FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

DOCTORAL THESIS

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Properties of Sobolev mappings

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Study programme: Mathematics
Study branch: Mathematical analysis

Prague 2017

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Title: Properties of Sobolev mappings

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Abstract: We study the properties of Sobolev functions and mappings, especially we study the violation of some properties. In the first part we study the Sobolev Embedding Theorem that guarantees $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ for some parameter $p^*(p, n, \Omega)$. We show that for a general domain this relation does not have to be smooth as a function of p and not even continuous and we give the example of the domain in question. In the second part we study the Cesari's counterexample of the continuous mapping in $W^{1,n}([-1,1]^n, \mathbf{R}^n)$ violating Lusin (N) condition. We show that this example can be constructed as a gradient mapping. In the third part we generalize the Cesari's counterexample and Ponomarev's counterexample for the higher derivative Sobolev spaces $W^{k,p}(\Omega, \mathbf{R}^n)$ and characterize the validity of the Lusin (N) condition in dependence on the parameters k and p and dimension.

Keywords: Sobolev space, Sobolev embedding, Lusin (N) condition, Irregular domain.

I wish to thank my supervisor, Stanislav Hencl, for his support. I specially thank him for huge amount of time and energy he spend helping all his students including me. I also wish to thank Jan Malý for many lessons and many good advises he gave me during my research.

I wish to thank my family for their support, specially my parents Bedřich and Blažena, who sacrificed a lot for me and my siblings, and my beloved wife Zuzana for creating my home. And I thank God for his blessing of my life and my work.

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- Roskovec, T.: *Sobolev homeomorphism in $W^{k,p}$ and the Lusin (N) condition.* Submitted. (2017).

The mathematical analysis was created as the tool to describe and study the phenomena in nature. Therefore mathematical definitions tries to describe the objects with as nice and as intuitive properties as possible. But since even in the nature our intuition often fails us, some objects are not as fine as we expected. We study the properties of the Sobolev spaces, which are the fundamental tool for solving PDEs in the weak sense. In this thesis, we focus on the failure of some properties and the highly unintuitive behaviour in some cases. So even though the Sobolev spaces are helping our understanding and modelling of natural processes, we often have to restrict them in order to guarantee the properties we expect and demand.

The Sobolev spaces itself are defined as the set of all the functions or mappings $f : \Omega \to \mathbf{R}^n$ with all its weak derivatives up to order k belonging to the Lebesgue spaces $L^p(\Omega)$. In the classical case, we are interested in the functions defined on some nice domain Ω, for example with a Lipschitz boundary. Then by the Sobolev Embedding Theorem, the most important derivatives would be the derivatives of the highest order. The property $|D^{k-1}f| \in L^p(\Omega)$ is the corollary of $|D^k f| \in L^p(\Omega)$, since

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \text{ for } p^* = \begin{cases} \frac{np}{n-p} & \text{for } p \in [1,n), \\ \infty & \text{for } p > n. \end{cases} \tag{1}$$

This classical result is widely studied, see [1] or many others. From the physical point of view, the highest derivative may represent some kind of energy and its key role is obvious.

In the first paper [7], we focus on the relation between p and $p^*(p,n,\Omega)$. It is not hard to show, that the introduced relation (1) does not hold for general set Ω. We can consider the sub-graph of x^α for $\alpha > 1$ as the counterexample, the Sobolev-type embedding characterization for this set can be found in [10]. We show that we can construct a domain Ω such that $p^*(p,n,\Omega)$ would not be continuous as a function of p. More precisely we can get a leap of any size in the graph of p^* in any single point inside $p \in (1,\infty)$. This is surprising, since any other discontinuity of p^* is not known and just the non-smoothness was presented by Gol'dshtein and Gurov [3]. On the other hand, the proof is based on the failure of highest derivative being the greatest contribution to the norm. Therefore we can not interpret this interesting case as the physical phenomena.

We sketch such a domain $\Omega_{\alpha,\beta}$ in Figure 1. The result is summarized as following, note that the function $q_{\Omega_{\alpha,\beta}}(p)$ is discontinuous at $\alpha+1$.

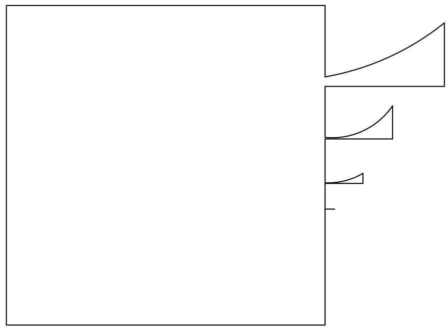

Figure 1: Domain $\Omega_{\alpha,\beta}$

Theorem 1 (Sobolev Embedding Theorem for an irregular domain). *Let $\alpha \geq 1$, $\beta > \alpha$, let $q_{\Omega_{\alpha,\beta}}(p) : [1, \beta+1) \to [1, \infty)$ be defined as*

$$q_{\Omega_{\alpha,\beta}}(p) := \begin{cases} p & \text{for } 1 \leq p < \alpha + 1, \\ \frac{(\beta+1)p}{\beta+1-p} & \text{for } \alpha + 1 \leq p < \beta + 1. \end{cases}$$

Then for p such that $1 \leq p < 1 + \beta$, $p \neq \alpha + 1$ we have

$$W^{1,p}(\Omega_{\alpha,\beta}) \subset L^{q_{\Omega_{\alpha,\beta}}(p)}(\Omega_{\alpha,\beta}).$$

Moreover, for every $q(p) > q_{\Omega_{\alpha,\beta}}(p)$ there exists a function $g : \Omega_{\alpha,\beta} \to \mathbf{R}$ satisfying

$$g \in W^{1,p}(\Omega_{\alpha,\beta}) \text{ and } g \notin L^{q(p)}(\Omega_{\alpha,\beta}).$$

We study the Lusin (N) condition in the second and the third part of thesis. We say that the mapping $f : \Omega \to \mathbf{R}^n$ satisfies Lusin (N) condition, if for any measurable set $A \subset \Omega$ we have

$$|A| = 0 \Rightarrow |f(A)| = 0. \tag{2}$$

The physical meaning of this property is that f does not create any new matter. This property also guarantees the validity of the area formula (or change of variables formula) for Sobolev mapping. Obviously we wish not to fail this condition in modelling of the deformation of solid body.

Although the positive results are more important for applications, we focus on the counterexamples to complete the characterization. The classical results by Cesari [2] and Malý, Martio [4] show that despite the Lusin (N) condition is valid for any mappings in $W^{1,p}(\Omega, \mathbf{R}^m)$ for $p > n$, it may fail in $W^{1,n}(\Omega, \mathbf{R}^m)$. If we consider Sobolev homeomorphisms instead of mappings, the counterexample by Ponomarev [6] shows that the condition is valid for any homeomorphism in $W^{1,n}(\Omega, \mathbf{R}^n)$, but may fail in $W^{1,p}(\Omega, \mathbf{R}^n)$ for $1 \leq p < n$. There are many works about the validity of the condition in the spaces near the limiting case $W^{1,n}$.

The second paper [8] is motivated by the works of Ulrich Menne [5]. In order to specify the optimal assumptions for his theory, the following question was asked: Is it possible to construct such a function, that its gradient would lie in $W^{1,n}(\Omega)$ and will fail the Lusin (N) condition? We construct this gradient mapping as the improvement of the Cesari's mapping, the proven theorem is following:

Theorem 2 (Cesari-type gradient mapping)**.** *There exists $f \in C^1([-1,1]^n)$, such that $Df \in W^{1,n}((-1,1)^n, \mathbf{R}^n)$ and $Df([-1,1] \times \{0\}^{n-1}) \supseteq [-1,1]^n$.*

The third paper [9] is dedicated to refinement of classical the Cesari's and Ponomarev's counterexamples and the validity of Lusin (N) condition in the higher order Sobolev spaces. By the Sobolev Embedding Theorem we expected, that the limiting case for higher order Sobolev spaces is $W^{k,\frac{n}{k}}$. This result is expected but not trivial, since both classical counterexamples are not smooth and lack even the second weak derivative. We prepare smoothing tools in order not to blow up the norms of the smoothed mapping and not to fail the one-to-one property in case of smoothed Ponomarev's construction. We manage to construct the examples of the possible failure of the condition in spaces $W^{k,p}(\Omega, \mathbf{R}^n)$ for $1 \leq p < \frac{n}{k}$. So the result is the guaranteed validity of the condition only in spaces $W^{k,p}(\Omega, \mathbf{R}^n)$ for $p > \frac{n}{k}$. If we consider a Sobolev homeomorphism, we get the validity even in case $W^{k,\frac{n}{k}}(\Omega, \mathbf{R}^n)$. If we consider a general Sobolev mapping, the Lusin (N) condition may fail in this space. The summarizing theorems are following:

Theorem 3 (Lusin (N) condition for Sobolev homeomorphisms)**.** *Let $k, n \in \mathbf{N}$, $n \geq k$ let $p \geq \frac{n}{k}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a homeomorphism $f \in W^{k,p}(\Omega, \mathbf{R}^n)$ satisfies Lusin (N) condition.*

On the other hand for every $k, n \in \mathbf{N}$, $n > k$ and $p \in [1, \frac{n}{k})$ there is a homeomorphism $f \in W^{k,p}((-1,1)^n, \mathbf{R}^n)$ which fails Lusin (N) condition.

Theorem 4 (Lusin (N) condition for Sobolev mappings)**.** *Let $k, m, n \in \mathbf{N}$, $n > k$, let $p > \frac{n}{k}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a mapping $f \in W^{k,p}(\Omega, \mathbf{R}^m)$ satisfies Lusin (N) condition.*

Moreover, let $m, n \in \mathbf{N}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a mapping $f \in W^{n,1}(\Omega, \mathbf{R}^m)$ satisfies Lusin (N) condition.

On the other hand for every $k, m, n \in \mathbf{N}$, $n > k$ and $p \in [1, \frac{n}{k}]$ there is a mapping $f \in W^{k,p}((-1,1)^n, \mathbf{R}^m)$ which fails Lusin (N) condition.

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1. Sobolev embedding theorem for irregular domains and discontinuity of $p \to p^*(p, n)$

Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 32 (2013), 301–315
DOI: 10.4171/ZAA/3445

Sobolev embedding theorem for irregular domains and discontinuity of $p \to p^*(p,n)$

Tomáš Roskovec

Abstract. For a domain $\Omega \subset \mathbb{R}^n$ we denote

$$q_\Omega(p) := \sup \left\{ r \in [1,\infty];\ \text{for all } f : \Omega \to \mathbb{R} : (f \in W^{1,p}(\Omega) \Rightarrow f \in L^r(\Omega)) \right\}.$$

Let $p_0 \in [2,\infty)$. We construct a domain $\Omega \subset \mathbb{R}^2$ such that $q_\Omega(p)$ is discontinuous at p_0.

Keywords. Sobolev space, Sobolev embedding

Mathematics Subject Classification (2010). Primary 46E35, secondary

1. Introduction

We study the Sobolev embedding theorem on irregular domains with non-Lipschitz boundary. The Sobolev embedding theorem on a domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary claims

$$f \in W^{1,p}(\Omega),\ p \neq n \Rightarrow f \in L^{p^*(p)}(\Omega),\ \text{where}$$

$$p^*(p) = \begin{cases} \frac{np}{n-p}, \text{ for } 1 \leq p < n, \\ \infty, \text{ for } n < p < \infty. \end{cases} \tag{1}$$

Inspired by this theorem, we can define the optimal embedding exponent for a domain $\Omega \subset \mathbb{R}^n$ as

$$q_\Omega(p) := \sup \left\{ r \in [1,\infty];\ \text{for all } f : \Omega \to \mathbb{R} : (f \in W^{1,p}(\Omega) \Rightarrow f \in L^r(\Omega)) \right\}. \tag{2}$$

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There are a lot of results in the field of characterization of $q_\Omega(p)$ for various classes of domains. For a Lipschitz domain Ω the function $p^*(p) = q_\Omega(p)$ is continuous and even smooth, (see (1)), this was proven by Sobolev in 1938 [12]. Later, embeddings ware examined on some more problematic classes of domains by V. G. Maz'ya [9, 10], O. V. Besov and V. P. Il'in [3], T. Kilpeläinen and J. Malý [5], D. A. Labutin [6, 7], B. V. Trushin [13, 14] and others. For further results and motivation we recommend the introduction by O. V. Besov [2].

For any domain Ω, it holds that $p \le q_\Omega(p) \le p^*(p)$. The "nicer" the domain Ω is, the greater the function $q_\Omega(p)$ is. The greatest possible values of the embedding exponent are $q_\Omega(p) = p^*(p)$. Even considering domains, which are irregular in some sense, the exponent $q_\Omega(p)$ has always been continuous and in most cases even smooth. We construct a domain Ω such that the function of the optimal embedding $q_\Omega(p)$ is continuous up to some point, jumps at this point and then it is continuous again. The point of discontinuity $p_0 \in [n, \infty)$ and the size of the jump can be chosen as desired.

Our work is inspired by the construction of a domain in [4], but our proof is completely different. The original article shows the construction of such a domain only in case $p_0 = n = 2$ and the proof is based on change of variables. We prove the same result by chaining Poincaré inequalities and we generalize the construction for the point of discontinuity anywhere in $[n, \infty)$. This result can be generalized to any dimension too, but for simplicity we show the calculations only in case $n = 2$.

An explicit example of a domain with the point of discontinuity under the point of dimension, i.e. $p_0 \in (1, n)$ would be of interest.

Our proof will be as follows. We choose a domain Ω and verify a given embedding. Then we continue the proof that the embedding is optimal by counterexamples.

1.1. Construction of $\Omega_{\alpha,\beta}$ and the embedding. Firstly, we construct a domain $\Omega_{\alpha,\beta} \subset \mathbb{R}^2$ for parameters $\alpha \ge 1, \beta > \alpha$. The point of discontinuity of $q_{\Omega_{\alpha,\beta}}(p)$ is $p_0 = \alpha + 1$, parameter β determinates the size of the jump $\lim_{t\to p_0+} q_{\Omega_{\alpha,\beta}}(t) - \lim_{t\to p_0-} q_{\Omega_{\alpha,\beta}}(t)$.

Let us denote by T_i the family of domains in $\mathbb{R}^2$

$$T_i := \big\{[x_1, x_2] \in \mathbb{R}^2 : \ x_1 \in \big(-2^{-i^2} i^{-1}, (-2^{-i^2} + 2^{-i}) i^{-1}\big), \\ x_2 \in \big(2^{-i+1}, 2^{-i+1} + (x_1 + 2^{-i^2} i^{-1})^\alpha 2^{-i(\beta-\alpha)} i^{-1+\alpha}\big)\big\}. \tag{3}$$

The shape of T_i is the sub-graph of $y(x) = x^\alpha$ function on some right neighbourhood of 0. By S we denote $S := (-4, 0) \times (-2, 2)$. Now we define

$$\Omega_{\alpha,\beta} := \bigcup_{i\in\mathbb{N}} T_i \cup S.$$

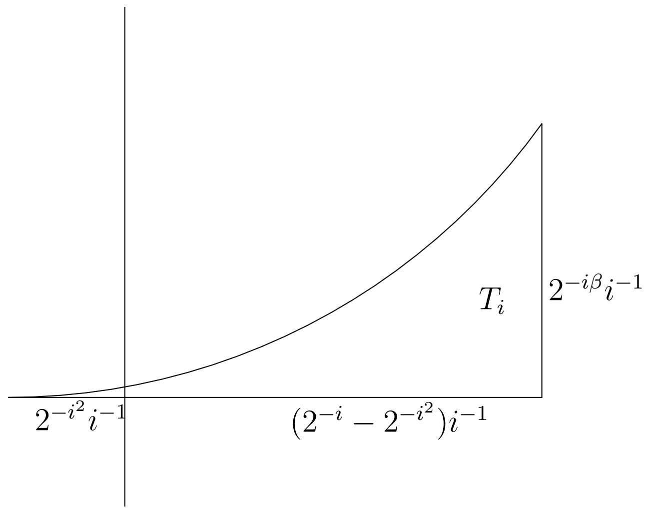

Figure 1: The domain T_i

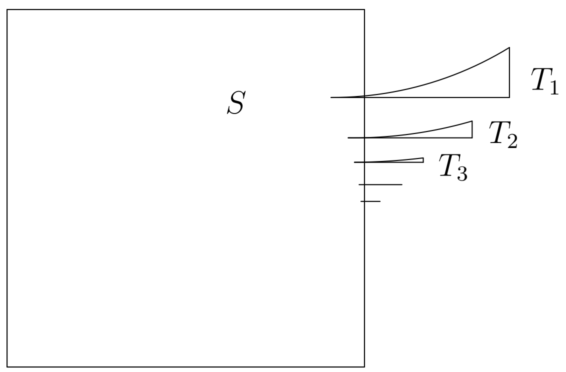

Figure 2: The domain $\Omega_{\alpha,\beta}$

We define $q_{\Omega_{\alpha,\beta}}(p) : [1, \beta+1) \to [1, \infty)$ by

$$q_{\Omega_{\alpha,\beta}}(p) := \begin{cases} p & \text{for } 1 \le p < \alpha+1, \\ \frac{(\beta+1)p}{\beta+1-p} & \text{for } \alpha+1 \le p < \beta+1. \end{cases}$$

The function $q_{\Omega_{\alpha,\beta}}(p)$ has a jump at $p_0 = \alpha+1$ of size

$$\lim_{t\to p_0+} q_{\Omega_{\alpha,\beta}}(t) - \lim_{t\to p_0-} q_{\Omega_{\alpha,\beta}}(t) = \frac{(\alpha+1)^2}{\beta-\alpha}.$$

Theorem 1.1 (Optimal Sobolev embedding Theorem for $\Omega_{\alpha,\beta}$)**.** *Let* $\alpha \ge 1$, $\beta > \alpha$ *and* $1 \le p < 1+\beta$, $p \ne \alpha+1$. *Then*

$$W^{1,p}(\Omega_{\alpha,\beta}) \subset L^{q_{\Omega_{\alpha,\beta}}(p)}(\Omega_{\alpha,\beta}).$$

Moreover, for every $q(p) > q_{\Omega_{\alpha,\beta}}(p)$ *there exists a function* $g : \Omega_{\alpha,\beta} \to \mathbb{R}$ *satisfying*

$$g \in W^{1,p}(\Omega_{\alpha,\beta}) \text{ and } g \notin L^{q(p)}(\Omega_{\alpha,\beta}).$$

We prove the first part of Theorem 1.1 in Section 3. The optimality part of Theorem 1.1 is proven in Section 4.

Remark 1.2 (Embedding for $p = \alpha + 1$)**.** We do not formulate previous Theorem for case $p = \alpha + 1$. However, if we use supremum definition (2) instead of $q(p) = \max\{r : r > 1, W^{1,p} \subset L^r\}$, then the Theorem would be valid even in case $p = \alpha + 1$ and it can be proven by the same means described in Section 3.

We decided to exclude the case $p = \alpha + 1$, so we show that the maximum and the supremum definition of $q(p)$ are equivalent for all cases $p \neq \alpha + 1$. The discontinuity of $q(p)$ is clear irrespective of precise value at this point and of the choice of the maximum or the supremum. We do not answer the question if $L^{q_{\Omega_{\alpha,\beta}}(\alpha+1)} \subset W^{1,\alpha+1}$ holds, we suppose that the answer is no.

2. Preliminaries

For simplicity we use the notation $\Omega = \Omega_{\alpha,\beta}$ and $q(p) = q_{\Omega_{\alpha,\beta}}(p)$. By C we denote a generic positive constant whose exact value may change at each occurrence. We write for example $C(a, b, c)$ if C may depend on parameters a, b and c.

We use standard notation for weak derivatives and Lebesgue and Sobolev spaces. We denote the Sobolev norm $\|f\|_{W^{1,p}(\Omega)}$ for the function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}, p \in [1, \infty]$ as

$$\|f\|_{W^{1,p}(\Omega)} = \begin{cases} \left(\|f\|^p_{L^p(\Omega)} + \sum_{i=1}^n \|D_i f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \text{ for } p \in [1, \infty) \\ \max\{\|f\|_{L^p(\Omega)}, \|D_1 f\|_{L^p(\Omega)}, \dots \|D_n f\|_{L^p(\Omega)}\} \text{ for } p = \infty. \end{cases} \tag{4}$$

We denote the Sobolev space $W^{1,p}(\Omega)$ as the set of all functions with finite norm $\|f\|_{W^{1,p}(\Omega)}$.

We use the notation $a_i \simeq b_i$, if there exists a constant $K > 0$ such that

$$\frac{1}{K} < \frac{a_i}{b_i} < K \text{ for every } i \in \mathbb{N}.$$

We denote the integral average by

$$f_A := \fint_A f = \frac{1}{|A|} \int_A f.$$

The following Poincaré-type inequality will be essential.

Lemma 2.1. *Let $b : B(0,r) \subset \mathbb{R}^n \to \mathbb{R}^n$ be a bi-Lipschitz mapping with a bi-Lipschitz constant $L > 0$, and $A = b(B(0,r))$. Let $1 \le p \le \infty$, $p \ne n$ and $1 \le m \le p^*(p)$. Then there exists a constant $C(n,p,m,L)$ such that for $f \in W^{1,p}(A)$ we have*

$$|A|^{-\frac{1}{m}} \|f - f_A\|_{L^m(A)} \le C(n,p,m,L) r |A|^{-\frac{1}{p}} \|Df\|_{L^p(A)}.$$

We use the convention $|A|^{-\frac{1}{\infty}} = 1$.

Let $p = n$ and $1 \le m < \infty$. Then there exists a constant $C(n,m,L)$, such that for $f \in W^{1,p}(A)$ it holds

$$|A|^{-\frac{1}{m}} \|f - f_A\|_{L^m(A)} \le C(n,m,L) r |A|^{-\frac{1}{n}} \|Df\|_{L^n(A)}.$$

Proof. In case b is the identity mapping and $p = q$ we get the classical result. The more difficult case $1 \le q \le p^*(p)$ follows from [8] as Theorem 12.23 and Exercise 12.24 and by applying Hölder's inequality. The general case where b is not the identity follows from a simple change of variables.

3. The proof of Sobolev embedding Theorem for $\Omega_{\alpha,\beta}$

In this section we prove the embedding part of Theorem 1.1 for the case $\alpha \ge 1$. We give the details for $\alpha > 1$ and the case $\alpha = 1$ is only sketched.

Let us suppose that $\alpha > 1$. Then for every $i \in \mathbb{N}$ we define the covering of $T_i \setminus S$ by domains, which are bi-Lipschitz equivalent to balls. The proof of $W^{1,p} \subset L^{q(p)}$ for $p < \alpha + 1$ is elementary and follows from (4), as every function in $W^{1,p}$ belongs to L^p. Further we suppose that $\beta + 1 > p > \alpha + 1$.

3.1. Covering of T_i. We define $k_\alpha = \frac{3}{2}\left(\frac{2}{\alpha-1}\right)^{\frac{1}{\alpha-1}}$,

$$\begin{aligned} s_{i,j} :=& k_\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j^{-\frac{1}{\alpha-1}}, \\ r_{i,j} :=& \frac{1}{2} k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}} i^{-1} j^{-\frac{\alpha}{\alpha-1}}. \end{aligned} \tag{5}$$

For fixed $i \in \mathbb{N}$ we define the sequence of domains $Q_{i,j}$, $j \in \mathbb{N}$

$$\begin{aligned} Q_{i,j} :=& \big\{ [x_1, x_2] \in T_i : \\ & x_1 + 2^{-i^2} i^{-1} \in \big(s_{i,j} - r_{i,j}, s_{i,j} + r_{i,j}\big) \cap \big(-2^{-i^2} i^{-1}, (-2^{-i^2} + 2^{-i}) i^{-1}\big) \big\}. \end{aligned} \tag{6}$$

Lemma 3.1 (Covering lemma)**.** *Let $i \in \mathbb{N}$, T_i be given by (3) and the sequence of domains $Q_{i,j}$ by (6). Then*

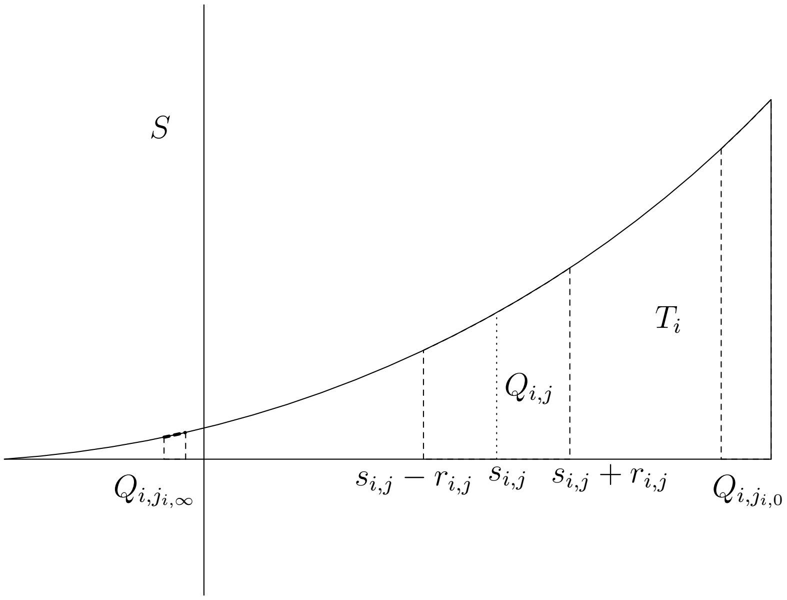

Figure 3: The covering of T_i

(i) *$Q_{i,j}$ are bi-Lipschitz equivalent to balls with radius $r_{i,j}$ with the same bi-Lipschitz constant L independent of i and j.*

(ii) *For fixed j_0 there exists only a finite number of domains $Q_{i,j}$ with non-empty intersection with Q_{i,j_0}. This number is bounded by some constant $C(\alpha,\beta)$.*

(iii) *For fixed j_0 let $A_{i,j_0} := Q_{i,j_0} \cap Q_{i,j_0+1}$. There exists some positive constant $C(\alpha,\beta)$ such that $C(\alpha,\beta) < \frac{|A_{i,j_0}|}{|Q_{i,j_0}|} < \frac{|A_{i,j_0}|}{|Q_{i,j_0+1}|}$.*

(iv) *There exists a smallest index $j_{i,\infty}$ satisfying $Q_{i,j_{i,\infty}} \subset S$, and there exists a biggest index $j_{i,0}$ satisfying $s_{i,j_{i,0}} + r_{i,j_{i,0}} \geq (-2^{-i^2} + 2^{-i})i^{-1} =$ "height of T_i". Estimated values are*

$$
\begin{aligned}
j_{i,\infty} &\simeq 2^{i(\beta-\alpha)+i^2(\alpha-1)},\\
j_{i,0} &\simeq 2^{i(\beta-1)}.
\end{aligned}
\tag{7}
$$

The proof is rather technical but straightforward and can be done by basic calculus, therefore we only outline it.

Sketch of the proof of Lemma 3.1. We define two bi-Lipschitz mappings:

$$
\begin{aligned}
b_{1,i,j} :& B(0,r_{i,j}) \to (-r_{i,j}, r_{i,j}) \times (0, r_{i,j}),\\
b_{2,i,j} :& (-r_{i,j}, r_{i,j}) \times (0, r_{i,j}) \to Q_{i,j},\\
b_{2,i,j}(x_1,x_2) :=& \big(x_1 + s_{i,j}, 2^{-i+1} + \frac{x_2}{r_{i,j}}(x_1 + s_{i,j} + 2^{-i^2} i^{-1})^{\alpha} 2^{-i(\beta-\alpha)} i^{-1+\alpha}\big).
\end{aligned}
$$

The mapping $b_{1,i,j}$ maps a ball to half a square and has bi-Lipschitz constant L_1 independent of i and j, its exact formula can be found easily. Let us consider Jacobi matrices of both $b_{2,i,j}$ and $b_{2,i,j}^{-1}$

$$D_{b_{2,i,j}}(x_1,x_2) = \begin{pmatrix} 1 & \frac{x_2}{r_{i,j}}\alpha(x_1+s_{i,j}+2^{-i^2}i^{-1})^{\alpha-1}2^{-i(\beta-\alpha)}i^{-1+\alpha} \\ 0 & r_{i,j}^{-1}(x_1+s_{i,j}+2^{-i^2}i^{-1})^{\alpha}2^{-i(\beta-\alpha)}i^{-1+\alpha} \end{pmatrix}$$
$$D_{b_{2,i,j}^{-1}}(b_{2,i,j}(x_1,x_2)) = \begin{pmatrix} 1 & -x_2\alpha(x_1+s_{i,j}+2^{-i^2}i^{-1})^{-1} \\ 0 & r_{i,j}(x_1+s_{i,j}+2^{-i^2}i^{-1})^{-\alpha}2^{i(\beta-\alpha)}i^{1-\alpha} \end{pmatrix}.$$

By a direct computation it is not difficult to check that all partial derivatives are bounded by a constant, i.e. the second mapping $b_{2,i,j}$ has bi-Lipschitz constant $L_{2,i,j}$ dependent on i and j, and it can be estimated by an same L_2 common for all i and j. The key observation is, that $L_{2,i,j}$ is a monotone sequence in both i and j. We have found a bi-Lipschitz mappings $b_{2,i,j}\circ b_{1,i,j} : B(0,r_{i,j}) \to Q_{i,j}$ with constant $L = L_1L_2$ and the first part is proven.

The second part can be proven by verifying the statement $\lim_{j\to\infty} s_{i,j} - s_{i,j+1} - r_{i,j} = 0$ for every $i \in \mathbb{N}$.

To prove the third part we define $P_{i,j} \subset A_{i,j}$,

$$P_{i,j} := (s_{i,j+1}, s_{i,j}) \times (2^{-i+1}, 2^{-i+1} + r_{i,j+1}).$$

We estimate $\frac{|P_{i,j}|}{|Q_{i,j}|}$ and we easily find $C(\alpha,\beta)$ such that $C(\alpha,\beta) < \frac{|P_{i,j}|}{|Q_{i,j}|}$.

The fourth part is important for further calculations. We estimate the indices $j_{i,0}$ and $j_{i,\infty}$ by the definition of $r_{i,j}$ (5). We observe that $\operatorname{diam}(Q_{i,j_{i,\infty}}) \simeq$ "height of $T_i \setminus S$ on the left edge" and $\operatorname{diam}(Q_{i,j_{i,\infty}}) \simeq r_{i,j_{i,\infty}}$ for $j_{i,\infty}$ and that $\operatorname{diam}(Q_{i,j_{i,0}}) \simeq$ "height of $T_i \setminus S$ on the right edge" and $\operatorname{diam}(Q_{i,j_{i,0}}) \simeq r_{i,j_{i,0}}$ for $j_{i,0}$. By this observation we have

$$\begin{aligned} 2^{-i\beta}i^{-1} \simeq r_{i,j_{i,0}} &\simeq \frac{1}{2}k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}}i^{-1}j_{i,0}^{-\frac{\alpha}{\alpha-1}}, \\ (2^{-i^2}i^{-1})^\alpha 2^{-i(\beta-\alpha)}i^{-1+\alpha} \simeq r_{i,j_{i,\infty}} &\simeq \frac{1}{2}k_\alpha^\alpha 2^{i\frac{\beta-\alpha}{\alpha-1}}i^{-1}j_{i,\infty}^{-\frac{\alpha}{\alpha-1}} \end{aligned} \tag{8}$$

which implies (7).

3.2. Proof of Theorem 1.1 for $\beta+1 > p > \alpha+1$, $\alpha \geq 1$.

Proof. We denote $q = q(p) = q_{\Omega_{\alpha,\beta}}(p)$. We estimate the power of the norm

$$\|f\|^q_{L^q(\Omega)} \leq \|f\|^q_{L^q(S)} + \sum_{i\in\mathbb{N}} \|f\|^q_{L^q(T_i\setminus S)} \leq \|f\|^q_{L^q(S)} + \sum_{i\in\mathbb{N}}\sum_{j=j_{i,0}}^{j_{i,\infty}} \|f\|^q_{L^q(Q_{i,j})}.$$

The part $\|f\|^q_{L^q(S)}$ is bounded for any $q \in [1,\infty)$ thanks to Sobolev embedding theorem for Lipschitz domains $W^{1,p}(S) \subset L^\infty(S), p > n = 2$. Therefore we have $\|f\|^q_{L^q(S)} \leq C$ and also $|f_{Q_{i,j_{i\infty}}}| < C$. We estimate

$$
\begin{aligned}
\|f\|^q_{L^q(\Omega)} \leq & \|f\|^q_{L^q(S)} \\
& + \sum_{i=1}^{\infty} \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(|f(x) - f_{Q_{i,j}}| + \sum_{k=j+1}^{j_{i,\infty}} |f_{Q_{i,k}} - f_{Q_{i,k-1}}| + C \Big)^q dx \\
\leq & C + C \sum_{i=1}^{\infty} \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \sum_{k=j+1}^{j_{i,\infty}} \Big(|f_{Q_{i,k}} - f_{Q_{i,k-1}}| \Big)^q dx \\
& + C \sum_{i=1}^{\infty} \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(|f(x) - f_{Q_{i,j}}| \Big)^q dx.
\end{aligned} \tag{9}
$$

By (3), Lemma 2.1 for $m = \infty$, Lemma 3.1 (ii) and $r_{i,j} \leq 1$ we have

$$
\begin{aligned}
\sum_{i=1}^{\infty} \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(|f(x) - f_{Q_{i,j}}| \Big)^q dx \leq & \sum_{i=1}^{\infty} \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(\|f(x) - f_{Q_{i,j}}\|_{L^\infty(Q_{i,j})} \Big)^q \\
\leq & \sum_{i=1}^{\infty} \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(C r_{i,j}^{\frac{p-2}{p}} \|Df\|_{W^{1,p}(Q_{i,j})} \Big)^q \\
\leq & C \sum_{i=1}^{\infty} |T_i| \|Df\|^q_{W^{1,p}(\Omega)} \leq C
\end{aligned} \tag{10}
$$

By Lemma 2.1 and Lemma 3.1 (iii) we have the following estimate

$$
\begin{aligned}
|f_{Q_{i,j}} - f_{Q_{i,j-1}}| \leq & \Big(\fint_{A_{i,j-1}} |f_{Q_{i,j}} - f(y)| dy + \fint_{A_{i,j-1}} |f_{Q_{i,j-1}} - f(y)| dy \Big) \\
\leq & C \Big(\fint_{Q_{i,j}} |f_{Q_{i,j}} - f(y)| dy + \fint_{Q_{i,j-1}} |f_{Q_{i,j-1}} - f(y)| dy \Big) \\
\leq & C \Big(r_{i,j}^{\frac{p-2}{p}} \Big(\int_{Q_{i,j}} |Df(y)|^p dy \Big)^{\frac{1}{p}} + r_{i,j-1}^{\frac{p-2}{p}} \Big(\int_{Q_{i,j-1}} |Df(y)|^p dy \Big)^{\frac{1}{p}} \Big).
\end{aligned} \tag{11}
$$

By this estimate and the Hölder inequality for sums and Lemma 3.1 (ii) we get

$$\begin{aligned}
&\sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(\sum_{k=j+1}^{j_{i,\infty}} |f_{Q_{i,k}} - f_{Q_{i,k-1}}|\Big)^q \\
&\qquad \le \sum_{j=j_{i,0}}^{j_{i,\infty}} \int_{Q_{i,j}} \Big(\sum_{k=j}^{j_{i,\infty}} r_{i,k}^{\frac{p-2}{p}} \Big(\int_{Q_{i,k}} |Df(y)|^p dy \Big)^{\frac{1}{p}} \Big)^q dx \\
&\qquad \le C|T_i| \Big(\sum_{k=j_{i,0}}^{j_{i,\infty}} r_{i,k}^{\frac{p-2}{p}} \Big(\int_{Q_{i,k}} |Df(y)|^p dy \Big)^{\frac{1}{p}} \Big)^q dx \\
&\qquad \le C|T_i| \Big(\sum_{k=j_{i,0}}^{j_{i,\infty}} r_{i,k}^{\frac{(p-2)p}{p(p-1)}} \Big)^{\frac{q(p-1)}{p}} \Big(\sum_{k=j_{i,0}}^{j_{i,\infty}} \Big(\int_{Q_{i,k}} |Df(y)|^p dy \Big)^{p\frac{1}{p}} \Big)^{\frac{q}{p}} dx \\
&\qquad \le C|T_i| \Big(\sum_{k=j_{i,0}}^{j_{i,\infty}} r_{i,k}^{\frac{p-2}{p-1}} \Big)^{\frac{q(p-1)}{p}}.
\end{aligned} \tag{12}$$

From (5), (9), (10), (12) and (3) we have

$$\begin{aligned}
\|f\|^q_{L^q(\Omega)} &\le C + C \sum_{i=1}^{\infty} i^{-2} 2^{-i(\beta+1)} \Big(\sum_{j=j_{i,0}}^{j_{i,\infty}} r_{i,j}^{\frac{p-2}{p-1}} \Big)^{\frac{q(p-1)}{p}} \\
&\le C + C \sum_{i=1}^{\infty} i^{-\frac{qp-2q+2p}{p}} 2^{i\left(\frac{(\beta-\alpha)q(p-2)}{(\alpha-1)p} - (\beta+1)\right)} \Big(\sum_{j=j_{i,0}}^{j_{i,\infty}} j^{-\frac{\alpha(p-2)}{(\alpha-1)(p-1)}} \Big)^{\frac{q(p-1)}{p}}.
\end{aligned}$$

We estimate the sum over j and by $p > \alpha + 1$ and (7) we get

$$\sum_{j=j_{i,0}}^{j_{i,\infty}} j^{-\frac{\alpha(p-2)}{(\alpha-1)(p-1)}} \le C 2^{i\frac{(\beta-1)(\alpha+1-p)}{(\alpha-1)(p-1)}}. \tag{13}$$

Finally we put the estimates together and we get

$$\begin{aligned}
\|f\|^q_{L^q(\Omega)} &\le C + C \sum_{i=1}^{\infty} i^{-\frac{qp-2q+2p}{p}} 2^{i\left(\frac{(\beta-\alpha)q(p-2)}{(\alpha-1)p} - (\beta+1)\right)} \Big(2^{i\frac{(\beta-1)(\alpha+1-p)}{(\alpha-1)(p-1)}} \Big)^{\frac{q(p-1)}{p}} \\
&\le C + C \sum_{i=1}^{\infty} i^{-\frac{qp-2q+2p}{p}} 2^{i\left(-(\beta+1) + q\frac{\beta+1-p}{p}\right)}.
\end{aligned}$$

The proof is complete, because the sum is finite if $q \le \frac{(\beta+1)p}{\beta+1-p}$.

Let us consider the case $\alpha = 1$. We have to change the definition (5) of $s_{i,j}$ and $r_{i,j}$ and the definition (6) of $Q_{i,j}$ as follows,

$$r_{i,j} := r_{i,0}(1 + 2^{-i(\beta-1)-1})^j, \text{ for } r_{i,0} = 2^{-i^2 - i(\beta-1)-1} i^{-1} \text{ and } s_{i,j} := \sum_{k=0}^{j-1} r_{i,j}. \tag{14}$$

We define $Q_{i,j}$ as trapezoids which have the average of basis equal to their width. We denote half of its width by $r_{i,j}$, that is

$$Q_{i,j} = T_i \cap \{x \in \mathbb{R}^2 : x_2 \in (s_{i,j} - r_{i,j}, s_{i,j} + r_{i,j})\}.$$

Let us denote, that the sequences $r_{i,j}$ and $s_{i,j}$ are strictly decreasing with

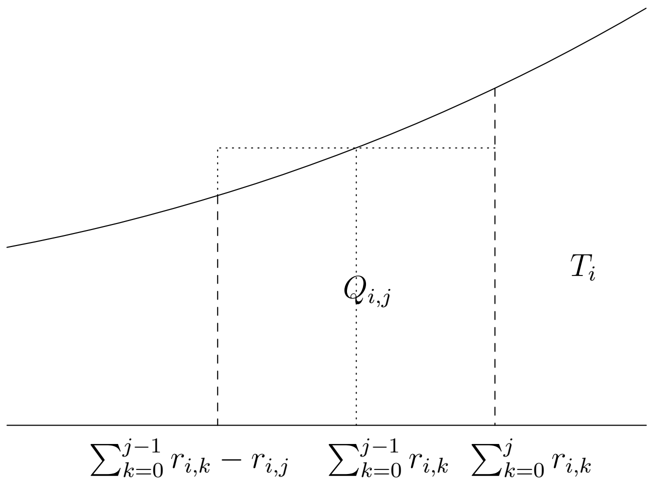

Figure 4: The domain $Q_{i,j}$

respect to index j in case $\alpha > 1$, but these sequences are strictly increasing in case $\alpha = 1$.

The Lemma 3.1 holds and is proven in the same way as for $\alpha > 1$, only the indices of border $Q_{i,j}$ are

$$j_{i,\infty} = -1$$

and analogously to (8)

$$2^{-i\beta} i^{-1} \simeq r_{i,j_{i,0}} = (1 + 2^{-i(\beta-1)-1})^{j_{i,0}} 2^{-i^2 - i(\beta-1)-1} i^{-1}$$

we get

$$j_{i,0} \simeq \frac{\ln(2)(i^2 - i)}{\ln(1 + 2^{-i(\beta-1)-1})}.$$

The idea of chaining the Poinceré inequality is analogous, and after an easy modification we get our result. We can copy all arguments and calculations from (9), (10), (12), then we use (3) for $\alpha = 1$, the new definition of $r_{i,j}$ and the previous estimates for $j_{i,0}, j_{i,\infty}$ and we get

$$\begin{aligned}\|f\|^q_{L^q(\Omega)} \leq & C + C\sum_{i=1}^{\infty} i^{-2} 2^{-i(\beta+1)} \Big(\sum_{j=j_{i,\infty}}^{j_{i,0}} r_{i,j}^{\frac{p-2}{p-1}} \Big)^{\frac{q(p-1)}{p}} \\ \leq & C + C\sum_{i=1}^{\infty} i^{-\frac{2p+q(p-2)}{p}} 2^{-i(1+\beta)+(-i^2-i(\beta-1))\frac{q(p-2)}{p}} \\ & \Big(\frac{(1+2^{-i(\beta-1)-1})^{\frac{p-2}{p-1}(\frac{\ln(2)(i^2-i)}{\ln(1+2^{-i(\beta-1)-1})}+2)} - 1}{(1+2^{-i(\beta-1)-1})^{\frac{p-2}{p-1}} - 1} \Big)^{\frac{q(p-1)}{p}},\end{aligned}$$

where the final term comes from the sum of geometric series. The right hand side can be estimated and after an easy calculation we have

$$\|f\|^q_{L^q(\Omega)} \leq C + C\sum_{i=1}^{\infty} i^{-\frac{2p+q(p-2)}{p}} 2^{i\frac{q(\beta+1-p)-(\beta+1)p}{p}}.$$

The right hand side is finite if $q \leq \frac{p(\beta+1)}{\beta+1-p}$ and the proof is complete.

The complete proof for $\alpha = 1$ with all details can be found in [11].

4. Optimality of $q(p)$ for $\Omega_{\alpha,\beta}$

Proof of the optimality. We fix $q > q(p)$. We define a function g by the choice of proper functions $g_i : T_i \to \mathbb{R}$ and the sequence d_i of positive numbers. We define

$$g(x_1,x_2) = \begin{cases} 0 & \text{for } (x_1,x_2) \in S, \\ d_i g_i(x_1,x_2) & \text{for } (x_1,x_2) \in T_i \setminus S, \forall i \in \mathbb{N}. \end{cases}$$

Clearly

$$\|g\|^p_{W^{1,p}(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{W^{1,p}(T_i)} \text{ and } \|g\|^q_{L^q(\Omega)} = \sum_{i=1}^{\infty} d_i^q \|g_i\|^q_{L^q(T_i)}. \tag{15}$$

The choice of g_i and d_i depends on p and $\alpha+1$, so we split the proof into two parts.

4.1. The case $p < \alpha + 1$.

Let us consider $p \in [1, \alpha+1)$. We define

$$\begin{aligned} g_i(x_1,x_2) &:= \Big(x_1 + 2^{-i^2} i^{-1}\Big)^{-\alpha} - (2^{-i^2} i^{-1})^{-\alpha} \text{ for } (x_1,x_2) \in T_i \setminus S, \\ d_i &:= (2^{-i^2} i^{-1})^{\alpha} 2^{\frac{i(\beta+1)}{q}} i^{\frac{2}{q}}. \end{aligned} \tag{16}$$

For fixed $i \in \mathbb{N}$ we estimate the norm in the space $L^q(T_i)$. By (3) the height of T_i for $x_1 \in \big(-i^{-1}2^{-i^2}, i^{-1}(2^{-i}-2^{-i^2})\big)$ is

$$l(x_1) = \Big(x_1 + 2^{-i^2}i^{-1}\Big)^{\alpha} 2^{i\alpha - i\beta} i^{\alpha-1} \tag{17}$$

and we get

$$\begin{aligned} \|g_i\|^q_{L^q(T_i)} &= \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \Big|\big(x_1 + 2^{-i^2}i^{-1}\big)^{-\alpha} - (2^{-i^2}i^{-1})^{-\alpha}\Big|^q l(x_1)dx_1 \\ &= \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \Big|\big(x_1 + 2^{-i^2}i^{-1}\big)^{-\alpha} - (2^{-i^2}i^{-1})^{-\alpha}\Big|^q \big(x_1 + 2^{-i^2}i^{-1}\big)^{\alpha} 2^{i\alpha-i\beta} i^{\alpha-1} dx_1. \end{aligned} \tag{18}$$

We want to estimate the size of the integral. The integrand is positive and concave, so we can estimate its value by its maximum. More precisely, we replace the integrand by constant function $(2^{-i^2}i^{-1})^{-\alpha q}\big(2^{-i}i^{-1}\big)^{\alpha}2^{i\alpha-i\beta}i^{\alpha-1}$, hence

$$\begin{aligned} \|g_i\|^q_{L^q(T_i)} &\simeq \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} (2^{-i^2}i^{-1})^{-\alpha q}\big(2^{-i}i^{-1}\big)^{\alpha}2^{i\alpha-i\beta}i^{\alpha-1}dx_1 \\ &\simeq i^{\alpha q-2}2^{i^2\alpha q-i-i\beta}. \end{aligned} \tag{19}$$

By (16) we get

$$\|g\|^q_{L^q(\Omega)} = \sum_{i=1}^{\infty} d_i^q \|g_i\|^q_{L^q(T_i)} \geq C \sum_{i=1}^{\infty} i^0 2^0 = \infty.$$

We need to prove the convergence of $\|g\|^p_{W^{1,p}(\Omega_\alpha)}$. First of all we estimate

$$\|g_i\|^p_{W^{1,p}(T_i)} \leq 2\max\{\|g_i\|^p_{L^p(T_i)}, \|Dg_i\|^p_{L^p(T_i)}\}.$$

The estimate of the norm of g_i in $L^p(T_i)$ is analogous to (19), by replacing p by q we get

$$\|g_i\|^p_{L^p(T_i)} \simeq i^{\alpha p-2}2^{i^2\alpha p-i-i\beta}.$$

We use $q > q(p) = p$ and we estimate the norm of g in $L^p(\Omega)$

$$\|g\|^p_{L^p(\Omega)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{L^p(T_i)} \leq C \sum_{i=1}^{\infty} i^{-2}(2^{-i-i\beta})^{\frac{q-p}{q}} < \infty.$$

Let us express the norm of g_i by a derivative

$$\|Dg_i(x_1,x_2)\|^p_{L^p(T_i)} = \int_{T_i} \left|\frac{\partial g_i(x_1,x_2)}{\partial x_1}\right|^p.$$

The estimate is similar to (18). The proof splits in two cases. Firstly, we consider $p > 1$ and we get

$$\begin{aligned}\int_{T_i} \left|\frac{\partial g_i}{\partial x_1}\right|^p \leq & C \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \left(x_1 + 2^{-i^2} i^{-1}\right)^{\alpha+p(-\alpha-1)} 2^{i(-\beta+\alpha)} i^{\alpha-1} dx_1 \\ \leq & C 2^{i(-\beta+\alpha)} i^{\alpha-1} \left[\left(x_1 + 2^{-i^2} i^{-1}\right)^{(1-p)(\alpha+1)}\right]_0^{i^{-1}(2^{-i}-2^{-i^2})} \\ \leq & C 2^{i(-\beta+\alpha)} i^{\alpha-1} (2^{-i^2} i^{-1})^{(1-p)(\alpha+1)}.\end{aligned} \tag{20}$$

It follows that

$$\begin{aligned}\|Dg\|^p_{L^p(\Omega)} = & \sum_{i=1}^{\infty} d_i^p \|Dg_i\|^p_{L^p(T_i)} \leq \sum_{i=1}^{\infty} C i^{2\frac{p-q}{q}+p(\alpha+1)} 2^{i(-\beta+\alpha+\frac{(\beta+1)p}{q})} 2^{i^2(p-\alpha-1)} \\ & < \infty.\end{aligned}$$

The proof of the finiteness of the norm in case $p = 1$ is similar, except the estimate in (20) involves $\int (x_1 + C)^{-1} = \log|x_1 + C|$. It is easy to finish the proof in this case too.

4.2. The case $p > \alpha + 1$. We define

$$\begin{aligned}g_i(x_1, x_2) := & \left(x_1 + 2^{-i^2} i^{-1}\right)^{\alpha} - (2^{-i^2} i^{-1})^{\alpha} \text{ for } (x_1, x_2) \in T_i \setminus S, \\ d_i := & 2^{i\left(\frac{\beta+1}{q}+\alpha\right)} i^{\alpha+\frac{2}{q}}.\end{aligned} \tag{21}$$

We use (15), (17) and we estimate the norms of g_i as in the previous case. Analogously to (18) and (19) we have

$$\begin{aligned}\|g_i\|^q_{L^q(T_i)} = & \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \left|\left(x_1 + 2^{-i^2} i^{-1}\right)^{\alpha} - (2^{-i^2} i^{-1})^{\alpha}\right|^q l(x_1) dx_1 \\ \simeq & \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \left(x_1 + 2^{-i^2} i^{-1}\right)^{(q+1)\alpha} 2^{i\alpha-i\beta} i^{\alpha-1} dx_1 \\ \simeq & \, i^{-\alpha q-2} 2^{-i(q\alpha+1+\beta)}.\end{aligned} \tag{22}$$

We estimate $\|g\|^q_{L^q(\Omega)}$ by

$$\|g\|^q_{L^q(\Omega)} = \sum_{i=1}^{\infty} d_i^q \|g_i\|^q_{L^q(T_i)} \geq C \sum_{i=1}^{\infty} i^0 2^0 = \infty.$$

Now we need to prove the convergence of the norms of g and Dg in $L^p(\Omega)$. Analogously to (22), by replacing the position of q by p we get

$$\|g_i\|^p_{L^p(T_i)} \simeq i^{-\alpha p-2} 2^{-i(p\alpha+1+\beta)}.$$

We use $q > q(p) > p$ and we estimate the norm of g in $L^p(\Omega)$

$$\|g\|^p_{L^p(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|g_i\|^p_{L^p(T_i)} \le C \sum_{i=1}^{\infty} (i^{-2} 2^{-i-i\beta})^{\frac{q-p}{q}} < \infty.$$

Let us express the norm of g_i by a derivative and we estimate

$$\begin{aligned}\int_{T_i} \Big|\frac{\partial g_i}{\partial x_1}\Big|^p \le & C \int_0^{\frac{2^{-i}-2^{-i^2}}{i}} \big(x_1 + 2^{-i^2} i^{-1}\big)^{(\alpha-1)p+\alpha} 2^{i(-\beta+\alpha)} i^{\alpha-1} dx_1 \\ \le & C 2^{i(-\beta+\alpha)} i^{\alpha-1} \Big[\big(x_1 + 2^{-i^2} i^{-1}\big)^{\alpha p - p + \alpha + 1}\Big]_0^{\frac{2^{-i}-2^{-i^2}}{i}} \\ \le & C i^{p(-\alpha+1)-2} 2^{i(-\alpha p + p - \beta - 1)}.\end{aligned}$$

It follows that

$$\|Dg\|^p_{L^p(\Omega_\alpha)} = \sum_{i=1}^{\infty} d_i^p \|Dg_i\|^p_{L^p(T_i)} \le \sum_{i=1}^{\infty} C i^{p-2\frac{q-p}{q}} 2^{i\left(p+(\beta+1)\frac{p-q}{q}\right)} < \infty,$$

where the finiteness follows from $q > q(p) = \frac{(\beta+1)p}{\beta+1-p}$.

Acknowledgement. We would like to thank to author's supervisor Stanislav Hencl for introducing this field, pointing out this problem and supporting and useful advices during work itself. We would like to thank to referees for careful reading and valuable comments.

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Received April 1, 2012; revised January 31, 2013

2. Construction of $W^{2,n}(\Omega)$ function with gradient violating Lusin (N) condition

Mathematische Nachrichten, 24 March 2017

Construction of $W^{2,n}(\Omega)$ function with gradient violating Lusin (N) condition

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Received XXXX, revised XXXX, accepted XXXX
Published online XXXX

Key words Sobolev space, Lusin condition, Peano curve
Subject classification 46E35

We construct $f \in C^1([-1,1]^n)$, such that $Df \in W^{1,n}((-1,1)^n, \mathbb{R}^n)$ maps $[-1,1] \times \{0\}^{n-1}$ onto $[-1,1]^n$. This shows that a mapping which maps a set of zero measure onto the set of positive measure can be a gradient mapping.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that the mapping $f\colon \Omega \to \mathbb{R}^n$ satisfies Lusin (N) condition if for every $A \subset \Omega, |A| = 0$, it holds $|f(A)| = 0$. We can see that this condition is needed for any natural physical model such as the deformation of a solid body in space. Otherwise we can make new material "from nothing", so the mapping is really unnatural for physical applications. Another important application is the connection between the area formula and the Lusin condition. If the mapping is a Sobolev mapping and satisfies the Lusin condition, the area formula holds, for more see [8].

Our work is focused on the so called Cesari example originally written in [2] for $n = 2$, later reminded and improved in [6]. That is a mapping in the Sobolev space, which maps the line segment onto a domain with positive n-dimension measure, so the (N) condition is violated. It is well known, that for $p < n$ and a domain $\Omega \subset \mathbb{R}^n$ with positive measure we can construct such an example in $W^{1,p}(\Omega, \mathbb{R}^n)$ and for $p > n$ the Lusin (N) condition holds in $W^{1,p}(\Omega, \mathbb{R}^n)$, see [3] or the original article [7]. The limiting case $W^{1,n}$ is the most important case and we study the violation of the condition in this case. This Cesari example is a special case of the Peano curve, the mapping such that the image of some line segment has non-empty interior. For other results concerning the study of the (N) condition for spaces close to $W^{1,n}$ see [5] and [4].

In this article we improve the previous results. We show an example of $f \in W^{2,n}$ such that Df can be used in Cesari construction. So the restriction to gradient mappings does not guarantee the Lusin (N) condition.

For the convenience of the reader we include all details of the construction and we use figures to illustrate the idea of the construction.

Theorem 1.1 *There exists* $f \in C^1([-1,1]^n)$*, such that* $Df \in W^{1,n}((-1,1)^n, \mathbb{R}^n)$ *and* $Df([-1,1] \times \{0\}^{n-1}) \supseteq [-1,1]^n$.

Remark 1.2 We can improve the Theorem 1.1 and construct f such that $|D^2 f| \in L^n \log^p L$ for $p < n-1$. We explain how to modify our construction in this way in the end of the proof.

This problem was originally motivated by the recent research of Menne in the field of varifolds. The theory of weakly differentiable functions on rectifiable varifolds with locally bounded first variation is introduced in [11] and it is extended as the theory of Sobolev functions on varifolds, which is presented in [10]. These papers are basis of framework for this new theory, but some results were already formulated and proven in advance. The

The author was supported by the ERC CZ grant LL1203 of the Czech Ministry of Education.
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area formula for Gauss map was formulated in [9] as Theorem 3, our result can be used as a counterexample to some relaxation of this Theorem. It can be translated to the theory of varifolds as follows.

Remark 1.3 If m is a positive integer $m \geq 2$, then there exists a C^1 submanifold M of dimension m of R^{m+1} such that

1. M gives rise to an integral varifold with mean curvature in L^{loc}_m; in fact, a curvature varifold with second fundamental form in L^{loc}_m.
2. The continuous Gauss map $\nu : M \to S^m$ does not satisfy the Lusin condition with respect to m dimensional Hausdorff measure on both sides.

This shows that the assumption in [9, p. 2254, Theorem 3] cannot be relaxed, i.e. the condition that $B \subseteq C$ cannot be replaced by $B \subseteq C'$ for

$$C' = (S \times S^m) \cap \{(a, u) \colon u = \nu(a) \text{ or } u = -\nu(a)\}$$

even if V corresponds to a C^1 submanifold M.

2 Preliminaries

We denote by $B(x, r)$ an open ball with the radius r and the center x. We denote by the annulus a set

$$A(x, s, t) = B(x, t) \setminus \overline{B(x, s)}.$$

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$.

We say $g(x) = D_u f(x)$ for vector $u \in \mathbb{R}^n$ if there exist $s > 0$ and line segment $(x - su, x + su)$ such that

$$\int_{x-su}^{x+su} g(y)\phi(|y - x|)dy = -\int_{x-su}^{x+su} g(y)\phi'(|y - x|)dy \text{ for any } \phi \in C_c^\infty(-s, s).$$

We denote

$$Df(x) = \big(D_{e_1} f(x), D_{e_2} f(x), \dots D_{e_n} f(x)\big).$$

We use some basic properties of Sobolev functions, especially we use the equality between the weak derivative and the classical derivative if the classical derivative exists and we use the Sobolev embedding theorem. For more details see [1].

We use following notation for Sobolev spaces. For a domain $\Omega \subseteq \mathbb{R}^n$, $f \colon \Omega \to \mathbb{R}^n$ measurable we define

$$\begin{aligned} W^{2,p}(\Omega) &= \{f \colon \Omega \to \mathbb{R}; \int_\Omega |f|^p < \infty, \int_\Omega |Df|^p < \infty, \int_\Omega |D^2 f|^p < \infty\}, \\ \|f\|_{2,p} &= \Big(\int_\Omega |f|^p + \sum_{i=1}^n \int_\Omega |D_i f|^p + \sum_{i,j=1}^n \int_\Omega |D_{i,j} f|^p \Big)^{\frac{1}{p}}. \end{aligned} \tag{2.1}$$

By C we denote the generic positive constant whose exact value may change at each occurrence. We write for example $C(a, b, c)$ if C may depend on parameters a, b and c.

3 Proof of Theorem 1.1

We split the proof of Theorem 1.1 into five steps. In the first step, we prepare some dense subset of $[-1, 1]^n$, in the second step we prepare the sequence of one dimensional functions f_i. In the third step we construct the sequence of the functions f_i^+ based on f_i and we use them to construct the function $f \colon [-1, 1]^n \to \mathbb{R}$. In the fourth step we verify the continuity of Df and in the last step we show that $f \in W^{2,n}((-1, 1)^n)$.

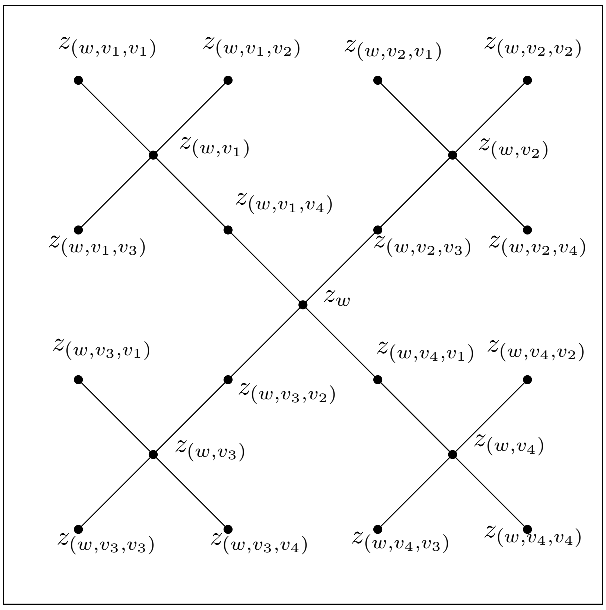

Fig. 1 Part of structure $z_w, w \in \mathbb{W}$ for $n = 2$.

3.1 First step: The tree of centres of dyadic sub-squares

By $\mathbb{V}$ we denote the set of the 2^n vertices of the cube $[-1,1]^n$ and by $\mathbb{V}^k$ we denote the product of k copies of the set $\mathbb{V}$.

We use the notation $w = (u, v)$ for case $w \in \mathbb{V}^k$, $u \in \mathbb{V}^{k-1}$ and $v \in \mathbb{V}$ if $w_i = u_i$ for $i = 1, 2, \dots k-1$ and $w_k = v$.

We decompose $[-1,1]^n$ into the dyadic sub-squares and by Z we denote the set of the centres of these squares. We begin by setting $z_0 = 0$ and we fix $v \in \mathbb{V}$. By z_v we denote

$$z_v = z_0 + 2^{-1}v.$$

Point z_v is clearly the centre of one dyadic sub-square of volume 1. For any $k \in \mathbb{N}$ we can fix $u \in \mathbb{V}^k$ and $w \in \mathbb{V}^{k+1}, w = (u, w_{k+1})$. By induction we define

$$z_w = z_u + 2^{-k-1}w_{k+1} \tag{3.1}$$

and if $z_u \in Z$, then clearly $z_w \in Z$. Moreover, by the same argument we can see that any element of Z can be identified with some finite sequence $w \in \mathbb{V}^m$ for some non negative integer m, see Firure 1. For simplicity we denote

$$\mathbb{W} = \bigcup_{k=0}^{\infty} \mathbb{V}^k,\ |w|_l = k \text{ for } w \in \mathbb{V}^k \text{ and } Z = \{z_w, w \in \mathbb{W}\}.$$

3.2 Second step: One dimension C^1 bump f_i

In this paper we will work with the function $t \log^{\varepsilon}(t^{-1}), \varepsilon < n^{-1}$.

We set

$$a_i^* = e^{-2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}}} \text{ and } b_i^* = e^{-2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}} - 2^{i\left(\frac{1+\varepsilon}{(1-\varepsilon)n-1-\varepsilon}+\varepsilon\right)}}, \text{ for } i > 0. \tag{3.2}$$

We can see that

$$\lim_{i\to\infty} \frac{2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}} - 2^{1+i\left(\frac{1+\varepsilon}{(1-\varepsilon)n-1-\varepsilon}+\varepsilon\right)}}{2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}}} = 1,$$

we choose some $i^*(n) \in \mathbb{N}$, depending only on n such that

$$\frac{2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}} - 2^{1+i\left(\frac{1+\varepsilon}{(1-\varepsilon)n-1-\varepsilon}+\varepsilon\right)}}{2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}}} > 2^{-1} \text{ for all } i \geq i^*(n). \tag{3.3}$$

We want to prove

$$a_i^* > b_i^* > b_i^{*2} a_i^{*-1} > 2^{n+2} a_{i+1}^* \text{ for } i \in \mathbb{N}, i \geq i_0(n) = \max\left\{\frac{\log(n)+3}{\log(2)}, i^*(n)\right\}. \tag{3.4}$$

The first and the second inequality are clear by (3.2). The inequality $b_i^{*2} a_i^{*-1} > 2^{n+2} a_{i+1}^*$ follows from the estimate

$$\log(b_i^{*2} a_i^{*-1} a_{i+1}^{*-1}) > \log(2)(n+2) \text{ for } i \geq i_0(n),$$

which can be shown for $i \geq i^*(n)$ and $i \geq \frac{\log(n)+3}{\log(2)}$ by

$$\begin{aligned}
\log(b_i^{*2} a_i^{*-1} a_{i+1}^{*-1}) &= (1-2) 2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}} + 2^{\frac{(i+1)}{1-\varepsilon-\frac{1+\varepsilon}{n}}} - 2^{1+i\left(\frac{1+\varepsilon}{(1-\varepsilon)n-1-\varepsilon}+\varepsilon\right)} \\
&> 2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}} - 2^{1+i\left(\frac{1+\varepsilon}{(1-\varepsilon)n-1-\varepsilon}+\varepsilon\right)} \\
&> 2^{-1} 2^{\frac{i}{1-\varepsilon-\frac{1+\varepsilon}{n}}} > 2^{i-1} > 4n > \log(2)(n+2).
\end{aligned}$$

We shift the index of (3.2) in order to satisfy (3.4) for all indices

$$a_i = a_{i+i_0}^* \text{ and } b_i = b_{i+i_0}^*. \tag{3.5}$$

We define function $q(t)$ and we calculate its derivatives for $t \in (0, a_1)$

$$\begin{aligned}
q(t) &= |\log(t^{-1})|^{\varepsilon} \\
q'(t) &= -\varepsilon t^{-1} |\log(t^{-1})|^{\varepsilon-1} \\
q''(t) &= \varepsilon t^{-2} |\log(t^{-1})|^{\varepsilon-1} + (\varepsilon-1)\varepsilon t^{-2} |\log(t^{-1})|^{\varepsilon-2}.
\end{aligned} \tag{3.6}$$

We also define the sequence

$$d_i = \frac{2^{-i-i_0}}{\big(q(b_i) - q(a_i) - q'(a_i)(b_i - a_i)\big)(1 - \frac{b_i}{a_i}) + |q'(b_i)|(a_i - b_i)\frac{b_i}{a_i}}. \tag{3.7}$$

By convexity of $q(t)$ we have

$$|q(b_i) - q(a_i) - q'(a_i)(b_i - a_i)| \leq |q(b_i) - q(a_i)| \leq |q'(b_i)|(a_i - b_i).$$

So d_i can be estimated as

$$\begin{aligned}
d_i &\leq \frac{2^{-i-i_0}}{q(b_i) - q(a_i) - q'(a_i)(b_i - a_i)} \\
&\leq \frac{2^{-i-i_0}}{|\log(b_i^{-1})|^{\varepsilon} - |\log(a_i^{-1})|^{\varepsilon} - \varepsilon a_i^{-1} |\log(a_i^{-1})|^{\varepsilon-1}(a_i - b_i)}.
\end{aligned}$$

Now we can use estimate $\varepsilon \frac{x-y}{\max\{x^{1-\varepsilon}, y^{1-\varepsilon}\}} \leq x^{\varepsilon} - y^{\varepsilon}$ based on mean value theorem for x^{ε}. We get

$$\begin{aligned}
d_i &\leq \frac{2^{-i-i_0}}{\varepsilon \frac{\log(a_i^{-1}) - \log(b_i^{-1})}{|\log(b_i^{-1})|^{1-\varepsilon}} - \varepsilon a_i^{-1} |\log(a_i^{-1})|^{\varepsilon-1}(a_i - b_i)} \\
&\leq \varepsilon^{-1} \frac{2^{-i-i_0} |\log(a_i^{-1}) + \log(a_i b_i^{-1})|^{1-\varepsilon}}{|\log(\frac{b_i}{a_i})| - (|\log(b_i^{-1})||\log(a_i^{-1})|^{-1})^{1-\varepsilon}}.
\end{aligned}$$

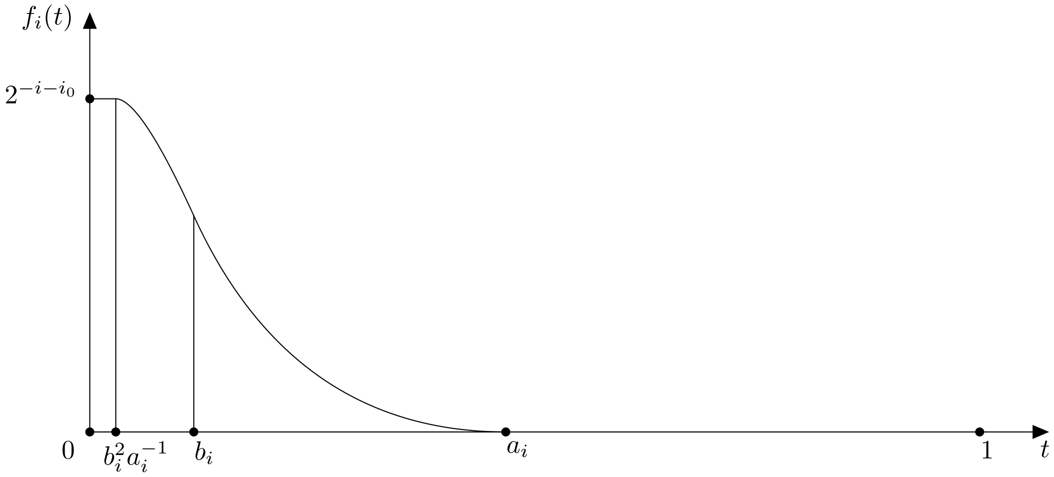

Fig. 2 The graph of f_i.

The first part of denominator is the sequence $|\log(\frac{b_i}{a_i})|$ and it goes to infinity by (3.5) and (3.2), the second part is the sequence $(|\log(b_i^{-1})||\log(a_i^{-1})|^{-1})^{1-\varepsilon}$ and it goes to one by (3.5), (3.2) and (3.3), therefore

$$\frac{1}{|\log(\frac{b_i}{a_i})| - (|\log(b_i^{-1})||\log(a_i^{-1})|^{-1})^{1-\varepsilon}} \le C\frac{1}{|\log(\frac{b_i}{a_i})|}.$$

By this and the previous estimate we have

$$d_i \le C\varepsilon^{-1}\frac{2^{-i-i_0}|2\log(a_i^{-1})|^{1-\varepsilon}}{|\log(\frac{b_i}{a_i})|} \le C\frac{2^{(i+i_0)\left(\frac{1-\varepsilon}{1-\varepsilon-\frac{1+\varepsilon}{n}}-1\right)+1-\varepsilon}}{2^{\frac{(i+i_0)(1+\varepsilon)}{(1-\varepsilon)n-(1+\varepsilon)}+\varepsilon(i+i_0)}} = C2^{-(i+i_0)\varepsilon}. \tag{3.8}$$

We define $p_i\colon (b_i, a_i) \to \mathbb{R}$ as

$$p_i(t) = q(t) - q'(a_i)t. \tag{3.9}$$

Now we define $f_i\colon \mathbb{R}^+ \to \mathbb{R}$ as

$$f_i = \begin{cases} 2^{-(i+i_0)} & \text{for } t \in [0, \frac{b_i^2}{a_i}) \\ C_1 + d_i\big(p_i'(b_i)t - \frac{b_i}{a_i}p_i(\frac{a_i}{b_i}t)\big) & \text{for } t \in [\frac{b_i^2}{a_i}, b_i) \\ C_2 + d_i p_i(t) & \text{for } t \in [b_i, a_i) \\ 0 & \text{for } t \in [a_i, \infty), \end{cases} \tag{3.10}$$

where C_1, C_2 are constants chosen such that $f_i \in C^1(0,1)$. Precisely

$$\begin{aligned} C_1 &= 2^{-(i+i_0)} - d_i\frac{b_i}{a_i}\big(q'(b_i)b_i - q(b_i)\big), \\ C_2 &= d_i\big(q'(a_i)a_i - q(a_i)\big). \end{aligned} \tag{3.11}$$

We check the continuity and the smoothness of f_i. The continuity at the points $b_i^2 a_i^{-1}$ and a_i is clear thanks to the formula describing C_1 and C_2 above. The continuity at the point b_i can be verified by (3.11), (3.9) and (3.7)

$$\begin{aligned} f_i(b_i) &= C_2 + d_i p_i(b_i) \\ &= d_i\big(q(b_i) - q(a_i) + (a_i - b_i)q'(a_i)\big) \\ &= 2^{-(i+i_0)} + d_i \frac{b_i}{a_i}\Big(q(b_i) - q(a_i) + (a_i - b_i)\big(q'(b_i) + q'(a_i)\big)\Big) \\ &= C_1 + d_i\big(p'(b_i)b_i - \frac{b_i}{a_i}p_i(a_i)\big) = \lim_{t\to(b_i)_-} f_i(t). \end{aligned}$$

The continuity of the first derivative is clear at points inside the intervals. At the endpoints of the intervals we have by (3.10) and (3.9)

$$\begin{aligned} f_i'(b_i^2 a_i^{-1}) =& d_i(p_i'(b_i) - p_i'(b_i)) = 0, \\ f_i'(b_i) =& \lim_{t\to b_i-} f_i'(t) = d_i p_i'(b_i) = d_i\big(p_i'(b_i) - p_i'(\frac{a_i}{b_i}b_i)\big) = \lim_{t\to b_i+} f_i'(t), \\ f_i'(a_i) =& d_i p_i'(a_i) = d_i(q'(a_i) - q'(a_i)) = 0. \end{aligned}$$

Let us note that $-f_i'(a_i) + f_i'(b_i) = f_i'(b_i) - f_i'(b_i^2 a_i^{-1})$. Moreover, we can see that f_i are C^∞ on each interval $(0, b_i^2 a_i^{-1})$, $(b_i^2 a_i^{-1}, b_i)$, (b_i, a_i), $(a_i, 1)$.

Remark 3.1 Although the definition of f_i is rather complicated, the smoothness and the figure will be enough to follow the idea. The precise formula is needed in the proof of the continuity of Df and in the proof of the finiteness of the norm.

3.3 Third step: Sequence of functions f_i, summation of f_i and the image of Df

Let us fix $v \in \mathbb{V}$, see subsection 3.1. We define

$$f_{i,v}^+(x) = |v|\langle x, v\rangle f_i(|x|) = \sqrt{n}\langle x, v\rangle f_i(|x|). \tag{3.12}$$

The support of $f_{i,v}^+$ is clearly the ball of radius a_i and the function is linear inside the smaller ball of radius $b_i^2 a_i^{-1}$. Precisely, it holds

$$Df_{i,v}^+(x) = 2^{-i_0-i}v \text{ for } |x| < b_i^2 a_i^{-1}. \tag{3.13}$$

Lemma 3.2 *Let $\{A_w\}_{w\in\mathbb{W}}$ be a set of line segments satisfying*

1. $A_w \subset [-1,1] \times \{0\}^{n-1}$ *for* $w \in \mathbb{W}$,
2. $A_w \cap A_v = \emptyset$ *for* $w, v \in \mathbb{W}, w \neq v, |w|_l = |v|_l$,
3. $A_{(w,u)} \subset A_w$ *for* $w \in \mathbb{W}, u \in \mathbb{V}$.

Let $f \in W^{2,n}((-1,1)^n) \cap C^1([-1,1]^n)$ be such that for all $w \in \mathbb{W}$ there exists $x \in A_w$ with $Df(x) = z_w$. Then $Df([-1,1] \times \{0\}^{n-1}) \supseteq [-1,1]^n$.

Proof. The image $Df([-1,1] \times \{0\}^{n-1})$ contains $Z = \{z_w, w \in \mathbb{W}\}$, but Df is also continuous. The continuous image of a compact set must be a compact set and the smallest compact set containing Z is $[-1,1]^n$. □

We have to define the system of subsets A_w and f from the statement of Lemma 3.2. We use the induction by the size of index $i = |w|_l$, we start with $i = 1$ and $\emptyset = \mathbb{V}^0$. We define

$$A_\emptyset = [-1,1] \times \{0\}^{n-1}.$$

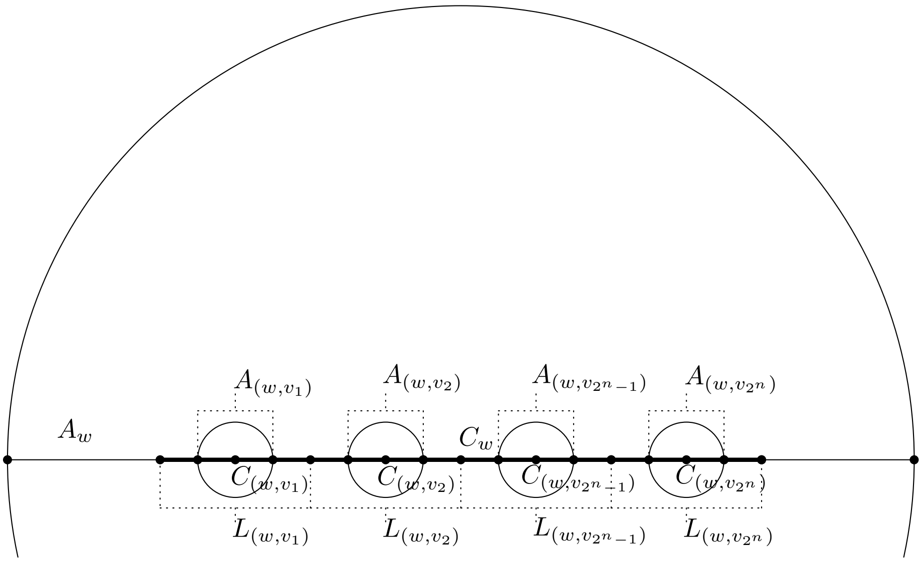

Fig. 3 One step in constriction A_w, case $n = 2$.

We denote

$$C_w = \text{ middle of line segment } A_w. \tag{3.14}$$

We fix some $w \in \mathbb{V}^{i-1}$. We take the line segment of length $2^{n+2}a_i$ with the middle in C_w. We split this line segment into 2^n sub-segments of length $4a_i$ denoted $L_{(w,v)}$ after 2^n vectors $v \in \mathbb{V}$. We can see that C_w is not contained inside any of these intervals, it lies exactly in between two middle ones $L_{(w,v)}$. Let us define

$$C_{(w,v)} = \text{ middle of line segment } L_{(w,v)} \text{ of length } 4a_{|w|_l+1}, \tag{3.15}$$

and by previous

$$A_{(w,v)} = \Big([-1,1] \times \{0\}^{n-1}\Big) \cap B(C_{(w,v)}, b_i^2 a_i^{-1}). \tag{3.16}$$

We check that the length of A_w is bigger than the length $2^{n+2}a_i$ of all $L_{(w,v)}$ together. But the length of A_w is $2b_{i-1}^2 a_{i-1}^{-1}$ by (3.16) and we get the estimate $b_{i-1}^2 a_{i-1}^{-1} > 2^{n+2}a_i$ by (3.4). We also get $A_w \subset L_w$ by (3.4), $2b_i^2 a_i^{-1} < 4a_i$. See Figure 3.

We can see that C_w is the middle of both L_w and A_w, we also see that length of A_w is $2a_{|w|_l}$, so by (3.4) property (3) from Lemma 3.2 holds.

Now we construct f such that properties from Lemma 3.2 will be satisfied. We recall three definitions (3.12), (3.14) and (3.16) and define

$$f(x) = 2^{i_0} \sum_{i=1}^{\infty} \sum_{w \in \mathbb{W}, |w|_l = i} f^+_{i,w_i}(x - C_w). \tag{3.17}$$

This function is clearly continuous and in the fourth step we will show that $f \in C^1([-1,1]^n)$. The property that there exists $x \in A_w$ such that $Df(x) = z_w$ can be proven by taking $x = C_w$. We remind that for $w \in \mathbb{W}$, $v \in \mathbb{V}$, $i = |w|_l$ the support of any function is $\operatorname{spt} f^+_{i+1,v}(x - C_{(w,v)}) = B(C_{(w,v)}, a_{i+1})$ and C_w does not lie inside any interval $L_{(w,v)}$ of length $4a_{i+1}$. So the distance between the $C_{(w,v)}$ =middle of $L_{(w,v)}$ and C_w satisfies $|C_w - C_{(w,v)}| \geq 2a_{i+1}$. We prove that C_w does not belong to this support by the triangle inequality

$$|C_w - x| \geq |C_w - C_{(w,v)}| - |C_{(w,v)} - x| \geq 2a_{i+1} - a_{i+1} > 0, \text{ for } x \in \operatorname{spt} f^+_{i+1,v}(x - C_{(w,v)}).$$

Because the supports of the functions are contained in the supports of the functions in previous step, C_w does not belong even to any support of function $f^+_{|u|_l+1,v}(x - C_{(u,v)})$, $|u|_l > |w|_l$, $u \in \mathbb{W}$, $v \in \mathbb{V}$. Indeed, by (3.13) and (3.1)

$$Df(C_w) = 2^{i_0} \sum_{i=1}^{|w|_l} Df^+_{i,w_i}(C_w - C_{(w_1,w_2,\dots,w_i)}) = 2^{i_0} \sum_{i=1}^{|w|_l} w_i 2^{-i-i_0} = z_w. \tag{3.18}$$

3.4 Fourth step: Verification of $f \in C^1([-1,1]^n)$

First, we observe that the supports of the gradients are disjoint in some way and therefore we can switch the summation and the derivative. Then we check that the gradient of f is bounded everywhere on $[-1,1]^n$ and it is continuous.

Our claim is, that for any $x \in [-1,1] \times \{0\}^{n-1}$ at most one function $f^+_{i,w_i}(x - C_w), w \in \mathbb{W}, i = |w|_l$ from definition of function f (3.17) is non linear in some small neighbourhood of x. First of all, we check that any two functions

$$f^+_{i,w_i}(x - C_w), f^+_{i,w^*_i}(x - C_{w^*}), w, u \in \mathbb{W}, i = |w|_l = |w^*|_l$$

have disjoint supports. This fact is obvious, because the supports of both functions are the balls with radius a_i and $|C_w - C_{w^*}| \geq 4a_i$ from (3.14). Then, we consider $u, w \in \mathbb{W}, v \in \mathbb{V}, w = (u,v), i = |u|_l$. We can see that the support of $f^+_{|w|_l,v}(x - C_w)$ belongs to the part of the support of $f^+_{|u|_l,u_i}(x - C_u)$, where it holds $Df^+_{|u|_l,u_i}(x - C_u) = 2^{-i_0} u_i 2^{-|u|_l}$. Indeed, by (3.4)

$$|x - C_u| \leq |C_u - C_w| + |C_w - x| \leq a_{i+1}2^n + a_{i+1} < b_i^2 a_i^{-1} \text{ for any } x \in B(C_w, a_{i+1}),$$

so $x \in B(C_u, b_i^2 a_i^{-1})$ and hence (3.13) gives $Df^+_{i,u_i}(x - C_u) = 2^{-i_0} u_i 2^{-i}$. Specially, for $x \in [-1,1]^n$ we have

$$\begin{aligned} Df(x) &= \sum_{i=1}^{\infty} \sum_{w \in \mathbb{V}^i} Df^+_{i,w_i}(x - C_w) \\ &= \sum_{i=1}^{\infty} Df^+_{i,w^*_i(x)}(x - C_{w^*(x)}), \text{ where } w^*(x) \in \mathbb{V}^i \text{ such that } |x - C_{w^*}| < a_i. \end{aligned} \tag{3.19}$$

For every $x \in [-1,1]^n$, $i \in \mathbb{N}$ at most one $w^*(x) \in \mathbb{W}, |w^*(x)|_l = i$ can satisfy the condition $|x - C_{w^*(x)}| < a_i$ and if there is no such $w^*(x)$ we set $f^+_{i,w^*_i(x)} \equiv 0$.

We want to estimate $|Df^+_{i,w_i}(x - C_w)|$ for fixed $f^+_{i,w_i}(x - C_w)$ and $x \in B(C_w, a_i)$. We change the coordinates $x_1, x_2 \dots x_n$ to a base containing the unit vector $y_1 = \frac{w_i}{|w_i|}$ and $n-1$ pairwise orthogonal unit vectors orthogonal to $\frac{w_i}{|w_i|}$ denoted $y_2 \dots y_n$. By (3.12) we estimate the gradient of $f^+_{i,w_i}(x - C_w)$

$$|Df^+_{i,w_i}(x - C_w)| \leq \max_{j \in \{1,2\dots n\}} \Big\{ \Big| \frac{\partial \sqrt{n} \langle x - C_w, w_i \rangle f_i(|x - C_w|)}{\partial y_j} \Big| \Big\}. \tag{3.20}$$

We use the formula for the derivative of the product and we get

$$\begin{aligned} \Big| \frac{\partial \langle x - C_w, w_i \rangle f_i(|x - C_w|)}{\partial y_j} \Big| \leq & \Big| \frac{\partial \langle x - C_w, w_i \rangle}{\partial y_j} f_i(|x - C_w|) \Big| \\ & + \Big| \langle x - C_w, w_i \rangle \frac{\partial f_i(|x - C_w|)}{\partial y_j} \Big|. \end{aligned}$$

We can see that $\frac{\partial \langle x - C_w, w_i \rangle}{\partial y_j} = 0$ for all $j \in \{2, 3 \dots n\}$ because $\langle y_j, w_i \rangle = 0$, in case of $y_1 = \frac{w_i}{|w_i|}$ we have

$$\frac{\partial \langle x - C_w, w_i \rangle}{\partial y_1} = \langle w_i, \frac{w_i}{|w_i|} \rangle = 1.$$

We estimate the second term

$$\left|\sqrt{n}\langle x-C_w,w_i\rangle\frac{\partial f_i(|x-C_w|)}{\partial y_j}\right|\leq\sqrt{n}|x-C_w|\big|f_i'(|x-C_w|)\big|,$$

because a directional derivative of radial mapping can be estimated by the biggest directional derivative. We estimate by (3.20) and by previous estimates applied for every $j=\{1,2\dots n\}$

$$|Df^+_{i,w_i}(x-C_w)|\leq\sqrt{n}\Big(\big|f_i(|x-C_w|)\big|+|x-C_w|\big|f_i'(|x-C_w|)\big|\Big). \tag{3.21}$$

By (3.10), (3.8), (3.9) and (3.6) we can estimate the value of function f_i and its derivative for $b_i<|x-C_w|<a_i$ as

$$\begin{aligned}&|f_i(|x-C_w|)|\leq 2^{-i-i_0},\\ |x-C_w||f_i'(|x-C_w|)|&=|x-C_w||d_ip_i'(|x-C_w|)|\leq\\ &\leq C2^{-i_0}|x-C_w|2^{-(i+i_0)\varepsilon}\varepsilon\big(|x-C_w|^{-1}|\log(|x-C_w|^{-1})|^{\varepsilon-1}-a_i^{-1}|\log(a_i^{-1})|^{\varepsilon-1}\big)\\ &\leq C2^{-i\varepsilon-i_0\varepsilon}b_i\big(b_i^{-1}|\log(b_i^{-1})|^{\varepsilon-1}\big)\leq C2^{-i\varepsilon}.\end{aligned} \tag{3.22}$$

These estimates are valid even for $0\leq|x-C_w|\leq b_i$, because by (3.10) for $0\leq|x-C_w|\leq b_i$ it holds $|f_i'(|x-C_w|)|\leq|f_i'(b_i)|$(see Figure 2). We apply (3.22) to (3.21) and we get

$$|Df^+_{i,w_i}(x-C_w)|\leq C2^{-i\varepsilon}.$$

By this estimate and (3.19) we can see that $Df(x)$ is the limit of the Cauchy sequence of continuous functions in the Banach space $C([-1,1]^n,\mathbb{R}^n)$.

3.5 Fifth step: Verification of $f\in W^{2,n}((-1,1)^n)$

We recall Sobolev embedding theorem, $(-1,1)^n$ is Lipschitz domain and therefore

$$|D^2f|\in L^n((-1,1)^n)\Rightarrow|Df|\in L^n((-1,1)^n)\text{ and }f\in L^n((-1,1)^n).$$

It remains to prove $|D^2f|\in L^n((-1,1)^n)$. Although f is defined as the sum of the functions with the non-disjoint supports, the supports of the second derivatives of these functions are pairwise disjoint thanks to the observation about the linearity of the functions on some part of the supports at the beginning of the fourth step. So by (3.17) and (3.13) we express

$$\int_{(-1,1)^n}|D^2f(x)|^n=2^{i_0n}\sum_{i=1}^{\infty}\sum_{w\in\mathbb{W},|w|_l=i}\int_{(-1,1)^n}|D^2f^+_{i,w_i}(x-p_w)|^n.$$

We can shift and rotate the function without changing the size of the integral. So we can identify all copies of function $|D^2f^+_{i,w_i}(x-C_w)|$ for the same size of index $|w|_l$ regardless on the orientation, we get $2^{n|w|_l}$ integrals with the same value. We consider any $v\in\mathbb{V}$ and we express by (3.12)

$$\sum_{i=1}^{\infty}\sum_{w\in\mathbb{W},|w|_l=i}\int_{(-1,1)^n}|D^2f^+_{i,w_i}(x-C_w)|^n=\sum_{i=1}^{\infty}2^{in}\int_{(-1,1)^n}|D^2(x_1|v|f_i(|x|))|^n. \tag{3.23}$$

We consider the function $x_1\sqrt{n}f_i(|x|)$ and split its support into the three parts following (3.10). The first part is the ball $B(0,b_i^2a_i^{-1})$, function is linear on this domain because $f_i(|x|)=2^{-i-i_0}$ from (3.10). The second part is the annulus $A(0,b_i^2a_i^{-1},b_i)$ and the third part is the annulus $A(0,b_i,a_i)$. The integral over the first part is clearly 0 because the second derivative of a linear function is constant 0. We look at the second part, by triangle inequality and (3.10) we estimate

$$\begin{aligned}
&\sum_{i=1}^{\infty} 2^{in} \int_{A(0,\frac{b_i^2}{a_i},b_i)} \Big|D^2\Big(x_1|v|f_i(|x|)\Big)\Big|^n = \\
&= \sum_{i=1}^{\infty} 2^{in} \int_{A(0\frac{b_i^2}{a_i},b_i)} \Big|D^2\Big(x_1|v|d_i\big(p'(b_i)|x| - \frac{b_i}{a_i}p_i(\frac{a_i}{b_i}|x|)\big)\Big)\Big|^n \qquad (3.24)\\
&\leq C\sum_{i=1}^{\infty} 2^{in}\Big(\int_{A(0,\frac{b_i^2}{a_i},b_i)} \big|D^2\big(x_1|v|d_ip'(b_i)|x|\big)\big|^n + \int_{A(0,\frac{b_i^2}{a_i},b_i)} \big|D^2\big(d_ix_1|v|\frac{b_i}{a_i}p_i(\frac{a_i}{b_i}|x|)\big)\big|^n\Big).
\end{aligned}$$

We estimate the first integral, by direct calculation

$$\Big|D^2\big(d_i|v|p_i'(b_i)x_1|x|\big)\Big| = d_i|v|p_i'(b_i)\max_{j,k\in\{1,2,\dots n\}}\Big\{\frac{|x_j|}{|x|}, \frac{|x_1x_jx_k|}{|x|^3}\Big\} \leq d_i|v|p_i'(b_i).$$

We apply this estimate on the first integral with (3.9), (3.8), (3.5) and (3.2) and hence

$$\begin{aligned}
\int_{A(0,b_i^2a_i^{-1},b_i)} \Big|D^2\big(x_1|v|d_ip_i'(b_i)|x|\big)\Big|^n &\leq Cb_i^n|d_ip_i'(b_i)|^n \leq Cb_i^n2^{-(i+i_0)\varepsilon n}|q'(b_i)-q'(a_i)|^n \\
&\leq Cb_i^n2^{-(i+i_0)\varepsilon n}|q'(b_i)|^n \qquad (3.25)\\
&\leq C2^{-(i+i_0)\varepsilon n}b_i^{n-n}|\log(b_i^{-1})|^{n(\varepsilon-1)}\\
&\leq C2^{-(i+i_0)\varepsilon n-(i+i_0)n}.
\end{aligned}$$

We can see that the sum of these integrals multiplied by 2^{in} is still finite.

Now we work with the second integral from (3.24) over the second part and we will see that there is a close connection with the integral over the third part. By (3.2) we have $b_ia_i^{-1} < 1$ and by (3.9) we get

$$\begin{aligned}
\Big|D^2\big(x_1d_i|v|\frac{b_i}{a_i}p_i(\frac{a_i}{b_i}|x|)\big)\Big| \leq Cd_i|v|\max_{j,k\in\{1,2,\dots n\}}\Big\{&\Big|\frac{x_j}{|x|}(q'(\frac{a_i}{b_i}|x|)-q'(a_i))\Big|,\\
&\Big|\frac{x_1x_jx_k}{|x|^3}\big((q'(\frac{a_i}{b_i}|x|)-q'(a_i)) + \frac{a_i}{b_i}|x|q''(\frac{a_i}{b_i}|x|)\big)\Big|\Big\}.
\end{aligned}$$

We use the simple inequality $\frac{x_j}{|x|} \leq 1$ and formula (3.6) and we estimate

$$\begin{aligned}
\Big|D^2\big(x_1d_i|v|\frac{b_i}{a_i}p_i(\frac{a_i}{b_i}|x|)\big)\Big| &\leq Cd_i|v|\max\Big\{\big|q'(\frac{a_i}{b_i}|x|)\big|, \big|q'(\frac{a_i}{b_i}|x|) + \frac{a_i}{b_i}|x|q''(\frac{a_i}{b_i}|x|)\big|\Big\}\\
&\leq Cd_i\frac{b_i}{a_i}|x|^{-1}\big|\log(\frac{b_i}{a_i}|x|^{-1})\big|^{\varepsilon-1}.
\end{aligned}$$

We estimate the integral by this and we transfer it into an one-dimensional case by the change of variables into the spherical system of coordinates. Then we use the change of variables $t = b_ia_i^{-1}s$ and we get

$$\begin{aligned}
&\int_{A\big(0,\frac{b_i^2}{a_i},b_i\big)} \Big|D^2\big(d_i|v|x_1\frac{b_i}{a_i}p_i(\frac{a_i}{b_i}|x|)\big)\Big|^n \leq \\
&\qquad\leq Cd_i^n\int_{A\big(0,\frac{b_i^2}{a_i},b_i\big)} \big(\frac{a_i}{b_i}|x|\big)^{-n}\Big|\log\big((\frac{a_i}{b_i}|x|)^{-1}\big)\Big|^{(\varepsilon-1)n}dx\\
&\qquad\leq C\int_{b_i^2a_i^{-1}}^{b_i} t^{n-1}\big(\frac{a_i}{b_i}t\big)^{-n}\Big|\log\big((\frac{a_i}{b_i}t)^{-1}\big)\Big|^{(\varepsilon-1)n}dt \qquad (3.26)\\
&\qquad\leq C\int_{b_i}^{a_i}\big(\frac{b_i}{a_i}s\big)^{n-1}s^{-n}\big|\log(s^{-1})\big|^{(\varepsilon-1)n}\frac{b_i}{a_i}ds\\
&\qquad\leq C\int_{b_i}^{a_i} s^{-1}\big|\log(s^{-1})\big|^{n(\varepsilon-1)}ds.
\end{aligned}$$

Now we skip the summation and we go back to the third integral over annulus $A(0,b_i,a_i)$. After a short calculation we will see, that the estimate goes in the same way as in (3.26). Precisely, we estimate the second partial derivative by the same arguments as before and we get

$$\begin{aligned}&\big|D^2\big(d_i|v|x_1p_i(|x|)\big)\big|\leq\\ &\quad\leq Cd_i|v|\max\Big\{\big|\frac{x_i}{|x|}(q'(|x|)-q'(a_i))\big|,\big|\frac{x_1x_ix_j}{|x|^3}(q'(|x|)-q'(a_i))+\frac{x_1x_ix_j}{|x|^2}q''(|x|)\big|\Big\}\\ &\quad\leq Cd_i|x|^{-1}\big|\log\big(|x|^{-1}\big)\big|^{\varepsilon-1}.\end{aligned}\tag{3.27}$$

We estimate the integral over the last annulus

$$\begin{aligned}\int_{A(0,b_i,a_i)}\big|D^2\big(d_i|v|x_1p_i(|x|)\big)\big|^n&\leq Cd_i^n\int_{A(0,b_i,a_i)}\Big(|x|^{-1}\big|\log\big(|x|^{-1}\big)\big|^{\varepsilon-1}\Big)^n dx\\ &\leq C\int_{b_i}^{a_i}t^{n-1}\big(t^{-1}|\log(t^{-1})|^{\varepsilon-1}\big)^n dt\\ &\leq C\int_{b_i}^{a_i}t^{-1}|\log(t^{-1})|^{n(\varepsilon-1)}dt.\end{aligned}\tag{3.28}$$

We get exactly the same integral as in (3.26). We use (3.2) and (3.5) to straightforward estimate

$$|\log(\frac{1}{t})|^{(1-\varepsilon)n-1-\varepsilon}2^{-i_0n}\geq 2^{in}\text{ for any }t\in(b_i,a_i).\tag{3.29}$$

By (3.23) and (3.24) we have

$$\begin{aligned}\int_{[-1,1]^n}|D^2f(x)|^n\leq C\sum_{i=1}^{\infty}2^{in}\Big(&\int_{A(0,b_i^2a_i^{-1},b_i)}\big|D^2(x_1|x||v|d_ip'(b_i))\big|^n\\ &+\int_{A(0,b_i^2a_i^{-1},b_i)}\big|D^2(d_ix_1\frac{b_i}{a_i}|v|p_i(\frac{a_i}{b_i}|x|))\big|^n\\ &+\int_{A(0,b_i,a_i)}|D^2(x_1|v|p_i(|x|))|^n\Big).\end{aligned}$$

Now by estimates (3.25), (3.26), (3.28) and (3.29) we get

$$\begin{aligned}\int_{[-1,1]^n}|D^2f(x)|^n&\leq C\sum_{i=1}^{\infty}2^{in}\Big(2^{-(i+i_0)\varepsilon n-(i+i_0)n}+\int_{b_i}^{a_i}t^{-1}|\log(t^{-1})|^{n(\varepsilon-1)}dt\Big)\\ &\leq C\sum_{i=1}^{\infty}2^{-i\varepsilon n}+C\sum_{i=1}^{\infty}2^{in}\int_{b_i}^{a_i}t^{-1}|\log(t^{-1})|^{n(\varepsilon-1)}dt\\ &\leq C+C\sum_{i=1}^{\infty}\int_{b_i}^{a_i}t^{-1}|\log(t^{-1})|^{n(\varepsilon-1)+(1-\varepsilon)n-1-\varepsilon}dt\\ &\leq C+C\int_0^1t^{-1}|\log(t^{-1})|^{-1-\varepsilon}dt<\infty.\end{aligned}\tag{3.30}$$

Remark 3.3 Let us consider any p, $p<n-1$ and $\varepsilon>0$ such that $p+1+\varepsilon<(1-\varepsilon)n$. We sketch that even $|D^2f|\in L^n\log^p(L)$ can be proven in this way. At first, we redefine (3.2) as

$$a_i^*=e^{-2^{\frac{in}{(1-\varepsilon)n-p-1-\varepsilon}}},\ b_i^*=a_i^*e^{-2^{i\left(\frac{1+\varepsilon}{(1-\varepsilon)n-p-1-\varepsilon}+\varepsilon\right)}}.$$

We follow the idea of the previous proof and only the fifth step of the proof would be slightly different. We sketch the calculation only on annulus $A(0,b_i,a_i)$, where we have the estimate (3.27).

In the last estimate (3.28) we replace the expression $|D^2 \cdot|^n$ by $|D^2 \cdot|^n |\log(|D^2 \cdot|)|^p$ and using nearly the same technique we get

$$\int_{A(0,b_i,a_i)} \left|D^2\big(d_i|v|x_1 p_i(|x|)\big)\right|^n \leq C \int_{b_i}^{a_i} t^{-1}|\log(t^{-1})|^{n(\varepsilon-1)+p} dt.$$

Then, insted of (3.29) we get

$$|\log(\frac{1}{t})|^{(1-\varepsilon)n-p-1-\varepsilon} 2^{-i_0 n} \geq 2^{in} \text{ for any } t \in (b_i, a_i).$$

If we use this pattern also on (3.25) and (3.26), we get the estimate analogous to (3.30), therefore

$$\begin{aligned} \int_{[-1,1]^n} |D^2 f(x)|^n |\log(|D^2 f(x)|)|^p \leq \\ &\leq C + C \sum_{i=1}^{\infty} \int_{b_i}^{a_i} t^{-1}|\log(t^{-1})|^{n(\varepsilon-1)+p+(1-\varepsilon)n-p-1-\varepsilon} dt \\ &\leq C + C \int_0^1 t^{-1}|\log(t^{-1})|^{-1-\varepsilon} dt < \infty. \end{aligned}$$

Acknowledgements We would like to thank Ulrich Menne, for pointing our interest to this problem. We also thank to Jan Malý, who gave us some useful ideas about the problem. And we thank to the author's supervisor, Stanislav Hencl, for valuable comments.

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3. Sobolev homeomorphism in $W^{k,p}$ and the Lusin (N) condition

Revista Matemática Complutense manuscript No.
(will be inserted by the editor)

Sobolev homeomorphism in $W^{k,p}$ and the Lusin (N) condition

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Received: date / Accepted: date

Abstract We characterize when the Luzin (N) condition holds in $W^{k,p}$ in dependence on k, p and dimension. We generalize well-known counterexamples in $W^{1,p}$ both for general mappings and for homeomorphisms.

Keywords Lusin condition, Sobolev space

Mathematics Subject Classification (2000) 46E35

1 Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open set. We say that the mapping $f\colon \Omega \to \mathbf{R}^m$ satisfies Lusin (N) condition if for every $A \subset \Omega, |A| = 0$, it holds that $|f(A)| = 0$. We can see that this condition is needed for any natural physical model such as the deformation of a solid body in space. Otherwise we can make new material "from nothing", so the mapping is really unnatural for physical applications. Another important application is the connection between the area formula and the Lusin condition. If the mapping is a Sobolev mapping and satisfies the Lusin (N) condition, the area formula holds, for more see [10].

Marcus and Mizel prove in [9], that the Lusin (N) condition is guaranteed for the mappings in $W^{1,p}(\Omega, \mathbf{R}^n)$ for $p > n$. If we consider Sobolev homeomorphisms, the condition is valid even in $W^{1,n}(\Omega, \mathbf{R}^n)$, see Reshetnyak [16]. These positive results are sharp and the main tool of the proof is the Sobolev Embedding Theorem. Our work is based on two classical counterexamples of the functions violating the (N) condition, which complete the characterization of validity in $W^{1,p}$.

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The first counterexample is the Cesari's construction originally written in [2] for $n = 2$, later reminded and improved by Malý and Martio in [8]. That is the continuous mapping in the Sobolev space $W^{1,p}([0,1]^n,[0,1]^m)$, $p \leq n$, which maps the line segment onto a domain with positive m-dimension measure, so the (N) condition is violated.

The second counterexample is the Ponomarev's construction [15]. This is the Sobolev homeomorphism in $W^{1,p}([0,1]^n,[0,1]^n)$, $p < n$, violating the (N) condition. The construction is essentially different from Cesari's construction and cannot be obtain as a simple modification.

The limiting case $W^{1,n}$ is the most important case in both settings. There is no work considering counterexamples for higher derivative spaces $W^{k,p}$, but by Sobolev embedding it is natural to expect that the space $W^{k,\frac{n}{k}}$ is the limiting case. The positive results can be easily obtained, but both counterexamples mentioned above lack the second derivative and cannot be used. In our work we fill this gap by careful smoothing the classical constructions and we show that the space $W^{k,\frac{n}{k}}$ is limiting in both cases. Our results can be summarized by these two theorems.

Theorem 1 *Let $k, n \in \mathbf{N}$, $n \geq k$ let $p \geq \frac{n}{k}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a homeomorphism $f \in W^{k,p}(\Omega,\mathbf{R}^n)$ satisfies Lusin (N) condition.*

On the other hand for every $k, n \in \mathbf{N}$, $n > k$ and $p \in [1,\frac{n}{k})$ there is a homeomorphism $f \in W^{k,p}((-1,1)^n,\mathbf{R}^n)$ which fails Lusin (N) condition.

If we consider general Sobolev mappings and not only homeomorphisms, then the scale shifts and a mapping in $W^{k,\frac{n}{k}}$ may fail to satisfy the condition. We give a counterexample violating the condition for this case.

Theorem 2 *Let $k, m, n \in \mathbf{N}$, $n > k$, let $p > \frac{n}{k}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a mapping $f \in W^{k,p}(\Omega,\mathbf{R}^m)$ satisfies Lusin (N) condition.*

Moreover, let $m, n \in \mathbf{N}$ and let $\Omega \subset \mathbf{R}^n$ be a domain. Then a mapping $f \in W^{n,1}(\Omega,\mathbf{R}^m)$ satisfies Lusin (N) condition.

On the other hand for every $k, m, n \in \mathbf{N}$, $n > k$ and $p \in [1,\frac{n}{k}]$ there is a mapping $f \in W^{k,p}((-1,1)^n,\mathbf{R}^m)$ which fails Lusin (N) condition.

The generalizations of the positive results can be proved by the Sobolev Embedding Theorem. However the counterexamples require a new approach because the classical counterexamples are defined as non-smooth mappings and they lack even the second weak derivative. The special question is the validity of the condition in case $n = k$, $p = 1$. We answer this question by the finer version of Sobolev Embedding Theorem by Peetre [14] for the Lorentz spaces and by the result by Kauhanen, Koskela and Malý [5].

For other results concerning the research of the (N) condition in spaces close to $W^{1,n}$ see [7] and [5]. Although the classical results are not new, there are fresh applications using these constructions as the limiting case for example in the varifold theory [12], [18] or in the field of the metric measure spaces [6]. There are also works concerning Lusin (N) condition using different methods of construction in order to get particular properties such that the Sobolev homeomorphism with $J_f = 0$ almost everywhere, see [3], or even the homeomorphism satisfying $\operatorname{rank}(Df) < n$, $\operatorname{rank}(Df^{-1}) < n$, see [13].

The paper is divided into two parts. In Section 3 we prove Theorem 1 and we give the example of the homeomorphism in $W^{k,p}$. In in Section 4 we prove Theorem 2 and we give the improved Cesari example of the mapping in $W^{k,\frac{n}{k}}$.

2 Preliminaries

We denote an open cube by

$$Q(x,r) = \{y \in \mathbf{R}^n, \|x-y\|_\infty < r\}.$$

We denote an open ball with the centre at x and radius r as $B(x,r)$. We denote a sign as $\operatorname{sgn}(t)$, i.e. $\operatorname{sgn}(t) = 1$ for $t > 0$, $\operatorname{sgn}(t) = -1$ for $t < 0$, $\operatorname{sgn}(t) = 0$ for $t = 0$.

By C we denote a generic positive constant whose exact value may change at each occurrence. We write for example $C(a,b,c)$ if C may depend on parameters a, b and c. Since we fix parameters n, k and p, the dependence of C on these parameters would not be mentioned at all.

Let us consider the convolution kernel $\phi : (-1,1) \to \mathbf{R}$ such that

$$\begin{aligned}
&(1)\ \ \phi(t) \geq 0,\\
&(2)\ \ \int_{-1}^{1} \phi(t) = 1,\\
&(3)\ \ |D^k \phi(t)| \leq C(k),\\
&(4)\ \ \int_{-1}^{1} |D^k \phi(t)|\, dt \leq C(k),\\
&(5)\ \ \phi(t) \in C_0^\infty((-1,1)).
\end{aligned}$$

For $r > 0$ we define $\phi_r(t) = r^{-1}\phi(r^{-1}t)$. Function $\phi_r(t)$ satisfies

$$\begin{aligned}
&(1)\ \ \phi_r(t) \geq 0,\\
&(2)\ \ \int_{-r}^{r} \phi_r(t) = 1,\\
&(3)\ \ |D^k \phi_r(t)| \leq r^{-k-1} C(k),\\
&(4)\ \ \int_{-r}^{r} |D^k \phi_r(t)|\, dt \leq r^{-k} C(k),\\
&(5)\ \ \phi_r(t) \in C_0^\infty((-r,r)).
\end{aligned} \tag{1}$$

Now we prepare a function for the smooth partition of unity. We consider $\lambda_{a,c,A} : [0,A] \to [0,1]$ for $0 < a < c < A$ such that

$$\begin{aligned}
&(1)\ \ \lambda(t) \in C^\infty((0,A)),\\
&(2)\ \ \lambda(t) \equiv 1 \text{ for } t \in [0,a),\\
&(3)\ \ \lambda(t) \equiv 0 \text{ for } t \in (c,A],\\
&(4)\ \ |D^k \lambda(t)| \leq (c-a)^{-k} C(k) \text{ for } t \in [a,c], k \in \mathbf{N}.
\end{aligned} \tag{2}$$

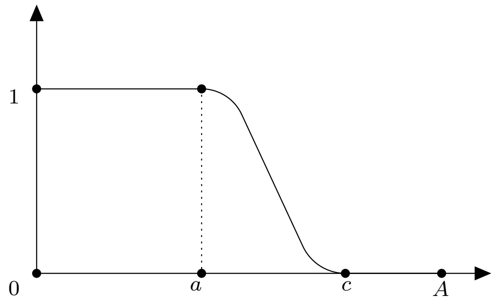

Fig. 1 Graph of $\lambda_{a,c,A}$.

We can construct such $\lambda(t)$ by connecting points $[0,1]$, $[a+\frac{c-a}{3},1]$, $[c-\frac{c-a}{3},0]$ and $[A,0]$ by the lines to form the piecewise affine function and then make this function smooth by convolution with $\phi_{\frac{c-a}{3}}$. The last estimate can be provided by Lemma 2 proven in the next part of the Section. The graph of $\lambda_{a,c,A}$ is sketched in Figure 1.

The n-dimesional Lebesgue measure of a measurable set A in $\mathbf{R}^n$ is denoted by $|A|$ or $\mathcal{L}^n(A)$. We use the formula for the derivative of the product

$$D^k(ab)(t) = \sum_{j=0}^{k} \binom{k}{j} D^j a(t) D^{k-j} b(t), \tag{3}$$

if both $D^j a(t)$, $D^j b(t)$ exist for all $j \in \{0, \dots k\}$ and the right-hand side term makes sense. For a measurable function f belonging to Lebesgue space L^p on some domain Ω we denote the norm

$$\|f\|_p = \|f\|_{L^p(\Omega)} = \Big(\int_\Omega |f(x)|^p \, dx\Big)^{\frac{1}{p}}.$$

Lemma 1 *Let $f : [-1,1]^n \to [-1,1]^n$ be a bijective continuous mapping. Then f is a homeomorphism.*

Proof In order to prove that open sets are mapped to open sets we want to prove that any closed set would be mapped to a closed set. A closed set in $[-1,1]^n$ is bounded and therefore it is a compact set and since f is continuous, the image of the compact set is a compact set or specially closed set. The mapping f is bijective, continuous and maps open sets to open sets and hence f is homeomorphism.

2.1 Convolution method

We use the convolution on some piecewise smooth function with some points of broken smoothness. We can control the value of its derivatives by the derivatives of the original function and by the radius of the convolution kernel. This control is described by the following Lemma.

Lemma 2 *Let* $\{a_i\}_{i=1}^{j+1} \subset \mathbf{R}$ *be a finite increasing sequence, let* $I = [a_1, a_{j+1}]$, $I_i = [a_i, a_{i+1}]$ *be closed intervals. Let* $h_i \in W^{k,\infty}(I_i, \mathbf{R}) \cap C^{k-1}(I_i, \mathbf{R})$. *We denote*

$$h(t) = h_i(t) \text{ for } t \in I_i.$$

Then for $t \in (a_1 + r, a_{j+1} - r)$ *we have*

$$\begin{aligned} D^k(h * \phi_r)(t) = &\sum_{i=1}^{j} \int_{B(t,r)\cap I_i} \phi_r(t-s) D^k h_i(s) ds \\ &+ \sum_{i=1}^{j-1}\Big(\sum_{l=0}^{k-1} D^{k-1-l}\phi_r(t-a_{i+1})\big(D^l h_{i+1}(a_{i+1}) - D^l h_i(a_{i+1})\big)\Big). \end{aligned} \tag{4}$$

Especially we can estimate by (1) (3) *and* (1) (4)

$$|D^k(h * \phi_r)(t)| \le C \max_{i\in\{1,\dots j\}, l=\{0,\dots k\}} r^{-k+l} \|D^l h_i\|_{L^\infty(I_i)}. \tag{5}$$

Moreover, it is sufficient to consider only i *such that* $a_i \in (t-r, t+r)$ *in* (5).

Proof Firstly we consider the case $j = 2$, $k = 2$. For $t \in (a_1 + r, a_3 - r)$ we consider a small positive u such that $t+u \in (a_1+r, a_3-r)$. We split $(t-r, t+r)$ into three (possible empty) subintervals

$$\begin{aligned} J_1(u) &= (t-r, t+r) \cap (a_1, a_2 - u), \\ J_2(u) &= (t-r, t+r) \cap (a_2, a_3), \\ S_2(u) &= (t-r, t+r) \cap (a_2 - u, a_2). \end{aligned}$$

By simple equality $\phi_r(t + u - s - u) = \phi_r(t - s)$ we calculate

$$\begin{aligned} \lim_{u\to 0_+} &\frac{(h*\phi_r)(t+u) - (h*\phi_r)(t)}{u} = \\ &= \lim_{u\to 0_+}\Big(\int_{J_1(u)} \frac{h_1(s+u) - h_1(s)}{u}\phi_r(t-s)\, ds \\ &\quad + \int_{J_2(u)} \frac{h_2(s+u) - h_2(s)}{u}\phi_r(t-s)\, ds \\ &\quad + \int_{S_2(u)} \frac{h_2(s+u) - h_1(s)}{u}\phi_r(t-s)\, ds\Big). \end{aligned}$$

Firstly we deal with the first and the second term. We can interchange the limiting and the integration process (the integrable dominating function is $\phi_r\|Dh_i\|_\infty$). The interval $J_i(u)$ slightly depends on u, but since we integrate the bounded function we can get rid of this dependence by simple calculation, we get

$$\lim_{u\to 0_+}\int_{J_i(u)} \frac{h_i(s+u) - h_i(s)}{u}\phi_r(t-s)\, ds = \int_{B(t,r)\cap I_i} Dh_i(s)\phi_r(t-s)\, ds.$$

We split the last term into two integrals and since we get the average integral of continuous function, we get

$$\lim_{u\to 0_+}\int_{S_2(u)}\frac{h_2(s+u)\phi_r(t-s)-h_1(s)\phi_r(t-s)}{u}\,ds = \\ = h_2(a_2)\phi_r(t-a_2)-h_1(a_2)\phi_r(t-a_2).$$

This gives us (4) for $k=1$. Now we iterate the previous to get the result for $k=2$. We rewrite the proven formula as

$$D(h*\phi_r)(t) = (Dh*\phi_r)(t) + (h_2-h_1)(a_2)\phi_r(t-a_2).$$

To get $D^2(h*\phi_r)$, we differentiate both terms. The first term gives us the same formula as for $D(h*\phi_r)$, we just replace h with Dh and we get

$$D(Dh*\phi_r)(t) = (D^2h*\phi_r)(t) + (Dh_2-Dh_1)(a_2)\phi_r(t-a_2).$$

The derivative of the second term is

$$D\big((h_2-h_1)(a_2)\phi_r(t-a_2)\big) = (h_2-h_1)(a_2)D\phi_r(t-a_2).$$

Together we have

$$\begin{aligned} D^2(h*\phi_r) =&(D^2h*\phi_r)(t)\\ &+(Dh_2-Dh_1)(a_2)\phi_r(t-a_2)+(h_2-h_1)(a_2)D\phi_r(t-a_2)\\ =&\sum_{i=1}^{2}\int_{B(t,r)\cap I_i}\phi_r(t-s)D^2h_i(s)\,ds\\ &+\sum_{i=1}^{1}\Big(\sum_{l=0}^{1}D^{1-l}\phi_r(t-a_{i+1})\big(D^lh_{i+1}(a_{i+1})-D^lh_i(a_{i+1})\big)\Big). \end{aligned}$$

That is (4) for $k=2$. The process how to get the formula for $k=2$ from $k=1$ can be used in general as the induction step, hence

$$\begin{aligned} &D(D^{k-1}h*\phi_r)(t) = (D^kh*\phi_r)(t)+\phi_r(t-a_2)(D^{k-1}h_2-D^{k-1}h_1)(a_2),\\ &D\big(D^m\phi_r(t-a_2)(D^lh_2-D^lh_1)(a_2)\big) = D^{m+1}\phi_r(t-a_2)(D^lh_2-D^lh_1)(a_2), \end{aligned} \tag{6}$$

as long as the we can interchange limiting and integration process by $\|D^kh_i\|_\infty<\infty$ and $D^{k-1}h_i$ is continuous. We suppose the validity of formula (4) for $k-1$

and by (6) we get

$$\begin{aligned} D^k(h*\phi_r) =& D(D^{k-1}h*\phi_r)(t) \\ &+ D\sum_{l=0}^{k-2} D^{k-2-l}\phi_r(t-a_2)(D^l h_2 - D^l h_1)(a_2) \\ =&(D^k h*\phi_r)(t) + \phi_r(t-a_2)(D^{k-1}h_2 - D^{k-1}h_1)(a_2) \\ &+ \sum_{l=0}^{k-2} D^{k-1-l}\phi_r(t-a_2)(D^l h_2 - D^l h_1)(a_2) \\ =&(D^k h*\phi_r)(t) + \sum_{l=0}^{k-1} D^{k-1-l}\phi_r(t-a_2)(D^l h_2 - D^l h_1)(a_2). \end{aligned}$$

This is formula (4) proven by induction in case $j=2$. In general case $j>2$ we have to consider more terms inside the sums of (4), but the proof is the same.

The last part (5) follows by (1), we use the estimates $|D^k\phi_r(t)| \le Cr^{-k-1}$, $\int |D^k\phi_r(t)|\,dt \le Cr^{-k}$ and the Hölder's inequality.

For later use we consider the mapping defined by several different smooth mappings on several different sub-domains of $(-1,1)^n$. We formulate an observation how we preserve the smoothness. In general we cannot assume the smoothness since the derivatives on the boundaries of the domains of the mappings does not have to be equal.

Observation 3 *Let $f_1 : \Omega_1 \to \mathbf{R}^n$ be a Sobolev mapping smooth inside Ω_1 and $f_2 : \Omega_2 \to \mathbf{R}^n$ be a Sobolev mappings smooth inside Ω_2, such that $f_1 = f_2$ for $x \in \Omega_1 \cap \Omega_2$. If $\partial\Omega_1 \subset \Omega_2$ (as in Figure 2) then we can define Sobolev mapping $f : \Omega_1 \cup \Omega_2 \to \mathbf{R}^n$ as $f = f_1$ in Ω_1 and $f = f_2$ in Ω_2 and this mapping is smooth.*

3 Proof of Theorem 1

3.1 Proof of the positive part

The proof of positive part can be found in [4, Chapter 4.2, p. 68, Theorem 4.5] for case $k=1$. For a domain $\Omega \subset \mathbf{R}^n$ the theorem claims that homeomorphism $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ satisfies Lusin (N) condition. To prove the general form we just use the Sobolev Embedding Theorem multiple times. Since the Lusin (N) condition is a local property, we can assume without loss of generality that Ω is a ball. We have

$$W^{k,\frac{n}{k}}_{\mathrm{loc}}(\Omega, \mathbf{R}^n) \subset W^{k-1,\frac{n}{k-1}}_{\mathrm{loc}}(\Omega, \mathbf{R}^n) \subset \cdots \subset W^{1,n}_{\mathrm{loc}}(\Omega, \mathbf{R}^n),$$

hence for any $k \le n$ a homeomorphism $f \in W^{k,\frac{n}{k}}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ satisfies the assumptions for the well-known version of the theorem.

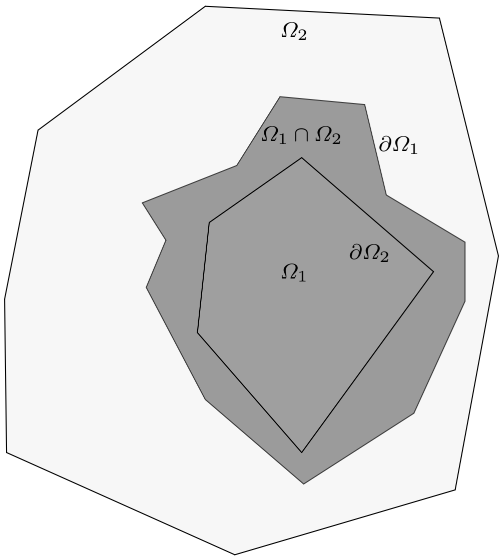

Fig. 2 Position of Ω_2 and $\partial\Omega_1$

3.2 Construction of Cantor sets for the homeomorphism f in $W^{k,p}$

In this subsection we prepare tools for the construction of a homeomorphism $f \in C \cap W^{k,p}([-1,1]^n, [-1,1]^n)$, $p < \frac{n}{k}$, such that f is an identity on the boundary and f does not satisfy Lusin (N) condition. Moreover, f is C^∞ a. e. in $[-1,1]^n$, in fact outside the Cantor type set of 0 measure.

Based on $k, n \in \mathbf{N}$, $p \in [1, \frac{n}{k})$ we choose A, B such that

$$A > \frac{kp}{n - kp} \text{ and } B > 1. \tag{7}$$

We denote

$$\begin{aligned} a_i &:= 2^{-i} 2^{-Ai}, \\ b_i &:= 2^{-i}(2^{-1} + 2^{-Bi-1}). \end{aligned} \tag{8}$$

We recall the construction used in [4, Chapter 4.3, p. 69, Theorem 4.10]. We will first give two Cantor-set constructions in $(-1,1)^n$. Our mapping f will be defined as the limit of the sequence of smooth homeomorphisms $f_i : (-1,1)^n \to (-1,1)^n$, where each f_i maps the i-th step of the first Cantor-set construction onto that of the second. Then the limit mapping f maps the first Cantor set onto the second one.

By $\mathbb{V}$ we denote the set of the 2^n vertices of the cube $[-1,1]^n$, we can index this set $\mathbb{V} = \{\vartheta^1, \vartheta^2, \ldots \vartheta^{2^n}\}$. The sets $\mathbb{V}^i = \mathbb{V} \times \ldots \times \mathbb{V}$, $i \in \mathbf{N}$, will serve as the sets of indices for our construction. Let us set $z_0 = \tilde{z}_0 = 0$.

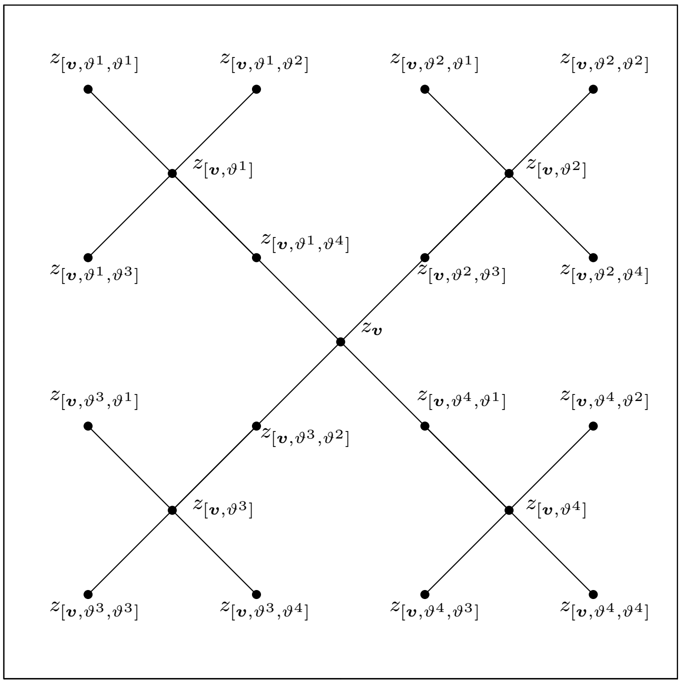

Fig. 3 Structure of centres $z_{\boldsymbol{v}}$

It follows that $(-1,1)^n = Q(z_0, a_0)$ and we proceed by induction. For $\boldsymbol{v} = [v_1, \dots, v_i] \in \mathbb{V}^i$ we denote $\boldsymbol{w} = [v_1, \dots, v_{i-1}]$ and we define

$$z_{\boldsymbol{v}} = z_{\boldsymbol{w}} + \frac{1}{2}(a_{i-1})v_i = z_0 + \sum_{j=1}^{i} \frac{1}{2}(a_{j-1})v_j,$$

$$Q'_{\boldsymbol{v}} = Q(z_{\boldsymbol{v}}, \frac{1}{2}a_{i-1}) \text{ and } Q_{\boldsymbol{v}} = Q(z_{\boldsymbol{v}}, a_i).$$

The decomposition of $Q_{\boldsymbol{v}}$ into the cubes with higher index is sketched in Figure 4. Formally we should write $\boldsymbol{w}(\boldsymbol{v})$ instead of $\boldsymbol{w}$ but for simplicity of notation we neglect this. The number of the cubes in $\{Q_{\boldsymbol{v}} : \boldsymbol{v} \in \mathbb{V}^i\}$ is 2^{ni}. It is not difficult to show that the resulting Cantor set

$$\bigcap_{i=1}^{\infty} \bigcup_{\boldsymbol{v} \in \mathbb{V}^i} Q_{\boldsymbol{v}} =: C_A = C_a \times \ldots \times C_a$$

is a product of n Cantor sets in $\mathbf{R}$. Moreover, $\mathcal{L}_n(C_A) = 0$ since

$$\mathcal{L}_n\big(\bigcup_{\boldsymbol{v} \in \mathbb{V}^i} Q_{\boldsymbol{v}}\big) \geq 2^{ni}(2^{-iA-i})^n \overset{i\to\infty}{\to} 0.$$

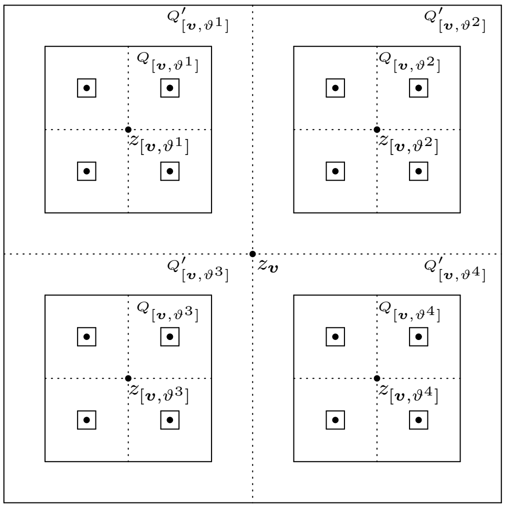

Fig. 4 $Q_{\boldsymbol{v}}$ and its decomposition

Analogously we define

$$\tilde{z}_{\boldsymbol{v}} = \tilde{z}_{\boldsymbol{w}} + \frac{1}{2}b_{i-1}v_i = \tilde{z}_0 + \frac{1}{2}\sum_{j=1}^{i} b_{j-1}v_j,$$

$$\tilde{Q}'_{\boldsymbol{v}} = Q(\tilde{z}_{\boldsymbol{v}}, \frac{1}{2}b_{i-1}) \text{ and } \tilde{Q}_{\boldsymbol{v}} = Q(\tilde{z}_{\boldsymbol{v}}, b_i).$$

The resulting Cantor set

$$\bigcap_{i=1}^{\infty} \bigcup_{\boldsymbol{v}\in\mathbb{V}^i} \tilde{Q}_{\boldsymbol{v}} =: C_B = C_b \times \ldots \times C_b$$

satisfies $\mathcal{L}_n(C_B) > 0$ since $\lim_{i\to\infty} 2^i b_i > 0$. It remains to find a homeomorphism f which maps C_A onto C_B and satisfies our assumptions. By $f(C_A) = C_B$, f does not satisfy the condition (N) since $\mathcal{L}_n(C_A) = 0$ and $\mathcal{L}_n(C_B) > 0$.

3.3 Basic functions for the construction of the homeomorphism f in $W^{k,p}$

In this part we prepare functions and mappings in order to construct the sequence of the suitable smooth homeomorphisms f_i converging uniformly to f. The desired property is $f_i(Q_{\boldsymbol{v}}) = \tilde{Q}_{\boldsymbol{v}}$ for every $\boldsymbol{v} \in \mathbb{V}^i, i \in \mathbf{N}$.

We denote a constant

$$\alpha_i = \frac{\frac{b_{i-1}}{2} - b_i - \left(\frac{b_{i-1}}{a_{i-1}} + \frac{b_i}{a_i}\right)\frac{a_{i+1}}{2}}{\frac{a_{i-1}}{2} - a_i - a_{i+1}}, \tag{9}$$

in order to define a function

$$l_i(t) := \alpha_i(t - a_i - \frac{a_{i+1}}{2}) + b_i + \frac{a_{i+1}b_i}{2a_i}.$$

The graph of this affine function connects the points $(a_i + \frac{a_{i+1}}{2}, b_i + \frac{a_{i+1}b_i}{2a_i})$ and $(\frac{a_{i-1}}{2} - \frac{a_{i+1}}{2}, \frac{b_{i-1}}{2} - \frac{a_{i+1}b_{i-1}}{2a_{i-1}})$, see Figure 5. We define the sequence of continuous functions $h_i^* : [-\frac{a_{i+1}}{2}, \frac{a_{i-1}+a_{i+1}}{2}] \to \mathbf{R}$ as

$$h_i^*(t) = \begin{cases} \frac{b_i}{a_i}t & \text{for } t \in [-\frac{a_{i+1}}{2}, a_i + \frac{a_{i+1}}{2}], \\ l_i(t) & \text{for } t \in [a_i + \frac{a_{i+1}}{2}, \frac{a_{i-1}-a_{i+1}}{2}], \\ \frac{b_{i-1}}{a_{i-1}}t & \text{for } t \in [\frac{a_{i-1}-a_{i+1}}{2}, \frac{a_{i-1}+a_{i+1}}{2}]. \end{cases} \tag{10}$$

We sketch the graph of $h_i^*(t)$ in Figure 5. The important property is the strict monotonicity of $h_i^*(t)$. We define the sequence of smooth functions

$$h_i(t) = (h_i^* * \phi_{\frac{a_{i+1}}{4}})(t) \text{ for } t \in \big[0, \frac{a_{i-1}}{2}\big). \tag{11}$$

We can see that these functions are smooth and strictly monotone, and its derivatives can be calculated and estimated by Lemma 2. Moreover, h_i is linear inside $[0, a_i)$ and it is linear inside $(\frac{a_{i-1}}{2} - \frac{a_{i+1}}{4}, \frac{a_{i-1}}{2})$. All these properties and the relation with $h_i^*(t)$ are sketched in Figure 5.

We use this function to define mapping $g_i^* : Q(0, \frac{a_{i-1}}{2}) \to Q(0, \frac{b_{i-1}}{2})$ by coordinates as

$$(g_i^*(x))_j = \operatorname{sgn}(x_j)h_i(|x_j|) \text{ for } j \in \{1, \dots n\}. \tag{12}$$

This mapping maps $Q(0, \frac{a_{i-1}}{2})$ onto $Q(0, \frac{b_{i-1}}{2})$ and its shifted version can map $Q'_{\boldsymbol{v}}$ onto $\tilde{Q}'_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^i$. It is clearly continuous, smooth and it is strictly monotone in every direction and therefore it is one-to-one and homeomorphism. Moreover, the j-th coordinate of $g_i(x)$ depends only on the j-th coordinate of x, so the only non-zero coordinate of partial derivatives of $D_j^k g_i$ may be the diagonal ones for $j \in \{1, \dots n\}$, $k \in \mathbf{N}$. We have to modify this mapping once more in order to get $g_i(x) = \frac{b_{i-1}}{a_{i-1}}x$ near the boundary of the cube $\|x\|_\infty = \frac{a_{i-1}}{2}$.

We recall (2) and we denote

$$\lambda(t) = \lambda_{\frac{a_{i-1}}{2} - \frac{a_{i+1}}{4}, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8}, \frac{a_{i-1}}{2}}(t).$$

For $x \in Q(0, \frac{a_{i-1}}{2})$ we define

$$g_i(x) = \prod_{j=1}^n \lambda(|x_j|)g_i^*(x) + \big(1 - \prod_{j=1}^n \lambda(|x_j|)\big)\frac{b_{i-1}}{a_{i-1}}x. \tag{13}$$

This mapping still maps $Q(0, \frac{a_{i-1}}{2})$ onto $Q(0, \frac{b_{i-1}}{2})$. Since both $g_i^*(x)$ and $\lambda(t)$ are smooth, $g_i(x)$ is also smooth. Moreover, g_i is equal to $\frac{b_{i-1}}{a_{i-1}}x$ near the

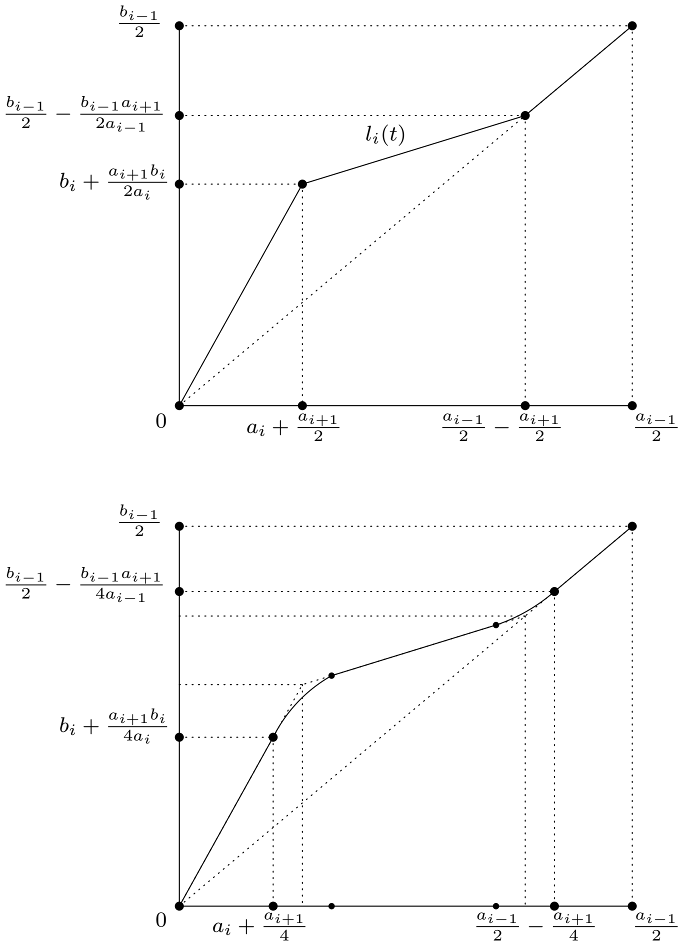

Fig. 5 Functions $h_i^*(t)$ and $h_i(t)$

boundary of $Q(0, \frac{a_{i-1}}{2})$, since $\prod_{j=1}^n \lambda(|x_j|) = 0$ if any $|x_j| > \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8}$. We note that $\prod_{j=1}^n \lambda(|x_j|) \in (0,1)$ only in the set

$$T_i = Q\Big(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8}\Big) \setminus Q\Big(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4}\Big). \tag{14}$$

Both $g_i^*(x)$ and $\frac{b_{i-1}}{a_{i-1}}x$ are strictly increasing in every coordinate and therefore bijective, we claim that $g_i(x)$ is also bijective. To prove this we have to focus only on the set T_i, elsewhere g_i is equal to one of the bijective mappings $g_i^*(x)$

or $\frac{b_{i-1}}{a_{i-1}}x$. We should also show that

$$g_i(T_i) = Q\Big(0, \frac{b_{i-1}}{2} - \frac{b_{i-1}a_{i+1}}{8a_{i-1}}\Big) \setminus Q\Big(0, \frac{b_{i-1}}{2} - \frac{b_{i-1}a_{i+1}}{4a_{i-1}}\Big),$$

we do not do the calculations here, it is straightforward after we check the bijectivity.

Remark 1 We can avoid the following discussion by the estimate of the Jacobian $|Jg_i| > 0$. Since this holds in whole $Q(0, \frac{a_{i-1}}{2})$, we can use advanced topological degree theory to claim the bijectivity, see [17, p. 17, Proposition 4.4]. We present the discussion for convenience of the reader not familiar with this theory.

We discuss if it is possible to have $g_i(x) = g_i(y)$ for $x \neq y$, $x, y \in T_i$ to get a contradiction. We split the indices of coordinates of $x \in T_i$ into two sets S_x and R_x such that

$$\begin{aligned}
&S_x \cup R_x = \{1, \dots n\}, \\
&x_s \in (-\tfrac{a_{i-1}}{2} + \tfrac{a_{i+1}}{8}, \tfrac{a_{i-1}}{2} - \tfrac{a_{i+1}}{8}) \setminus (-\tfrac{a_{i-1}}{2} - \tfrac{a_{i+1}}{4}, \tfrac{a_{i-1}}{2} - \tfrac{a_{i+1}}{4}) \qquad \text{for } s \in S_x, \\
&x_r \in (-\tfrac{a_{i-1}}{2} + \tfrac{a_{i+1}}{4}, \tfrac{a_{i-1}}{2} - \tfrac{a_{i+1}}{4}) \qquad \text{for } r \in R_x.
\end{aligned}$$

This sets can provide us the decomposition of T_i based on the belonging of indices of $x \in T_i$ to S_x and R_x. This decomposition in case $n = 2$ is sketched in Figure 6. In general, there are n-dimensional cubes by the vertices of $Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4})$, where $R_x = \emptyset$ and $S_x = \{1, \dots n\}$. The rest of T_i is split into n-dimensional hyper-rectangles with both R_x and S_x non-empty. We intend to prove that $g(x)$ is a homeomorphism in each component of decomposed T_i and maps this component onto the corresponding component in the image. This imply $g(x)$ is a homeomorphism in whole T_i. Further we consider only $x, y \in T_i, x \neq y$ lying inside the same component, it means $R_x = R_y, S_x = S_y$.

For x_s, $s \in S_x$, we have by (13), (12), (11) and (10) that $h_i(|x_s|) = \frac{b_{i-1}}{a_{i-1}}|x_s|$ and hence

$$\begin{aligned}
(g_i(x))_s &= \prod_{j=1}^n \lambda(|x_j|)(g_i^*(x))_s + (1 - \prod_{j=1}^n \lambda(|x_j|))\frac{b_{i-1}}{a_{i-1}}x_s \\
&= \prod_{j=1}^n \lambda(|x_j|)\operatorname{sgn}(x_s)h_i(|x_s|) + (1 - \prod_{j=1}^n \lambda(|x_j|))\frac{b_{i-1}}{a_{i-1}}x_s = \frac{b_{i-1}}{a_{i-1}}x_s.
\end{aligned} \tag{15}$$

This imply that $g_i(x) = \frac{b_{i-1}}{a_{i-1}}x$ in parts where $R_x = \emptyset$ and it is clearly a homeomorphism. Let us consider a part where $R_x \neq \emptyset$. Let $s \in S_x$, then we get $(g_i(x))_s = \frac{b_{i-1}}{a_{i-1}}x_s$ so we can suppose that $x_s = y_s$ for all $s \in S_x = S_y$. Now we discuss if $g(x) = g(y)$ is possible for some x and y belonging to the same $|R_x|$-dimensional rectangle defined by

$$\{z \in T_i : R_z = R_x; z_s = x_s \text{ for } s \in S_x\}. \tag{16}$$

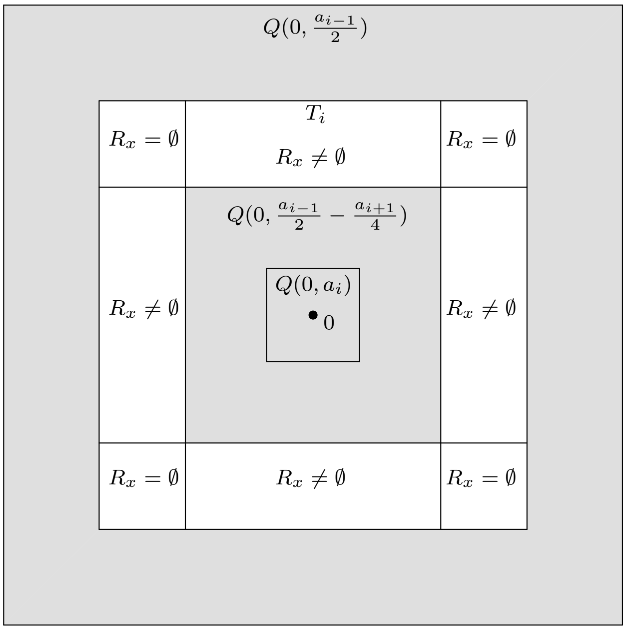

Fig. 6 Set T_i and its splitting

Specially for $n = 2$, we get line segments since both S_x and R_x are sets of one element.

By the definition of R_x we have $\lambda(|x_r|) = 1$ for $r \in R_x$, hence

$$(g_i(x))_r = \prod_{s \in S_x} \lambda(|x_s|)(g_i^*(x))_r + \Big(1 - \prod_{s \in S_x} \lambda(|x_s|)\Big)\frac{b_{i-1}}{a_{i-1}} x_r.$$

For $m, r \in R_x$, $m \neq r$ we get

$$\begin{aligned} D_r(g_i(x))_r &= \prod_{s \in S_x} \lambda(|x_s|) Dh_i(|x_r|) + \Big(1 - \prod_{s \in S_x} \lambda(|x_s|)\Big)\frac{b_{i-1}}{a_{i-1}}, \\ D_m(g_i(x))_r &= 0. \end{aligned} \tag{17}$$

The derivative $D_r(g_i(x))_r$ is positive, since it is a convex combination of two positive numbers. This observation gives us the contradiction for case $|R_x| = 1$ as $(g_i(x))_r$ is increasing and $(g_i(x))_r = (g_i(y))_r$ implies $x_r = y_r$. Hence the proof is finished in case $n = 2$. In general case we study the rectangle defined by (16). Restriction of g_i on this rectangle has a diagonal Jacobi matrix with positive numbers on diagonal positions and hence it is a strictly monotone mapping in every direction and therefore it is a bijection. This gives us the contradiction and g_i is bijective in every part of T_i, therefore in whole T_i and in whole $Q(0, \frac{a_{i-1}}{2})$.

For later use we estimate the derivatives of $g_i(x)$ inside the set T_i defined in (14). We recall (8) to get the comparability of a_i and a_{i+1} by estimates $a_{i+1} < a_i$ and $a_{i+1} > Ca_i$. For $x \in T_i$ we estimate by (2)(4) and (3)

$$\Big|D^m\Big(\prod_{s\in S}\lambda(|x_s|)\Big)\Big| \le C\max\Big\{\prod_{s\in S}\big|D^{\gamma_s}\lambda(|x_s|)\big|; \gamma_s\in\mathbf{N}_0, \sum_{s\in S}\gamma_s = m)\Big\} \le Ca_i^{-m}.$$

By (15) and (17) we estimate $|D^k(g_i(x))|$ for $k>1$, for $x\in T_i$ as

$$\begin{aligned}
|D^k(g_i(x))| \le &\max_{r\in R_x, d\in S_x}\Big\{\big|D^k((g_i(x))_d)\big|, \big|D^k((g_i(x))_r)\big|\Big\}\\
\le &\max_{r\in R_x, d\in S_x}\Big\{\big|D^k(\frac{b_{i-1}}{a_{i-1}}x_d)\big|,\\
&\qquad \Big|D^k\Big(\prod_{s\in S}\lambda(|x_s|)h_i(|x_r|) + \big(1-\prod_{s\in S}\lambda(|x_s|)\big)\frac{b_{i-1}}{a_{i-1}}x_r\Big)\Big|\Big\}\\
\le &C\max_{r\in R_x, 0\le m\le k}\Big\{0, \big|D^m\big(\prod_{s\in S}\lambda(|x_s|)\big)D^{k-m}h_i(|x_r|)\big|, \Big|D^{k-1}\prod_{s\in S}\lambda(|x_s|)\frac{b_{i-1}}{a_{i-1}}\Big|,\\
&\qquad \Big|D^k\big(\prod_{s\in S}\lambda(|x_s|)\big)\frac{b_{i-1}}{a_{i-1}}x_r\Big|\Big\}\\
\le &C\max_{r\in R_x, 0\le m\le k}\Big\{|a_i^{-m}D^{k-m}h_i(|x_r|)|, \big|a_i^{-k+1}\frac{b_{i-1}}{a_{i-1}}\big|, |a_i^{-k}b_{i-1}|\Big\}.
\end{aligned} \tag{18}$$

3.4 Construction of the homeomorphism f in $W^{k,p}$

We will give the sequence of smooth homeomorphisms $f_i : (-1,1)^n \to (-1,1)^n$. We set $f_0(x) = x$ and we proceed by induction. Firstly we give a mapping f_1 which stretches each cube Q_v, $v\in\mathbb{V}^1$, homogeneously so that $f_1(Q_v) = \tilde{Q}_v$. We also need $f_1(Q'_v\setminus Q_v) = \tilde{Q}'_v\setminus\tilde{Q}_v$. We define

$$f_1(x) = \begin{cases} f_0(x) = x & \text{for } x\in(-1,1)^n\setminus\bigcup_{\boldsymbol{v}\in\mathbb{V}}Q'_{\boldsymbol{v}},\\ z_{\boldsymbol{v}} + g_1(x - z_{\boldsymbol{v}}), & \text{for } x\in Q'_{\boldsymbol{v}},\ \boldsymbol{v}\in\mathbb{V}. \end{cases}$$

We check the desired properties, f_1 is bijective and smooth inside both parts of the domain of function. The bijectivity inside $Q'_{\boldsymbol{v}}$ is given by the bijectivity of g_1 proven in previous Subsection 3.3, the property $f_1(Q'_{\boldsymbol{v}}) = \tilde{Q}'_{\boldsymbol{v}}$ follows from (13). In order to prove the smoothness we check the assumptions of Observation 3. We have $h_1^*(t) = \frac{a_0}{b_0}t = t$ for $t\in(\frac{a_0-a_2}{2}, \frac{a_0+a_2}{2})$, $h_1(t) = t$ for $t\in(\frac{a_0}{2}-\frac{a_2}{4}, \frac{a_0}{2})$. We have $z_v = \tilde{z}_v$ for $v\in\mathbb{V}^1$ and therefore we get $f_1(x) = f_0(x) = x$ inside annuli $Q(z_v, \frac{a_0}{2})\setminus Q(z_v, \frac{a_0}{2}-\frac{a_2}{8})$. So we can extend the set where $f_1(x) = f_0(x) = x$ to the set $(-1,1)^n\setminus\bigcup_{\boldsymbol{v}\in\mathbb{V}}Q(z_{\boldsymbol{v}}, \frac{a_0}{2}-\frac{a_2}{4})$ as is required in Observation 3 and hence f_1 is smooth in whole $(-1,1)^n$.

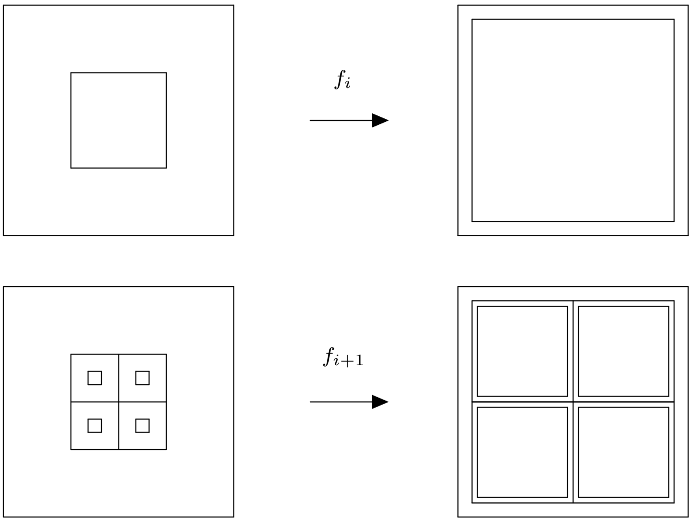

Fig. 7 Modifying f_i into f_{i+1}

This first step also shows us the idea of the induction process. We define

$$f_i(x) = \begin{cases} f_{i-1}(x) & \text{for } x \in (-1,1)^n \setminus \bigcup_{\boldsymbol{v}\in\mathbb{V}^i} Q'_{\boldsymbol{v}}, \\ f_{i-1}(z_{\boldsymbol{v}}) + g_i(x - z_{\boldsymbol{v}}) & \text{for } x \in Q'_{\boldsymbol{v}},\ \boldsymbol{v} \in \mathbb{V}^i. \end{cases} \tag{19}$$

In the general step we change the $f_i(x)$ only inside the cubes $Q'_{\boldsymbol{v}}$, $\boldsymbol{v} \in \mathbb{V}^i$. This f_i stretches each cube $Q_{\boldsymbol{v}}$ homogeneously onto $\tilde{Q}_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^i$. Moreover our definition of f_i will coincide with f_{i-1} on some neighbourhood of the boundary of $\bigcup_{\boldsymbol{v}\in\mathbb{V}^i} Q'_{\boldsymbol{v}}$ and hence f_i is smooth in whole $(-1,1)^n$ by Observation 3, as we have showed for f_1.

It is not difficult to check that each f_i is a homeomorphism by the bijectivity of g_i. Moreover, f_i satisfies

$$f_i\Big(\bigcup_{\boldsymbol{v}\in\mathbb{V}^i} Q_{\boldsymbol{v}}\Big) = \bigcup_{\boldsymbol{v}\in\mathbb{V}^i} \tilde{Q}_{\boldsymbol{v}}.$$

We illustrate the induction step in Figure 7, we sketch how the squares $Q_{\boldsymbol{v}}$ are homogeneously mapped and how we place the new generations of $Q_{\boldsymbol{v},v_{i+1}}$ and $Q'_{\boldsymbol{v},v_{i+1}}$ inside them.

We define

$$f(x) = \lim_{i\to\infty} f_i(x).$$

The mapping satisfies $f(C_A) = C_B$ from previous step and from the definition of these Cantor sets. Moreover, it is continuous as the uniform limit of the continuous mappings. Mapping $f(x)$ is clearly one-to-one inside these Cantor sets. To check the one-to-one property outside these Cantor sets, we may consider every set $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$ for the same index $\boldsymbol{v} \in \mathbb{V}^i, i \in \mathbf{N}$ separately. These sets cover whole $(-1,1)^n \setminus C_A$. We know that $f = f_i$ is homeomorphism inside $Q'_{\boldsymbol{v}} \setminus Q_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \mathbb{V}^i, i \in \mathbf{N}$. Since f is bijective in every part we consider and the images of these sets do not collide and fill $(-1,1)^n$, f is bijective.

The limit mapping f is continuous and one-to-one, therefore it is a homeomorphism by Lemma 1. It remains to show that the norm $\|D^k f\|_p$ is finite.

3.5 Finiteness of the norm of the homeomorphism f in $W^{k,p}$

Before the proof, we estimate the derivatives of h_i in $[0, \frac{a_{i-1}}{2}]$, $i \in \mathbf{N}$. By Lemma 2 and (11) we have

$$|D^k h_i| \leq C \max_{l=\{0,\dots k\}} (a_{i+1})^{-k+l} \|D^l h_i^*\|_\infty.$$

We consider only the case $k > 1$. Since h_i^* is defined as the continuous piecewise affine function, the only non-zero derivatives would be the zeroth and the first, all others would be zero. We estimate the pointwise values of h_i^* and Dh_i^* by definition (10). By definition of α_i (9) we can see that $\alpha_i \leq \min\{\frac{b_{i-1}}{a_{i-1}}, \frac{b_i}{a_i}\}$ from the meaning of α_i in the graph as is shown in Figure 5. We get

$$\begin{aligned} |h_i^*(t)| &\leq Cb_{i-1} \text{ for } t \in [-\frac{a_{i+1}}{2}, \frac{a_{i-1}+a_{i+1}}{2}], \\ |Dh_i^*(t)| &\leq \frac{b_i}{a_i} \text{ for } t \in [-\frac{a_{i+1}}{2}, \frac{a_{i-1}+a_{i+1}}{2}]. \end{aligned}$$

Together with previous estimates and (8) we get

$$|D^k h_i| \leq C \max_{l=\{0,1\}} (a_{i+1})^{-k+l} \|D^l h_i^*\|_\infty \leq C(A) a_i^{-k} b_i \leq C 2^{(A+1)ik-i}. \tag{20}$$

We estimate the norm of $D^k f$ by the sum

$$\|D^k f\|_p^p \leq \|D^k f_0\|_p^p + \sum_{i=1}^{\infty} \|D^k f_i - D^k f_{i-1}\|_p^p = \sum_{i=1}^{\infty} \|D^k f_i - D^k f_{i-1}\|_p^p, \tag{21}$$

since f_0 is identity. We consider one member of the sum and we observe by (19), that the set where $f_i \neq f_{i-1}$ can be covered by 2^{ni} cubes $Q'_{\boldsymbol{v}}, \boldsymbol{v} \in \mathbb{V}^i$ of measure

$$|Q'_{\boldsymbol{v}}| = (a_{i-1})^n = 2^{-n(i-1)-An(i-1)}. \tag{22}$$

For $y \in Q'_{\boldsymbol{v}}$ by (19) we have $f_i(y) = f_{i-1}(z_{\boldsymbol{v}}) + g_i(y - z_{\boldsymbol{v}})$. We know by (19) that $f_{i-1}(y)$ is an affine function inside $Q'_{\boldsymbol{v}}$, so its higher derivatives are 0. We denote $x = y - z_{\boldsymbol{v}} \in Q(0, \frac{a_{i-1}}{2})$, we get

$$|D^k f_i(y) - D^k f_{i-1}(y)| = |D^k f_i(y) - 0| = |D^k f_i(x + z_{\boldsymbol{v}})| = |D^k g_i(x)|. \tag{23}$$

Now we discuss the possible values of $D^k g_i$ inside $Q(0, \frac{a_{i-1}}{2})$. By (13), we split $Q(0, \frac{a_{i-1}}{2})$ into three subsets, inner cube $Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4})$, middle part T_i and outer annulus $Q(0, \frac{a_{i-1}}{2}) \setminus Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8})$. Inside the inner part $Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4})$, all $\lambda(|x_j|) \equiv 1$ and we have

$$g_i(x) = g_i^*(x) = \big(\mathrm{sgn(x_1)}\, h_i(|x_1|), \mathrm{sgn(x_2)}\, h_i(|x_2|), \dots \mathrm{sgn(x_n)}\, h_i(|x_n|)\big).$$

Since every coordinate $(g_i(x))_j$ depends only on the coordinate x_j, the only nonzero partial derivatives would be the diagonal ones. We have to differentiate multiple times in the same direction only so we get

$$|D^k g_i(x)| = \max_{j=1,\dots n}\{|D^k (g_i^*(x))_j|\} = \max_{j=1,\dots n}\{|D^k h_i(|x_j|)|\}.$$

We use (20) to estimate this term

$$|D^k g_i(x)| \leq C 2^{(A+1)ik-i}.$$

The middle part of $Q(0, \frac{a_{i-1}}{2})$ is the annulus $T_i = Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8}) \setminus Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{4})$ described by (14) and we have prepared the estimate (18). We use (20) and (8) and we get

$$\begin{aligned}
|D^k (g_i(x))| \leq& C \max_{r\in R_x, 0\leq m\leq k}\Big\{|a_i^{-m} D^{k-m} h_i(|x_r|)|, \Big|a_i^{-k+1}\frac{b_{i-1}}{a_{i-1}}\Big|, |a_i^{-k} b_{i-1}|\Big\}.\\
\leq& C \max_{0\leq m\leq k}\big\{|2^{(A+1)im} 2^{(A+1)i(k-m)-i}, 2^{(A+1)ik} 2^{-(i-1)}\big\}\\
\leq& C 2^{(A+1)ik-i}.
\end{aligned}$$

The outer part of $Q(0, \frac{a_{i-1}}{2})$ is annulus $Q(0, \frac{a_{i-1}}{2}) \setminus Q(0, \frac{a_{i-1}}{2} - \frac{a_{i+1}}{8})$, where at least one $\lambda(|x_j|)$ from formula (13) is zero. So we get

$$g_i(x) = 0 g_i^*(x) + 1 \frac{b_{i-1}}{a_{i-1}} x = \frac{b_{i-1}}{a_{i-1}} x,$$

and $D^k(g_i(x)) = 0$ for any $k > 1$. We combine these pointwise estimates for all three parts of $Q(0, \frac{a_{i-1}}{2})$ and we get for any $x \in Q(0, \frac{a_{i-1}}{2})$

$$|D^k g_i(x)| \leq C \max\{2^{(A+1)ik-i}, 2^{(A+1)ik-i}, 0\} \leq C 2^{(A+1)ik-i}.$$

Together with (22) and (23) we get

$$\begin{aligned}
\sum_{i=1}^{\infty} \|D^k f_i - D^k f_{i-1}\|_p^p &\leq \sum_{i=1}^{\infty} \sum_{\boldsymbol{v}\in\mathbb{V}^i} \int_{Q'_{\boldsymbol{v}}} |D^k f_i(y) - D^k f_{i-1}(y)|^p \, dy\\
&\leq C \sum_{i=1}^{\infty} \sum_{\boldsymbol{v}\in\mathbb{V}^i} \int_{Q'_{\boldsymbol{v}}} (2^{Aik+ik-i})^p \, dy\\
&\leq C \sum_{i=1}^{\infty} 2^{ni} 2^{-n(i-1)-An(i-1)} (2^{Aik+ik-i})^p\\
&\leq C \sum_{i=1}^{\infty} 2^{(A(kp-n)+kp)i-ip}.
\end{aligned}$$

As we apply the condition for A (7), we see that the term $A(kp-n)+kp$ is negative and the sum is finite. Together with (21) we get

$$\|D^k f\|_p^p \le \sum_{i=1}^{\infty} \|D^k f_i - D^k f_{i-1}\|_p^p \le C \sum_{i=1}^{\infty} 2^{(A(kp-n)+kp)i-ip} < \infty.$$

4 Proof of Theorem 2

4.1 Proof of the positive part

Let us remind, that the Lusin (N) condition is guaranteed for the continuous mappings in $W^{1,p}(\Omega,\mathbf{R}^n)$ for $p>n$ by the result of Marcus and Mizel [9]. To prove the general form we just use the Sobolev Embedding Theorem multiple times as we have done in Subsection 3.1. As we study the local property, once again we suppose without the loss of generality that Ω is a ball. We have

$$W^{k,\frac{p}{k}}(\Omega,\mathbf{R}^n) \subset W^{k-1,\frac{p}{k-1}}(\Omega,\mathbf{R}^n) \subset \cdots \subset W^{1,p}(\Omega,\mathbf{R}^n),$$

hence for any $k<n$, $p>n$ a mapping $f \in W^{k,\frac{p}{k}}(\Omega,\mathbf{R}^n)$ satisfies the assumptions for the well-known version of the theorem.

Now we consider the special case $n=k$, $p=1$. We consider domain $\Omega \subset \mathbf{R}^n$. We recall the result by Peetre [14] that Sobolev and Lorentz spaces are embedded as

$$W^1L^{p,q}(\Omega) \subset L^{p^*,q}(\Omega).$$

We repeat this argument $n-1$ times to get

$$W^nL^{1,1}(\Omega) \subset W^{n-1}L^{1^*,1}(\Omega) \subset \cdots \subset W^1L^{n,1}(\Omega).$$

It follows that

$$D^n f \in L^1 \Rightarrow f \in WL^{n,1}.$$

By the paper [5], mappings in $WL^{n,1}$ have continuous representative and even satisfy the Lusin (N) condition.

4.2 Classical counterexample in $W^{1,n}$

We recall the classic counterexample by Malý and Martio [8] on $W^{1,n}$, the continuous mapping that maps a line segment $[-1,1]\times\{0\}^{n-1}$ onto the whole $[-1,1]^m$. We briefly remind this construction and then we improve the properties of the mapping. The key step is finding the function such that

$$\begin{aligned}&\forall\varepsilon>0\,\forall r>0\,\exists\varrho>0\,\exists h\in W_0^{1,n}(B(0,r))\cap\overline{C(B(0,r))},\\&\text{such that } h(x)\equiv 1 \text{ on } B(0,\varrho) \text{ and } \|Dh\|_n^n<\varepsilon.\end{aligned}$$

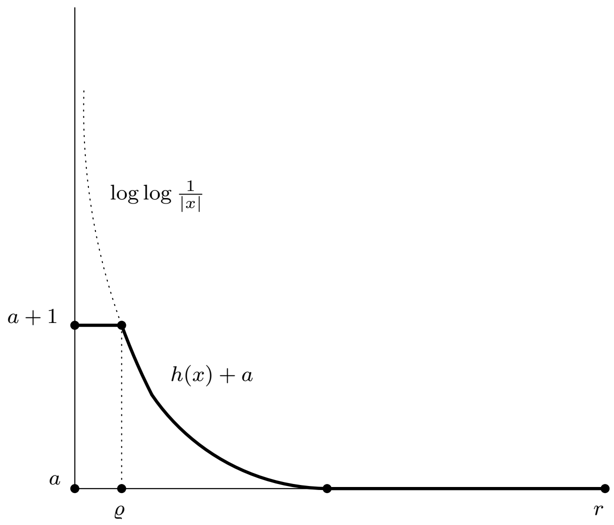

Fig. 8 Function $h(x)$

There is such a function, for a big parameter $a > \log\log\frac{1}{r}$ we set

$$h(x) = \min\Big\{1, \Big(\log\big(\log(\frac{1}{|x|})\big) - a\Big)^+\Big\}.$$

We sketch the graph of $h(x)$ at Figure 8, it is obvious that the function is continuous but not smooth. We estimate its norm as

$$\begin{aligned}\|Dh\|_n^n = \int_{B(0,r)} |Dh(x)|^n\, dx &\leq C \int_0^r t^{n-1} \Big|\big(\log\log\frac{1}{|t|}\big)'\Big|^n\, dt \\ &\leq C \int_0^r t^{-1} \frac{1}{\log^n(\frac{1}{t})}\, dt.\end{aligned}$$

The term on the right hand side tends to zero as r tends to zero and hence it is smaller then ε for r small enough.

Let us consider a sequence of such mappings $\{h_i\}_{i=1}^{\infty}$ so that corresponding parameters satisfy

- $r_1 < 2^{-m}$,
- $r_i < 2^{-m}\varrho_{i-1}$,
- $\varepsilon_i < 4^{-mi}$.

Let $m \in \mathbf{N}$ and define $\mathbb{V}$ the vertices of the cube $[-1,1]^m$ similarly to the beginning of Subsection 3.2. For $\boldsymbol{v} \in \mathbb{V}^i$, $\boldsymbol{w} \in \mathbb{V}^{i-1}$ such that $\boldsymbol{v} = [\boldsymbol{w}, v_i]$ we redefine

$$z_0 = 0 \text{ and } z_{\boldsymbol{v}} = z_{\boldsymbol{w}} + 2^{-i}v_i = z_0 + \sum_{j=1}^{i} 2^{-j}v_j.$$

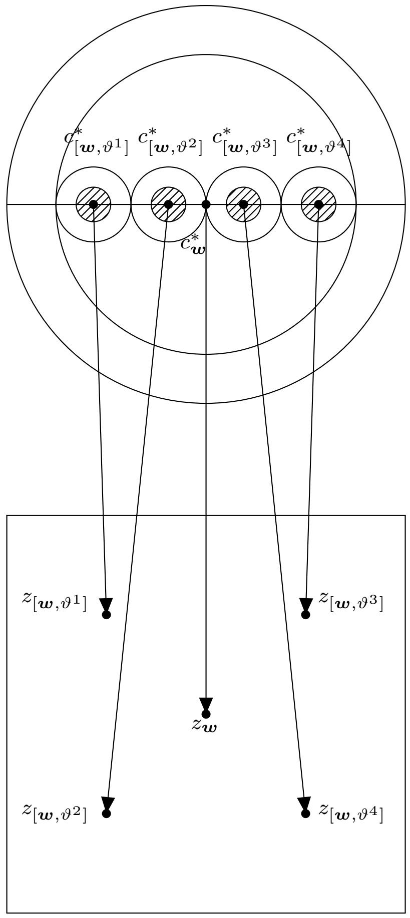

Fig. 9 Step in mapping a line into the structure $z_{\boldsymbol{v}}$

The set $\bigcup_{i=1}^{\infty} \bigcup_{\boldsymbol{v}\in\mathbb{V}^i} z_{\boldsymbol{v}}$ is dense in $[-1,1]^m$ as it is sketched in Figure 3. We will construct the continuous mapping such that it maps the line segment $[-1,1]\times\{0\}^{n-1}$ onto the set which contains $\{z_{\boldsymbol{v}}\}_{i\in\mathbf{N},\boldsymbol{v}\in\mathbb{V}^i}$. Since the continuous image of the compact set is a compact set, the image of $[-1,1]\times\{0\}^{n-1}$ has to be at least $[-1,1]^m$.

We define the set of points $\{c_{\boldsymbol{v}}\}_{i\in\mathbf{N},\boldsymbol{v}\in\mathbb{V}^i} \in (-1,1)$ by induction. We set $c_0 = 0$. By induction, in every interval $B(c_w, \varrho_{i-1})$ we choose non-overlapping intervals $\{B(c_{[\boldsymbol{w},v]}, r_i)\}_{v\in\mathbb{V}}$ around 2^m chosen centres $\{c_{[\boldsymbol{w},v]}\}_{v\in\mathbb{V}}$. We use the inner interval $B(c_{[\boldsymbol{w},v]}, \varrho_i)$ of every of these intervals in the next step. This process generate 2^{mi} centres $\{c_{\boldsymbol{v}}\}_{\boldsymbol{v}\in\mathbb{V}^i}$ in i-th step (see Figure 9). We define

$$c^*_{\boldsymbol{v}} = c_{\boldsymbol{v}} \times \{0\}^{n-1}.$$

Now we define the sequence of continuous mapping $f_i : [-1,1]^n \to [-1,1]^m$. First step is $f_0(x) \equiv z_0 = 0^n$, then by induction we define

$$f_i(x) = \begin{cases} f_{i-1}(x) + h_i(x - c^*_{\boldsymbol{v}})(z_{\boldsymbol{v}} - z_{\boldsymbol{w}}) & \text{for } \boldsymbol{v} = [\boldsymbol{w}, v_i] \in \mathbb{V}^i, |x - c_{\boldsymbol{v}}| \leq r_i, \\ f_{i-1}(x) & \text{otherwise.} \end{cases}$$

By this construction we get the uniformly converging sequence of continuous functions. We claim, that for f_i, $\boldsymbol{v} \in \mathbb{V}^i$ the small subinterval of $x_1 \times \{0\}^{n-1}$ around the point $c_{\boldsymbol{v}} \times \{0\}^{n-1}$ is mapped to the point $z_{\boldsymbol{v}}$ as we sketch in Figure 9 (even a small ball around $c^*_{\boldsymbol{v}}$ is mapped to the point $z_{\boldsymbol{v}}$). We use $h_i \equiv 1$ in some small neighbourhood of $c^*_{\boldsymbol{v}}$ and the fact that this small neighbourhood $B(c^*_{\boldsymbol{v}}, \varrho_i)$ is also the part of neighbourhoods in previous step $B(c^*_{\boldsymbol{w}}, \varrho_{i-1})$. We get for $x \in B(c^*_{\boldsymbol{v}}, \varrho_i) \cap (-1,1) \times \{0\}^{n-1}$ the claim

$$f_i(x) = f_{i-1}(x) + h_i(x - c^*_{\boldsymbol{v}})(z_{\boldsymbol{v}} - z_{\boldsymbol{w}}) = z_{\boldsymbol{w}} + (z_{\boldsymbol{v}} - z_{\boldsymbol{w}}) = z_{\boldsymbol{v}}.$$

Altogether we get

$$f_i\Big(\bigcup_{\boldsymbol{v} \in \mathbb{V}^i} B(c_{\boldsymbol{v}}, \varrho_i) \times \{0\}^{n-1}\Big) = \{z_{\boldsymbol{v}}\}_{\boldsymbol{v} \in \mathbb{V}^i}.$$

In order to finish the proof, we check that $f = \lim_{i\to\infty} f_i$ is still a continuous mapping and we verify that its $W^{1,n}$ norm is finite.

Let us remark, that the presented result can cover even some finer scale of spaces and we can improve the $W^{1,n}$ upto $WL^n \log^{n-1} L$. We can also see this result as the corollary of the capacity theory, since we can consider any space where points have zero capacity.

4.3 The improved result in $W^{k,\frac{n}{k}}$

Before we begin, we prepare two estimates. We claim, that, for any $F : \mathbf{R} \to \mathbf{R}$ smooth enough and $x \neq 0$ we have

$$|D^k(F(|x|))| \leq C \max_{j=\{0,\dots k-1\}} \{|x|^{-j} |D^{k-j} F(|x|)|\}. \tag{24}$$

We can find a small positive $T_k > 0$, such that for $t \in (0, T_k)$ the derivatives of $\log\log\frac{1}{t}$ of order up to k can be estimated as

$$|D^j(\log\log\tfrac{1}{t})| \leq C \frac{1}{t^j \log\frac{1}{t}} \leq C|D^j(\log\log\tfrac{1}{t})| \text{ for } 0 \leq j \leq k. \tag{25}$$

We do not prove these two estimates, the proofs are straightforward and elementary.

Now we improve the classical construction. We reuse all steps from the classical case, we only have to find some finer function instead of h. Precisely we search for a function g which satisfies

$$\begin{aligned} &\forall \varepsilon > 0\, \forall R > 0\, \exists \varrho > 0\, \exists g \in W_0^{k,\frac{n}{k}}(B(0,R)) \cap \overline{C(B(0,R))} \\ &\text{such that } g(B(0,\varrho)) \equiv 1 \text{ and } \|D^k g\|_{\frac{n}{k}}^{\frac{n}{k}} < \varepsilon. \end{aligned} \tag{26}$$

We use the previous function $h(t)$ as the one dimensional function. We choose the key parameter a big enough so that $h(t) = \min\{1, \big(\log(\log(\frac{1}{t})) - a\big)^+\}$ satisfies following

- $h(B(0, 2\varrho)) \equiv 1$ for some $\varrho > 0$,
- $a = \log\log\frac{1}{r}$ for some $r < \frac{1}{2}R$, $r < 12T_k$,
- $$C\int_0^{2r} \frac{1}{t\log^{\frac{n}{k}}\frac{1}{t}}\,dt < \varepsilon$$

  for given $C > 0$ depending only on p, k and the dimension n.

We can find a formula for C, but we just present the estimates leading to it.

We make $h(t)$ smooth by convolution, but we have to use two different radii, because ϱ and r are incomparable. Because of that we use (2), the partition of unity,

$$\lambda(t) = \lambda_{\frac{r}{4}, \frac{3r}{4}, R}(t), \tag{27}$$

as it is introduced in the preliminaries. We denote one dimensional function

$$f(t) = \lambda(t)(\phi_{\frac{\varrho}{2}} * h)(t) + (1 - \lambda(t))(\phi_{\frac{r}{8}} * h)(t).$$

We define

$$g(x) = f(|x|).$$

Our claim is, that $g(x) = f(|x|)$ satisfies all conditions of (26). Obviously, $g(x)$ is smooth, $\mathrm{spt}(g) \subset B(0,R)$, $g(x) \equiv 1$ on $B(0,\varrho)$. The only remaining and the most important part is the smallness of the norm.

We calculate the derivatives of $f(t)$ in order to estimate them. The support of the function $f(t)$ is $[0, r + \frac{r}{8}]$ and we have by (3) the derivative of the product formula

$$D^k f(t) = \sum_{j=0}^{k}\binom{k}{j} D^j\lambda(t) D^{k-j}(\phi_{\frac{\varrho}{2}} * h)(t) + \sum_{j=0}^{k}\binom{k}{j} D^j(1-\lambda(t)) D^{k-j}(\phi_{\frac{r}{8}} * h)(t). \tag{28}$$

Firstly, we consider only the member with $D^j\lambda(t)$ for $j > 0$. This derivative is non-zero only inside $(\frac{r}{4}, \frac{3r}{4})$ by (27). For such $t \in (\frac{r}{4}, \frac{3r}{4})$ by (2) we estimate

$$\sum_{j=1}^{k}\big|D^j\lambda(t)D^{k-j}(\phi_{\frac{\varrho}{2}} * h)\big| \le \sum_{j=1}^{k} Cr^{-j}\big|D^{k-j}(\phi_{\frac{\varrho}{2}} * h)(t)\big|. \tag{29}$$

We apply Lemma 2 for $I = I_1 = \{\frac{r}{4} - \frac{\varrho}{2}, \frac{3r}{4} + \frac{\varrho}{2}\}$ on $\phi_{\frac{\varrho}{2}} * h$. There is no point of broken smoothness of $h(t)$ since neither 2ϱ nor r lies inside $(\frac{r}{4} - \frac{\varrho}{2}, \frac{3r}{4} + \frac{\varrho}{2})$, so we get

$$D^{k-j}\int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}} \phi_{\frac{\varrho}{2}}(t-s)h(s)\,ds = \int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}} \phi_{\frac{\varrho}{2}}(t-s)D^{k-j}h(s)\,ds.$$

Since the estimates of the derivatives of $\log\log\frac{1}{t}$ are bigger for smaller t by (25), we estimate by the Hölder's inequality

$$\int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}} \left|\phi_{\frac{\varrho}{2}}(t-s)D^{k-j}h(s)\right| ds \leq |D^{k-j}h(t-\tfrac{\varrho}{2})| \leq C\frac{1}{(t-\frac{\varrho}{2})^{k-j}\log\frac{1}{t-\frac{\varrho}{2}}}.$$

We apply this to the estimate (29) and by (25) we get

$$\begin{aligned}\sum_{j=1}^{k}\left|D^j\lambda(t)D^{k-j}(\phi_{\frac{\varrho}{2}}*h)\right| &\leq \sum_{j=1}^{k} Cr^{-j}|D^{k-j}h(t-\tfrac{\varrho}{2})| \\ &\leq C\sum_{j=1}^{k}\left|r^{-j}\frac{1}{(t-\frac{\varrho}{2})^{k-j}\log\frac{1}{t-\frac{\varrho}{2}}}\right|.\end{aligned}$$

Since we consider only $t\in(\frac{r}{4},\frac{3r}{4})$, there exist some C such that

$$\sum_{j=1}^{k}\left|D^j\lambda(t)D^{k-j}(\phi_{\frac{\varrho}{2}}*h)\right| \leq \sum_{j=1}^{k} C\left|t^{-j}\frac{1}{t^{k-j}\log\frac{1}{t}}\right| \leq C\left|D^k(\log\log\frac{1}{t})\right|. \quad (30)$$

Analogously for $t\in(\frac{r}{4},\frac{3r}{4})$ we get

$$\sum_{j=1}^{k}\left|D^j(1-\lambda(t))D^{k-j}(\phi_{\frac{r}{8}}*h)(t)\right| \leq C\left|D^k(\log\log\frac{1}{t})\right|. \quad (31)$$

Secondly, we estimate the members of the sums in (28) for $j=0$. We consider $t\in[0,\frac{3r}{4}]$, we estimate $\lambda(t)\leq 1$ inside this interval, otherwise we have $\lambda(t)=0$. Inside this interval lies 2ϱ, the point of broken smoothnes of $h(t)$. By Lemma 2 and triangle inequality we have

$$\begin{aligned}\left|\lambda(t)D^k(\phi_{\frac{\varrho}{2}}*h)\right| \leq &\int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}}\left|\phi_{\frac{\varrho}{2}}(t-s)D^kh(s)\right| ds \\ &+\sum_{l=1}^{k-1}\left|D^{k-1-l}(\phi_{\frac{\varrho}{2}})(t-2\varrho)\right|\left|D^l1-D^l(\log\log\tfrac{1}{2\varrho})\right|.\end{aligned} \quad (32)$$

In the second term we missed the member of the sum for $l=0$, but this member is zero, since $h(t)$ is continuous at 2ϱ. For $t\in(2\varrho-\frac{\varrho}{2},2\varrho+\frac{\varrho}{2})$ we estimate the second member by (1)(3) as

$$\begin{aligned}\sum_{l=1}^{k}\left|D^{k-1-l}(\phi_{\frac{\varrho}{2}})(t-2\varrho)\right|\left|D^l(\log\log\tfrac{1}{2\varrho})\right| &\leq \sum_{l=1}^{k} C\left|\varrho^{-k+l}\right|\left|\frac{1}{\varrho^l\log\frac{1}{\varrho}}\right| \\ &\leq C\left|D^k(\log\log\frac{1}{\varrho})\right| \leq C\left|D^k(\log\log\frac{1}{t})\right|,\end{aligned} \quad (33)$$

anywhere else this member is zero. We estimate the first member of (32) for $t \in [2\varrho - \frac{\varrho}{2}, \frac{3r}{4}]$ as

$$\int_{t-\frac{\varrho}{2}}^{t+\frac{\varrho}{2}} \big|\phi_{\frac{\varrho}{2}}(t-s) D^k h(s)\big|\, ds \leq \big|D^k(\log\log \frac{1}{t-\frac{\varrho}{2}})\big| \leq C\big|D^k(\log\log \frac{1}{t})\big|. \quad (34)$$

Analogously to these two estimates (33) and (34) we estimate for $t \in [\frac{r}{4}, r + \frac{r}{8}]$

$$\sum_{j=0}^{k} \big|D^j(1-\lambda(t)) D^{k-j}(\phi_{\frac{r}{8}} * h)(t)\big| \leq C\big|D^k(\log\log \frac{1}{t})\big|. \quad (35)$$

By estimating members of formula for $D^k f$ written in (28), we get a pointwise estimate for every member anywhere on its support. Altogether by (30), (31), (33), (34) and (35) we get

$$|D^k(f(t))| \leq C\big|D^k(\log\log \frac{1}{t})\big|.$$

By (24), (25), spherical coordinates and the condition for r we get

$$\begin{aligned} \|D^k g(x)\|_{\frac{n}{k}}^{\frac{n}{k}} &\leq \int_{B(0,2r)} |D^k f(|x|)|^{\frac{n}{k}}\, dx \\ &\leq C \int_0^{2r} t^{n-1} \Big(\max_{i=0,\dots,k-1} \big\{t^{-i} |D^{k-i}(\log\log \frac{1}{t})|\big\}\Big)^{\frac{n}{k}}\, dt \\ &\leq C \int_0^{2r} \frac{1}{t^{-n+1+n} \log^{\frac{n}{k}} \frac{1}{t}}\, dt < \varepsilon. \end{aligned}$$

All the properties of $g(x)$ and checked. We use it the same way as $h(x)$ is used in the classical case and get the counterexample in $W^{k,\frac{n}{k}}$.

Remark 2 There was a partial result on smoothing of Cesari counterexample, Matějka has proven the case for $k = 2$ in [11]. He smoothed $h(x)$ by redefining it explicitly near the points of discontinuity and his example is C^1 but not C^2.

Acknowledgements We would like to thank to Jan Malý, who gave us some useful ideas about the problem. And we thank to the author's supervisor, Stanislav Hencl, for valuable comments.

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