function) is a limit along a non-principal ultrafilter $\mathcal{U}$ ($\mathcal{U}$-limit) of the sequence $x = (x_i)_{i\in\mathbb{N}}$, i.e. :

$$f(x) = L \in \mathbb{R} \Leftrightarrow \forall \varepsilon \in \mathbb{R}^+ : \left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} x_i - L \right| < \varepsilon \right\} \in \mathcal{U}.$$

The proof is again straightforward:
Let $f(x) = L$, $f(y) = K$ and fix $\varepsilon > 0$. Then the sets

$$X = \left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} x_i - L \right| < \frac{\varepsilon}{2} \right\} \text{ and } Y = \left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} y_i - K \right| < \frac{\varepsilon}{2} \right\}$$

are in $\mathcal{U}$ and so is their intersection. For $n \in X \cap Y$ we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} (x_i + y_i) - (L + K) \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} x_i - L \right| + \left| \frac{1}{n} \sum_{i=1}^{n} y_i - K \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence,

$$\left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} (x_i + y_i) - (L + K) \right| < \varepsilon \right\} \supseteq X \cap Y \in \mathcal{U}$$

and $f(x + y) = f(x) + f(y)$.

Let $f(x) = L$ and $K \in \mathbb{R}$. If $K = 0$, then the equation

$$\frac{1}{n} \sum_{i=1}^{n} 0 \cdot x_i = 0 = 0 \cdot L$$

holds for every $n \in \mathbb{N}$, so the condition $f(0 \cdot x) = 0 \cdot L$ is satisfied. Now suppose that $K \neq 0$ and fix $\varepsilon > 0$. We know that the set

$$X = \left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} x_i - L \right| < \frac{\varepsilon}{|K|} \right\}$$

is in $\mathcal{U}$. For $n \in X$ the following equalities hold

$$\left| \frac{1}{n} \sum_{i=1}^{n} K \cdot x_i - K \cdot L \right| = |K| \cdot \left| \frac{1}{n} \sum_{i=1}^{n} x_i - L \right| < |K| \cdot \left(\frac{\varepsilon}{|K|} \right) = \varepsilon,$$

so we get that

$$\left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} K \cdot x_i - K \cdot L \right| < \varepsilon \cdot |K| \right\} \supseteq X \in \mathcal{U}.$$

Therefore, $f(K \cdot x) = K \cdot f(x)$.

We continue with the other conditions from Definition 3.2.2:

(i) If $\forall i \in \mathbb{N} : x_i = c$, then

$$\forall \varepsilon \in \mathbb{R}^+ : \left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} x_i - c \right| < \varepsilon \right\} = \mathbb{N},$$

so this condition is satisfied.

(ii) Suppose that $\forall i \in \mathbb{N}: \ x_i \geq 0$ and $f(x) < 0$. We can find $\varepsilon \in (0, -f(x))$. Then for this ε we have

$$\left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} x_i - f(x) \right| < \varepsilon \right\} = \emptyset.$$

This is a contradiction with the fact that $\emptyset \notin \mathcal{U}$.

(iii) Fix $\varepsilon > 0$. First, we compute $f\left((x_i)_i - (x_{i+1})_i\right)$:

$$\left| \frac{1}{n} \sum_{i=1}^{n} (x_i - x_{i+1}) \right| = \left| \frac{1}{n} (x_1 - x_{n+1}) \right| \xrightarrow[n \to \infty]{} 0.$$

So, we can find $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_0$ we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} (x_i - x_{i+1}) \right| < \varepsilon.$$

The set $\{n_0, n_0 + 1, n_0 + 2, \cdots\}$ is in $\mathcal{U}$ because its complement is a finite set and $\mathcal{U}$ is non-principal, hence cannot contain any finite set. Altogether, we showed that for arbitrary $\varepsilon \in \mathbb{R}^+$ the set

$$\left\{ n \in \mathbb{N} : \left| \frac{1}{n} \sum_{i=1}^{n} (x_i - x_{i+1}) \right| < \varepsilon \right\}$$

is in $\mathcal{U}$. Hence, $f\left((x_i)_i - (x_{i+1})_i\right) = 0$ and by linearity we get $f\left((x_i)_i\right) = f\left((x_{i+1})_i\right)$.

△

4. The Čech-Stone Compactification

4.1 Definitions

In this section, we introduce basic definitions from general topology. For more details, see Willard [2004]. Throughout this chapter, suppose that $\mathbb{R} \subseteq \mathbb{X}$ and $\mathbb{U}(\mathbb{X}) \xrightarrow{*} \mathbb{U}({}^*\mathbb{X})$ is an enlargement.

Definition 4.1.1. *Given a set X, a family $\tau \subseteq \mathcal{P}(X)$ of subsets of X is a **topology** if the following conditions hold:*

(i) $X \in \tau$, $\emptyset \in \tau$,

(ii) $U, V \in \tau \Rightarrow U \cap V \in \tau$,

(iii) $\mathcal{V} \subseteq \tau \Rightarrow \bigcup_{U \in \mathcal{V}} U \in \tau$.

*A **topological space** (X, τ) is a set X equipped with a topology τ. A set $A \subseteq X$ is called **open** if $A \in \tau$. It is **closed** if $A^C \in \tau$.*

Definition 4.1.2. *Let (X, τ) be a topological space. A **base of** the topology τ is a collection of sets $\mathcal{B} \subseteq \tau$ such that*

$$\tau = \left\{ \bigcup_{B \in \mathcal{S}} B : \mathcal{S} \subseteq \mathcal{B} \right\}.$$

Lemma 4.1.3. *A family $\mathcal{B}$ is a base for a topology on X iff:*

(i) $X = \bigcup_{B \in \mathcal{B}} B$,

(ii) $\forall A, B \in \mathcal{B}\ \forall p \in A \cap B\ \exists C \in \mathcal{B} : p \in C \subseteq A \cap B$.

Proof. If $\mathcal{B}$ is a base of topology τ on X, then $X \in \tau$, so $X = \bigcup_{B \in \mathcal{B}} B$. Moreover, if $A, B \in \mathcal{B}$, then $A, B \in \tau$ and their intersection is in τ as well, so there is a subcollection $\mathcal{S} \subseteq \mathcal{B}$ such that $A \cap B = \bigcup_{B \in \mathcal{S}}$. Therefore, if $p \in A \cap B$, we can find $B \in \mathcal{S}$ such that $p \in B$ and that is the sought-for element of $\mathcal{B}$.

Conversely, let $\mathcal{B}$ be a set with properties (i) and (ii). We define

$$\tau = \left\{ \bigcup_{B \in \mathcal{S}} B : \mathcal{S} \subseteq \mathcal{B} \right\}.$$

Then $X \in \tau$ due to (i) and $\emptyset \in \tau$ because it is a union of empty subcollection of $\mathcal{B}$. If $U, V \in \tau$, then

$$U = \bigcup_{B \in \mathcal{U}} B \text{ and } V = \bigcup_{C \in \mathcal{V}} C$$

for some $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}$. Then

$$U \cap V = \left(\bigcup_{A \in \mathcal{U}} A \right) \cap \left(\bigcup_{B \in \mathcal{V}} B \right) = \bigcup_{A \in \mathcal{U}} \bigcup_{B \in \mathcal{V}} (A \cap B)$$

By the property (ii) we can write

$$A \cap B = \bigcup_{p \in A \cap B} C_p,$$

for $C_p \in \mathcal{B}$, so

$$U \cap V = \bigcup_{A \in \mathcal{U}} \bigcup_{B \in \mathcal{V}} \bigcup_{p \in A \cap B} C_p$$

is a union of sets from $\mathcal{B}$. Therefore, $U \cap V \in \tau$.

If $\mathcal{V} \subseteq \tau$, then $\bigcup_{v \in \mathcal{V}} \mathcal{S}_v \subseteq \mathcal{S}$, so the union is also in τ. Altogether, τ is a topology on X. $\square$

Definition 4.1.4. *Let (X, τ) be a topological space and $A \subseteq X$. We define the* ***closure*** *of A (and denote $\overline{A}$) as the intersection of all nonempty closed supersets of A. The set A is called* ***dense*** *if $\overline{A} = X$.*

Definition 4.1.5. *A topological space (X, τ) is called a* ***Hausdorff space*** *(or* T_2***-space****) if for every two distinct points $x, y \in X$, $x \neq y$ we can find $U, V \in \tau$ such that $U \cap V = \emptyset$, $x \in U$ and $y \in V$.*

Definition 4.1.6. *A space (X, τ) is* ***compact*** *if each open cover of X has a finite subcover, i. e. for all $T_i \in \tau$, $i \in I$ with $X \subseteq \bigcup_{i \in I} T_i$ there exists a finite set $J \subseteq I$ such that $X \subseteq \bigcup_{i \in J} T_i$.*

Definition 4.1.7. *A* ***compactification*** *of the topological space X is an ordered pair (K, h) where K is a compact Hausdorff space and h is a homeomorphic embedding of X as a dense subset of K, i.e. $\overline{h(X)} = K$.*

Definition 4.1.8. *The* ***Čech-Stone compactification*** *of the topological space X is a compactification $(\beta X, h)$ satisfying the* ***universal property****:*
Let K be a compact Hausdorff space. Then any continuous map $f : X \to K$ extends uniquely to a continuous map $\beta f : \beta X \to K$, i.e. $\beta f\big|_X = f$.

Theorem 4.1.9. *The Čech-Stone compactification is unique up to a homeomorphism.*

Proof. Suppose that $(\beta_1 X, h_1)$ and $(\beta_2 X, h_2)$ are two Čech-Stone compactifications of a topological space X. Then $\beta_2 X$ is compact, so by the universal property there exists a unique continuous map $f_1 : \beta_1 X \to \beta_2 X$ such that $h_2 = f_1 \circ h_1$. Analogously, there exists a continuous map $f_2 : \beta_2 X \to \beta_1 X$ such that $h_1 = f_2 \circ h_2$. Then

$$f_1 \circ f_2 = (h_2 \circ h_1^{-1}) \circ (h_1 \circ h_2^{-1}) = \mathrm{id}\big|_{\beta_2 X}$$

and

$$f_2 \circ f_1 = (h_2^{-1} \circ h_1) \circ (h_1 \circ h_2^{-1}) = \mathrm{id}\big|_{\beta_1 X}.$$

Hence, f_1 is a continuous map with a continuous inverse f_2, so it is a homeomorphism. $\square$

Definition 4.1.10. *A point x is called a* ***limit of a filter*** $\mathcal{F}$ *if every neighbourhood of x belongs to $\mathcal{F}$. We then say that the filter $\mathcal{F}$* ***converges*** *to x.*

Lemma 4.1.11. *Every filter in a Hausdorff topological space (X, τ) has at most one limit.*

Proof. Take a filter $\mathcal{F}$ and suppose that there exist two different limit points $x, y \in X$ of $\mathcal{F}$. Since X is Hausdorff, there exists a neighbourhood $U \in \tau$ of x and a neighbourhood $V \in \tau$ of y with an empty intersection. Because both x and y are limits of $\mathcal{F}$, necessarily $U \in \mathcal{F}$ and $V \in \mathcal{F}$. Hence, their intersection must also lie in $\mathcal{F}$ but this is a contradiction to the fact that $\emptyset \notin \mathcal{F}$. □

Definition 4.1.12. *We say that a point $x \in {}^*X$ is **infinitely close** to $y \in {}^*X$ if for every open set $\mathcal{O} \subseteq X$ the following condition holds:*

$$y \in {}^*\mathcal{O} \Rightarrow x \in {}^*\mathcal{O}.$$

The following theorem is known as Robinson's Compactness Criterion and comes from [Robinson, 1966], Theorem 4.1.13.

Theorem 4.1.13. *A set $A \subseteq X$ is compact if and only if every point of ${}^*A$ is infinitely close to some (standard) point of A.*

Proof. Suppose that A is compact and for contradiction suppose that there exist a point $x \in {}^*A$ that is not infinitely close to any $y \in A$. Therefore, for every $y \in A$ we can find an open set $\mathcal{O}_y$ such that $y \in {}^*\mathcal{O}_y$ and $x \notin {}^*\mathcal{O}_y$. Then

$$\{\mathcal{O}_y : y \in A\}$$

is an open cover of A. Since A is compact, we can find a finite subcover, i.e.

$$A \subseteq \mathcal{O}_{y_1} \cup \mathcal{O}_{y_2} \cup \cdots \cup \mathcal{O}_{y_n}.$$

By Lemma 1.2.8, we get

$${}^*A \subseteq {}^*\left(\mathcal{O}_{y_1} \cup \mathcal{O}_{y_2} \cup \cdots \cup \mathcal{O}_{y_n}\right) = {}^*\mathcal{O}_{y_1} \cup {}^*\mathcal{O}_{y_2} \cup \cdots \cup {}^*\mathcal{O}_{y_n}.$$

Hence, $x \in {}^*\mathcal{O}_{y_i}$ for some $i \in \{1, \ldots, n\}$ which contradicts the construction of the sets $\mathcal{O}_y$.

Conversely, suppose that A is not compact. It means that there exists an open cover $\mathcal{T} = \{T_i \in \tau : i \in I\}$ such that no finite subset covers A. We define a binary relation R on $\mathcal{T} \times A$:

$$TRx \Leftrightarrow x \notin T.$$

Then R is finitely satisfiable, so by Theorem 2.4.2, there exists $x \in {}^*A$ such that $x \notin {}^*T$ for all $T \in \mathcal{T}$. If we take any $y \in A$, then $y \in T \subseteq {}^*T$ for some $T \in \mathcal{T}$ but $x \notin {}^*T$. Hence, x is not infinitely close to any $y \in A$ and the proof is completed. □

Theorem 4.1.14. *For a topological space (X, τ), the following are equivalent:*

(i) X is compact,

(ii) each ultrafilter in X converges.

Proof.

Method 1. The first method is the standard one. For the implication $(i) \Rightarrow (ii)$ suppose that X is compact and that there is an ultrafilter $\mathcal{U}$ in X that is not convergent, i.e.

$$\forall x \in X \; \exists U_x \in \tau : U_x \notin \mathcal{U}.$$

The collection $\{U_x : x \in X\}$ is an open cover of X, so there exists a finite subcover $\{U_1, \dots U_n\}$. Take any $A \in \mathcal{U}$. Then we can express A as

$$A = A \cap (U_1 \cup \dots \cup U_n) = (A \cap U_1) \cup \dots \cup (A \cap U_n).$$

From Lemma 1.1.3 we get that there exists unique $i \in \{1, \dots, n\}$ such that $A \cap U_i \in \mathcal{U}$. Necessarily, $U_i \in \mathcal{U}$ and that is in contradiction to the definition of U_i.

To prove the other implication, suppose that $\{U_i : i \in I\}$ is an open cover of X that does not have a finite subcover. Then the collection

$$\{X \setminus U_i : i \in I\}$$

has the finite intersection property. By Theorem 1.1.8, there exists an ultrafilter $\mathcal{U}$ satisfying $\{X \setminus U_i : i \in I\} \subseteq \mathcal{U}$. By the assumption, there exists $x \in X$ such that $\mathcal{U}$ converges to x. But then $x \in U_i$ for some $i \in I$, so U_i is a neighbourhood of x and necessarily $U_i \in \mathcal{U}$. However, that is not possible since $X \setminus U_i \in \mathcal{U}$.

Method 2. In the non-standard proof, we make use of Theorem 4.1.13 and, in fact, we prove the equivalence

$$\text{Every point of } x \in {}^*X \text{ is infinitely close to some } y \in X \Leftrightarrow \\ \Leftrightarrow \text{ every ultrafilter in } X \text{ converges.}$$

First, suppose that

$$\forall x \in {}^*X \; \exists y \in X \; \forall T \in \tau : y \in {}^*T \Rightarrow x \in {}^*T$$

and let $\mathcal{U}$ be an ultrafilter in X. The ultrafilter $\mathcal{U}$ has the finite intersection property, so from the definition of an enlargement,

$$\bigcap_{U \in \mathcal{U}} {}^*U \neq \emptyset.$$

Take $x \in \bigcap_{U \in \mathcal{U}} {}^*U$. By assumption, there exists $y \in X$ that is infinitely close to x. Hence, for every neighbourhood U of y we have

$$y \in {}^*U \Rightarrow x \in {}^*U.$$

Our aim is to show that $\mathcal{U}$ converges to y. Let U be a neighbourhood of y. Then also $x \in {}^*U$. We can conclude that $U \in \mathcal{U}$, because otherwise $X \setminus U \in \mathcal{U}$ but $x \notin {}^*(X \setminus U)$. So, every neighbourhood of y is in $\mathcal{U}$ and y is a limit of $\mathcal{U}$.

Next, suppose that every ultrafilter in X converges and suppose that X is not compact, i.e. there exists $x \in {}^*X$ such that for all $y \in X$ there exists $U_y \in \tau$ such that $y \in {}^*U_y$ and $x \notin {}^*U_y$. Then, similarly as in the standard proof, the collection

$$\{X \setminus U_y : y \in X\}$$

has the finite intersection property, so it is contained in some ultrafilter $\mathcal{U}$. But this ultrafilter doesn't converge because for every $y \in X$ the neighbourhood U_y is not in $\mathcal{U}$. Hence, the proof is finished.

□

4.2 Construction of The Čech-Stone Compactification Via Ultrafilters

Let us consider the space of natural numbers equipped with the discrete topology, i.e. the one-element sets $\{n\}$, $n \in \mathbb{N}$ form the basis of the topology. We define the space

$$\beta\mathbb{N} := \{\mathcal{U} : \mathcal{U} \text{ is ultrafilter on } \mathbb{N}\}.$$

The basis of its topology are the sets of the form

$$\widehat{X} = \{\mathcal{U} \in \beta\mathbb{N} : X \in \mathcal{U}\} \text{ for } X \subseteq \mathbb{N}.$$

Lemma 4.2.1. *For every $A, B \subseteq \mathbb{N}$ it holds that:*

(i) $\widehat{A \cup B} = \widehat{A} \cup \widehat{B}$,

(ii) $\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$,

(iii) $\widehat{A^C} = \widehat{A}^C$,

(iv) $\widehat{A} = \emptyset \iff A = \emptyset$.

Proof. (i) We need to prove two inclusions. First, take $\mathcal{U} \in \widehat{A \cup B}$. By the definition, it means that $A \cup B \in \mathcal{U}$. From the properties of ultrafilters (Lemma 1.1.3) we have $A \in \mathcal{U}$, or $B \in \mathcal{U}$, in other words $\mathcal{U} \in \widehat{A}$, or $\mathcal{U} \in \widehat{B}$.

Now take $\mathcal{U} \in \widehat{A} \cup \widehat{B}$. It means that $\mathcal{U}$ belongs either in $\widehat{A}$, or $\widehat{B}$. Without loss of generality suppose that $\mathcal{U} \in \widehat{A}$. So we have $A \in \mathcal{U}$ and by closedness of ultrafilters under supersets (Definition 1.1.1(iv)) we have $A \cup B \in \mathcal{U}$, so $\mathcal{U} \in \widehat{A \cup B}$.

(ii) The proof is similar to the first part. If $\mathcal{U} \in \widehat{A \cap B}$, then $A \cap B \in \mathcal{U}$, so again by Definition 1.1.1(iv) $A \in \mathcal{U}$ and $B \in \mathcal{U}$, i.e. $\mathcal{U} \in \widehat{A} \cap \widehat{B}$. The other inclusion holds due to the closedness of ultrafilters under finite intersections (Definition 1.1.1(iii)).

(iii) For an ultrafilter $\mathcal{U}$ the following sequence of equivalences holds:

$$\mathcal{U} \in \widehat{A}^C \iff A \notin \mathcal{U} \overset{(1)}{\iff} A^C \in \mathcal{U} \iff \mathcal{U} \in \widehat{A^C},$$

where the implication $\Rightarrow$ of (1) follows from Definition 1.1.1(v) and the other from Definition 1.1.1(iii) and (ii), because $A \cap A^C = \emptyset$.

(iv) Let A be an empty set. By Definition 1.1.1(ii), no ultrafilter can contain an empty set. Therefore, $\widehat{A}$ is also an empty set. On the other hand, suppose that A is not empty and fix an element $a \in A$. We can define $\mathcal{U} = \{X \subseteq \mathbb{N} : a \in X\}$. Clearly, $\mathcal{U}$ is a principal ultrafilter. Hence, $\widehat{A}$ is not empty.

□

Lemma 4.2.2. *The family $\mathfrak{B} = \{\widehat{X} : X \subseteq \mathbb{N}\}$ forms the basis of a topology on $\beta\mathbb{N}$.*

Proof. We need to check the two conditions from Lemma 4.1.3:

(i) $\beta\mathbb{N} = \bigcup_{B \in \mathfrak{B}} B$ is trivial.

(ii) Take $\mathcal{A}, \mathcal{B} \in \mathfrak{B}$, $P \in \mathcal{A} \cap \mathcal{B}$. $\mathcal{A}$ and $\mathcal{B}$ are families of ultrafilters on $\mathbb{N}$ of the form $\widehat{A}$ and $\widehat{B}$ for some $A, B \subseteq \mathbb{N}$ and P is an ultrafilter lying in both of them. Hence, $A \cap B \neq \emptyset$. From Lemma 4.2.1(ii), we know that $\widehat{A \cap B} = \widehat{A} \cap \widehat{B}$. We can take $\mathcal{C} = \widehat{A \cap B}$. Then $\mathcal{C} \in \mathfrak{B}$ and $P \in \mathcal{C} = \mathcal{A} \cap \mathcal{B}$.

Hence, $\mathfrak{B}$ is the basis of a topology on $\beta\mathbb{N}$. □

Remark. Lemma 4.2.1(iii) implies that the sets of the form $\widehat{X}$ are open and closed at the same time.

We can construct the embedding $h : \mathbb{N} \to \beta\mathbb{N}$ as follows: to every $n \in \mathbb{N}$ we assign the principal ultrafilter $\mathcal{U}_n = \{M \subseteq \mathbb{N} : n \in M\}$. Clearly, the set $\beta\mathbb{N} \setminus h(\mathbb{N})$ consists of free ultrafilters on $\mathbb{N}$.

We also need to define the unique extension function. Let K be a compact Hausdorff space and $f : \mathbb{N} \to K$ a continuous map. Then the extension function $\beta f : \beta\mathbb{N} \to K$ is defined as follows: for any ultrafilter $\mathcal{U} \in \beta\mathbb{N}$, the image

$$\bar{f}(\mathcal{U}) = \{X \subseteq K : f^{-1}(X) \in \mathcal{U}\}$$

is an ultrafilter (the proof will be given in Theorem 4.2.3). Since K is compact Hausdorff, by Theorem 4.1.14 this ultrafilter has a unique limit $x \in K$. We define

$$\beta f(\mathcal{U}) = x.$$

Theorem 4.2.3. *Space $\beta\mathbb{N}$ with the topology generated by $\mathfrak{B}$ is a Čech-Stone compactification of the discrete space $\mathbb{N}$.*

Proof. We divide this proof into several parts.

1. $\beta\mathbb{N}$ **is a Hausdorff space**: Take $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N}$, $\mathcal{U} \neq \mathcal{V}$. Then there exists a set $A \subset \mathbb{N}$ such that $A \in \mathcal{U}$ but $A \notin \mathcal{V}$. Since $\mathcal{V}$ is an ultrafilter, we get from Definition 1.1.1 (v) that $\mathbb{N} \setminus A \in \mathcal{V}$. Moreover, $\mathbb{N} \setminus A \notin \mathcal{U}$. Define

$$U = \{\mathcal{W} \in \beta\mathbb{N} : A \in \mathcal{W}\} \text{ and } V = \{\mathcal{W} \in \beta\mathbb{N} : (\mathbb{N} \setminus A) \in \mathcal{W}\}.$$

Clearly, U and V are open in the topology of $\beta\mathbb{N}$, $\mathcal{U} \in U$, $\mathcal{V} \in V$ and $U \cap V = \emptyset$, so the space $\beta\mathbb{N}$ is Hausdorff.

2. $\beta\mathbb{N}$ **is compact**: Let $\{T_i\}_{i\in I}$ be an open cover of $\beta\mathbb{N}$, i.e.

$$\beta\mathbb{N} \subseteq \bigcup_{i\in I} T_i.$$

Without loss of generality, suppose that every T_i is a member of the basis of open sets. It means that for every T_i we can find $X_i \subseteq \mathbb{N}$ such that $T_i = \widehat{X_i}$.

For contradiction suppose that there is no finite subset J of I such that $\{T_i\}_{i\in J}$ is an open cover of $\beta\mathbb{N}$. Denote by $\mathcal{P}_F(I)$ the finite subsets of I. In other words, for every $J \in \mathcal{P}_F(I)$ it holds that

$$\beta\mathbb{N} \nsubseteq \bigcup_{i\in J} T_i.$$

Take

$$\mathcal{U} = \left\{X_i^C\right\}_{i\in I}.$$

We aim to show that $\mathcal{U}$ has the finite intersection property. Take $F \in \mathcal{P}_F(I)$. Then, by Lemma 4.2.1, we have

$$\widehat{\left(\bigcap_{i\in F} X_i^C\right)} = \bigcap_{i\in F} \left(\widehat{X_i}\right)^C = \bigcap_{i\in F} (T_i)^C = \left(\bigcup_{i\in F} T_i\right)^C \neq \emptyset$$

and from part (iv), it follows that

$$\bigcap_{i\in F} X_i^C \neq \emptyset,$$

so the family $\{X_i^C\}_{i\in I}$ has the finite intersection property.

By Theorem 1.1.8, $\mathcal{U}$ is contained in some ultrafilter $\mathcal{U}' \in \beta\mathbb{N}$. By construction, $\mathcal{U}' \notin \bigcup_{i\in I} T_i$, but that is a contradiction since $\{T_i\}_{i\in I}$ was supposed be an open cover of $\beta\mathbb{N}$.

3. **The universal property**: First, we show that

$$\bar{f}(\mathcal{U}) = \{X \in K : f^{-1}(X) \in \mathcal{U}\}$$

is an ultrafilter if $\mathcal{U}$ is an ultrafilter. We check the conditions from Lemma 4.2.1. They follow from properties of preimages and properties of $\mathcal{U}$:

   (i) It is not empty because $f^{-1}(K) = X \in \mathcal{U}$.

   (ii) It does not contain an empty set since $f^{-1}(\emptyset) = \emptyset \notin \mathcal{U}$.

(iii) Take $X, Y \in \bar{f}(\mathcal{U})$. Then both $f^{-1}(X)$ and $f^{-1}(Y)$ are contained in $\mathcal{U}$, so is their intersection. Hence,

$$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y) \in \mathcal{U}$$

and $X \cap Y \in \bar{f}(\mathcal{U})$.

(iv) If $X \in \bar{f}(\mathcal{U})$ and $X \subseteq Y$, then

$$f^{-1}(Y) \supseteq f^{-1}(X) \in \mathcal{U},$$

so $Y \in \bar{f}(\mathcal{U})$.

(v) Suppose that $X \subseteq K$ and $X \notin \bar{f}(\mathcal{U})$. Then $f^{-1}(X) \notin \mathcal{U}$, therefore $(f^{-1}(X))^C \in \mathcal{U}$. The proof is finished by the observation that

$$f^{-1}(K \setminus X) = f^{-1}(K) \setminus f^{-1}(X) = \mathbb{N} \setminus f^{-1}(X).$$

Hence, $\bar{f}(\mathcal{U})$ is an ultrafilter. The limit of this ultrafilter is unique, so we defined a unique function.

Next, we need to show that this function is continuous and that it is an extension of f. Take $n \in \mathbb{N}$. To show that βf is an extension of f, i.e.

$$f(n) = \beta f(h(n)),$$

we need to show that every neighbourhood U of $f(n)$ in K belongs to the ultrafilter $\bar{f}(h(n))$. This means that $f^{-1}(U)$ lies in the principal ultrafilter generated by n. But this is obvious because $f(n) \in U$, so $n \in f^{-1}(U)$ and $f^{-1}(U)$ belongs to $h(n)$.

To prove the continuity of βf, take $U \subseteq K$ an open neighbourhood of $\beta f(\mathcal{U})$ for some $\mathcal{U} \in \beta\mathbb{N}$. We would like to find an open neighbourhood G of $\mathcal{U}$ such that $\beta f(G) \subseteq U$. Take V and open neighbourhood of $\beta f(\mathcal{U})$ such that $\overline{V} \subseteq U$. Since $\beta f(\mathcal{U})$ is a limit of $\bar{f}(\mathcal{U})$, we have $V \in \bar{f}(\mathcal{U})$. It means that $f^{-1}(V) \in \mathcal{U}$. Define

$$G = \widehat{f^{-1}(V)}.$$

Then G is an open neighbourhood of $\mathcal{U}$ in $\beta\mathbb{N}$. Moreover, for every $\mathcal{F} \in G$ we have

$$\mathcal{F} \in \widehat{f^{-1}(V)} \iff f^{-1}(V) \in \mathcal{F}.$$

Let W be an open neighbourhood of $\beta f(\mathcal{F})$. Then $W \in \bar{f}(\mathcal{F})$, so $f^{-1}(W) \in \mathcal{F}$. We have

$$f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W) \in \mathcal{F},$$

hence $V \cap W \neq \emptyset$. We have proved that any open neighbourhood of $\beta f(\mathcal{F})$ has nonempty intersection with V. It follows that $\beta f(\mathcal{F}) \in \overline{V} \subseteq U$.

Hence, $\beta f(G) \subseteq U$ and βf is continuous.

4. $h : \mathbb{N} \to h(\mathbb{N}) \subseteq \beta\mathbb{N}$ **is a homeomorphism**: By definition, it is a bijection. Moreover, it is continuous because $\mathbb{N}$ is a discrete space. It remains to show that f^{-1} is continuous, i.e. for every open $U \subseteq \mathbb{N}$, the set $f(U)$ is open in the topology of $\beta\mathbb{N}$. We can express U as $U = \bigcup_{n \in I} \{n\}$ for some (possibly infinite)

index set I.Then $h(U) = \bigcup_{n \in I} h(n)$. But $h(n)$ is the principal ultrafilter, i.e. it is of the form $h(n) = \widehat{\{n\}}$, which is an open set. Altogether, $h(U)$ is a union of open sets, hence also open.

5. $h(\mathbb{N})$ **is dense in** $\beta\mathbb{N}$: Suppose that

$$\beta\mathbb{N} \setminus \overline{h(\mathbb{N})} \neq \emptyset.$$

Take

$$\mathcal{U} \in \beta\mathbb{N} \setminus \overline{h(\mathbb{N})}.$$

Since $\mathcal{U} \notin \overline{h(\mathbb{N})}$, there exists V a neighbourhood of $\mathcal{U}$ in $\beta\mathbb{N}$ such that

$$V \cap h(\mathbb{N}) = \emptyset.$$

From the definition of a neighbourhood and the basis of the topology on $\beta\mathbb{N}$, there exists a set $X \subseteq \mathbb{N}$ such that $\widehat{X} \subseteq V$. This is a contradiction because for every $n \in X$ the principal ultrafilter

$$h(n) = \{M \subseteq \mathbb{N} : n \in M\}$$

contains also the set X and hence lies in $\widehat{X}$, so it follows that

$$h(n) \in V \cap h(\mathbb{N}).$$

We proved that space $\beta\mathbb{N}$ with the topology generated by $\mathfrak{B}$ is Hausdorff, compact, satisfies the universal property and the mapping h is a homeomorphic embedding of the space $\mathbb{N}$ into $\beta\mathbb{N}$. Altogether, $(\beta\mathbb{N}, h)$ is the Čech-Stone compactification of $\mathbb{N}$. □

Remark. This construction of the Čech-Stone compactification is valid for every discrete topological space. We restrict ourselves to the space of natural numbers because of the following sections.

4.3 Relation to the Hypernatural Numbers

This part is mainly based on Di Nasso [2015] with many details filled in.

Definition 4.3.1. *The **ultrafilter generated** by a hypernatural number $\alpha \in {}^*\mathbb{N}$ is the family*

$$\mathfrak{U}_\alpha = \{X \subseteq \mathbb{N} : \alpha \in {}^*X\}.$$

Lemma 4.3.2. *The family $\mathfrak{U}_\alpha$ is an ultrafilter on* $\mathbb{N}$.

Proof. We need to check the conditions from Definition 1.1.1.

(i) The set $\mathbb{N}$ is always in $\mathfrak{U}_\alpha$ because $\alpha \in {}^*\mathbb{N}$.

(ii) Since ${}^*\emptyset = \emptyset$ and $\alpha \notin \emptyset$, the empty set cannot lie in $\mathfrak{U}_\alpha$.

(iii) Let $K, L \in \mathfrak{U}_\alpha$. Then $\alpha \in {}^*K$ and $\alpha \in {}^*L$, hence it is also in their intersection, which precisely means that $K \cap L \in \mathfrak{U}_\alpha$.

(iv) Let $K \in \mathfrak{U}_\alpha$ and $K \subseteq L$. By the definition, $\alpha \in {}^*K$ and ${}^*K \subseteq {}^*L$, so $\alpha \in {}^*L$. Hence, $L \in \mathfrak{U}_\alpha$.

(v) Let $K \subseteq \mathbb{N}$ and suppose that $K \notin \mathfrak{U}_\alpha$. It means that $\alpha \notin {}^*K$. Therefore, $\alpha \in {}^*\mathbb{N} \setminus {}^*K = {}^*(\mathbb{N} \setminus K)$ and $\mathbb{N} \setminus K \in \mathfrak{U}_\alpha$.

Altogether, $\mathfrak{U}_\alpha$ is an ultrafilter. □

Definition 4.3.3. *We say that $\alpha, \beta \in {}^*\mathbb{N}$ are **equivalent** (and write $\alpha \sim \beta$) if $\mathfrak{U}_\alpha = \mathfrak{U}_\beta$.*

Remark. From the definition of the ultrafilter generated by a natural number, we can easily deduce that

$$\alpha \sim \beta \iff (\forall X \subseteq \mathbb{N} : \alpha \in {}^*X \Leftrightarrow \beta \in {}^*X).$$

Definition 4.3.4. *We define the **S-topology** on ${}^*\mathbb{N}$ in such a way that the family $\{{}^*A : A \subseteq \mathbb{N}\}$ forms a basis of open sets.*

Remark. It is called S-topology because the open sets are the standard sets of ${}^*\mathbb{N}$.

Now, let us consider the quotient space ${}^*\mathbb{N}/\sim$. The topology on $({}^*\mathbb{N})/\!\sim$ is the quotient topology, i.e. the set $U \in ({}^*\mathbb{N})/\!\sim$ is open if and only if

$$\{n \in {}^*\mathbb{N} : [n]_\sim \in U\}$$

is open. If we consider the basis of S-topology, $\{{}^*A : A \subseteq \mathbb{N}\}$, then we can deduce that the basis of the quotient topology consists of sets of the form

$$({}^*A)/\!\sim \text{ for } A \subseteq \mathbb{N}.$$

For the sake of simplicity, we will call this topology also S-topology.

Theorem 4.3.5. *If the model of nonstandard analysis is an enlargement, then the space $({}^*\mathbb{N})/\!\sim$ equipped with the S-topology is the Čech-Stone compactification of $\mathbb{N}$, i.e. there exists a homeomorphism $f : ({}^*\mathbb{N})/\!\sim \to \beta\mathbb{N}$.*

Proof. We define $f(\alpha) = \mathfrak{U}_\alpha$. We proved in Lemma 4.3.2 that $\mathfrak{U}_\alpha$ is an ultrafilter, hence, by Section 4.2, the mapping is well-defined. It is injective because every equivalence class of ${}^*\mathbb{N}$ generates different ultrafilter.

To prove surjectivity, let $\mathcal{U} \in \beta\mathbb{N}$ be an ultrafilter on $\mathbb{N}$. We define

$$\mathfrak{B} = \{X : X \in \mathcal{U}\}.$$

Then $\mathfrak{B}$ has the FIP, since $\bigcap_{X \in F} X \in \mathcal{U}$ for any finite $F \subseteq \mathcal{U}$, so $\bigcap_{X \in F} X \neq \emptyset$. From the enlargement property we get

$$\bigcap_{X \in \mathfrak{B}} {}^*X \neq \emptyset.$$

Therefore, there exists $\alpha \in {}^*\mathbb{N}$ such that $\forall X \in \mathcal{U} : \alpha \in {}^*X$. But that is exactly the definition of $\mathfrak{U}_\alpha$, so $\mathcal{U} = f([\alpha]_\sim)$ and f is surjective.

One equivalent definition of a homeomorphism is that it is a bijection such that a set A is open if and only if $f(A)$ is open. It is enough to prove that elements of bases of topologies map one to each other. Recall that the basis of the quotient topology are the sets

$$({}^*A)\,/\!\sim \ \text{ for } A \subseteq \mathbb{N}$$

and the basis of the Čech-Stone topology are the sets

$$\{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\} \text{ for } A \subseteq \mathbb{N}.$$

Fix $A \subseteq \mathbb{N}$ and consider $({}^*A)\,/\!\sim$. For any $a \in ({}^*A)\,/\!\sim$, the image $f(a)$ is an ultrafilter $\mathfrak{U}_a$. Hence,

$$f\left(({}^*A)\,/\!\sim\right) = \{\mathfrak{U}_a : a \in ({}^*A)\,/\!\sim\}.$$

We may notice that this is actually equal to $\{\mathfrak{U}_a : a \in {}^*A\}$ – we do not add any new ultrafilters since the elements in ${}^*A$ generate the same ultrafilters as elements in $({}^*A)\,/\!\sim$. Moreover,

$$\{\mathfrak{U}_a : a \in {}^*A\} = \{\{X \subseteq \mathbb{N} : a \in {}^*X\} : a \in {}^*A\}.$$

Now we prove

$$\{\mathfrak{U}_a : a \in {}^*A\} = \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}.$$

To prove the inclusion $\subseteq$, take $b \in {}^*A$ and $\mathfrak{U}_b \in \{\mathfrak{U}_a : a \in {}^*A\}$. Then

$$\mathfrak{U}_b = \{X \subseteq \mathbb{N} : b \in {}^*X\},$$

so $A \in \mathfrak{U}_b$ and we already know that $\mathfrak{U}_b \in \beta\mathbb{N}$.

On the other hand, take $\mathcal{U} \in \{\mathcal{V} \in \beta\mathbb{N} : A \in \mathcal{V}\}$. We know that $A \in \mathcal{U}$ and that f is surjective, so there is some $b \in {}^*\mathbb{N}$ such that

$$A \in \mathcal{U} = f([b]_\sim) = \mathfrak{U}_b = \{X \subseteq \mathbb{N} : b \in {}^*X\}.$$

Hence, $b \in {}^*A$ and the inclusion is proven.

We have shown that elements of the basis of topology on $({}^*A)\,/\!\sim$ are mapped to elements of the basis of the Čech-Stone topology on $\beta\mathbb{N}$. Therefore, the map f is a homeomorphism and space $({}^*\mathbb{N})\,/\!\sim$ equipped with the S-topology is the Čech-Stone compactification of $\mathbb{N}$. □

Remark. We proved that if the model of nonstandard analysis is an enlargement, then the quotient space of hypernatural numbers is the Čech-Stone compactification of $\mathbb{N}$. This justifies the observation in Section 2.3 that the nonstandard and invisible dictators are in fact the same.

Conclusion

We studied several applications of ultrafilters in nonstandard analysis, social choice theory and topology.

First, we introduced ultrafilters and showed their properties. Then we defined the nonstandard framework, used ultrafilters to establish the ultrapower construction and proved some properties of the transfer principle and hyperreal numbers.

Next, we studied the ordinal preference relations. The Arrow's impossibility theorem was proved using previously seen properties of ultrafilters – the sets of decisive sets form an ultrafilter. If the set of voters is finite, then every ultrafilter on this set is principal, so the dictator exists. On the other hand, if we have infinitely many voters, then (assuming the axiom of choice) a non-principal ultrafilter on this set exists, hence the system is non-dictatorial. We have seen two nonstandard proofs of the existence of such non-dictatorial social welfare functions in the infinite case.

We continued by defining the cardinal preference relations and established the examples of a cardinal welfare function that is not cardinally dictatorial.

Finally, we studied a connection between the Čech-Stone compactification of natural numbers and ultrafilters. We have seen that the set of all ultrafilters on $\mathbb{N}$ with appropriately chosen topology is the Čech-Stone compactification of $\mathbb{N}$. Moreover, if the model of nonstandard analysis is an enlargement, then the quotient space of hypernatural numbers with so-called S-topology is also the Čech-Stone compactification of $\mathbb{N}$.

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List of Symbols

$\mathbb{U}(\mathbb{X})$	universe over the set $\mathbb{X}$, Definition 1.2.2
${}^*a$	nonstandard embedding of a, Definition 1.2.3
$[[f = g]]$	$\{i \in I : f(i) = g(i)\}$, Section 1.2.2
$[f]$	the equivalence class of f
$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of natural numbers
$\mathbb{I}$	the set of infinitesimal numbers, Definition 1.2.10(i)
$a \simeq b$	a is infinitely close to b, Definition 1.2.11
$\mathrm{st}(x)$	the standard part of x, Lemma 1.2.12
$\mathcal{P}(X)$	the power set of a set X
A	a set of alternatives
V	a set of voters
F	a set of situations
σ	a social welfare function
Σ	a set of social welfare functions
$\mathscr{P}$	a set of preferences
$\underline{X}$	a cardinal profile, Section 3.1
$\overline{A}$	closure of A, Definition 4.1.4
βX	the Čech-Stone compactification of X, Definition 4.1.8

A. Appendix

A.1 Zorn's lemma

Definition A.1.1. *Let P be a set. We say that a binary relation $\leq$ on a set P is a **partial order** (and $(P, \leq)$ is a **partially ordered set**) if it is:*

(i) ***reflexive**: $\forall a \in P : a \leq a$;*

(ii) ***antisymmetric**: $\forall a, b \in P : (a \leq b \wedge b \leq a) \Rightarrow a = b$;*

(iii) ***transitive**: $\forall a, b, c \in P : (a \leq b \wedge b \leq c) \Rightarrow a \leq c$.*

Definition A.1.2. *Let $(P, \leq)$ be a partially ordered set. A **chain** in P is a subset $S \subseteq P$ such that*

$$\forall a, b \in S : a \leq b \vee b \leq a$$

(i. e. all its elements are comparable).

Definition A.1.3. *An element $u \in P$ is an **upper bound** of a set $S \subseteq P$ if for all $s \in S$ we have $s \leq u$.*

Definition A.1.4. *An element $m \in P$ of a partially ordered set $(P, \leq)$ is **maximal** if there is no element $n \in P$ such that $m \leq n$.*