Charles University in Prague

Faculty of Mathematics and Physics

DOCTORAL THESIS

Lenka Slavíková

Weighted inequalities and properties of operators and embeddings on function spaces

Department of Mathematical Analysis

Supervisor of the doctoral thesis: prof. RNDr. Luboš Pick, CSc., DSc.

Study programme: Mathematics

Study branch: Mathematical Analysis

Prague 2016

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University in Prague has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In Bilbao on April 15, 2016 Lenka Slavíková

Název práce: Váhové nerovnosti a vlastnosti operátorů a vnoření na prostorech funkcí

Autor: Lenka Slavíková

Katedra: Katedra matematické analýzy

Vedoucí disertační práce: prof. RNDr. Luboš Pick, CSc., DSc., Katedra matematické analýzy

Abstrakt: Tato disertační práce je věnována studiu nejrůznějších vlastností Banachových prostorů funkcí se zvláštním zřetelem k aplikacím v teorii Sobolevových prostorů a v harmonické analýze. Práce sestává ze čtyř článků. V prvním z nich zkoumáme vnoření vyššího řádu prostorů Sobolevova typu vybudovaných nad Banachovými prostory funkcí s normou invariantní vůči nerostoucímu přerovnání. Mimo jiné ukážeme, že optimální Sobolevova vnoření vyššího řádu plynou z izoperimetrických nerovností. Ve druhém článku se zabýváme otázkou, kdy je výše zmíněný prostor Sobolevova typu Banachovou algebrou vzhledem k bodovému násobení funkcí. Dokážeme, že vnoření Sobolevova prostoru do prostoru esenciálně omezených funkcí je odpovědí na tuto otázku v mnoha standardních i nestandardních případech. Třetí článek je věnován problému platnosti Lebesgueovy věty o derivování v kontextu Banachových prostorů funkcí s normou invariantní vůči nerostoucímu přerovnání. Nalezneme nutnou a postačující podmínku pro platnost této věty vyjádřenou pomocí konkavity jistého funkcionálu závisejícího na dané normě a poskytneme rovněž několik alternativních charakterizací zadaných pomocí vlastností maximálního operátoru vybudovaného nad danou normou. Poslední článek se týká omezenosti Hardyova-Littlewoodova maximálního operátoru na váhových Lebesgueových prostorech s různými váhami. Zaměříme se na studium zesílených verzí Muckenhouptovy A_p-podmínky, v literatuře označovaných jako "bump podmínky". Je známo, že tyto podmínky jsou postačují pro dvojváhovou maximální nerovnost; v článku dokážeme, že ovšem nejsou nutné.

Klíčová slova: Banachův prostor funkcí, Sobolevův prostor, izoperimetrická nerovnost, Banachova algebra, Lebesgueova věta o derivování, maximální operátor, bump podmínka.

Title: Weighted inequalities and properties of operators and embeddings on function spaces

Author: Lenka Slavíková

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Luboš Pick, CSc., DSc., Department of Mathematical Analysis

Abstract: The present thesis is devoted to the study of various properties of Banach function spaces, with a particular emphasis on applications in the theory of Sobolev spaces and in harmonic analysis. The thesis consists of four papers. In the first one we investigate higher-order embeddings of Sobolev-type spaces built upon rearrangement-invariant Banach function spaces. In particular, we show that optimal higher-order Sobolev embeddings follow from isoperimetric inequalities. In the second paper we focus on the question when the above-mentioned Sobolev-type space is a Banach algebra with respect to a pointwise multiplication of functions. An embedding of the Sobolev space into the space of essentially bounded functions is proved to be the answer to this question in several standard as well as nonstandard situations. The third paper is devoted to the problem of validity of the Lebesgue differentiation theorem in the context of rearrangement-invariant Banach function spaces. We provide a necessary and sufficient condition for the validity of this theorem given in terms of concavity of certain functional depending on the norm in question and we find also alternative characterizations expressed in terms of properties of a maximal operator related to the norm. The object of the final paper is the boundedness of the Hardy-Littlewood maximal operator between weighted Lebesgue spaces with different weights. We focus on strengthenings of the Muckenhoupt A_p-condition, called "bump conditions" in the literature. These conditions are known to be sufficient for the two-weighted maximal inequality; we prove that they are however not necessary.

Keywords: Banach function space, Sobolev space, isoperimetric inequality, Banach algebra, Lebesgue differentiation theorem, maximal operator, bump condition.

I would like to express my deep gratitude to Luboš Pick, whom I started to meet when I was a bachelor student writing the thesis under his supervision and who afterwards became also my diploma supervisor and PhD advisor. I am very grateful to him for introducing me into mathematics research already at the early stages of our collaboration and for his constant effort to help me to find the direction of my research later on. I also really enjoyed working with him as with a coauthor of our joint papers. His endless optimism and great enthusiasm for everything he does helped to make my study at Charles University an unforgettable experience.

I would like to thank Andrea Cianchi and Paola Cavaliere for very pleasant collaboration on our joint papers and for wonderful visits of Florence and Salerno.

Special thanks belong to my family for their never-ceasing support of all kinds.

Contents

Introduction

1 Prologue

Integrability properties of a function can be described via its membership into a *Lebesgue space* L^p of all measurable functions whose absolute value is integrable when raised to the power p (if $p \in [1, \infty)$), or essentially bounded (if $p = \infty$). The significance of Lebesgue spaces in various branches of analysis, including harmonic analysis or the analysis of partial differential equations, is unquestionable, and Lebesgue spaces definitely constitute one of the basic tools in these areas. However, it turns out that the scale of Lebesgue spaces is often not rich enough to provide a satisfactory solution of a particular problem, and other, more delicate function spaces have to be called into play. Let us now present two examples of such problems. Both of them are due to E. M. Stein.

The *Hardy–Littlewood maximal operator* M, defined for every measurable function f on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|, \quad x \in \mathbb{R}^n, \tag{1}$$

where the supremum is taken over all cubes Q containing x and having their sides parallel to coordinate axes, plays a significant role in harmonic analysis. It is well known that if $p \in (1, \infty]$, then Mf belongs to L^p if and only if f belongs to L^p. On the other hand, Mf does not belong to L^1 unless $f = 0$ a.e., since $Mf(x) \geq \frac{C}{|x|^n}$ near infinity for f nontrivial. Fortunately, we can often obtain at least the local integrability of Mf, but the right condition on f in order to ensure this cannot be expressed in terms of Lebesgue spaces. In fact, it was proved in [54] that whenever f is a measurable function supported in a ball B, then Mf is integrable on B if and only if

$$\int_B |f| \log_+ |f| < \infty,$$

which is equivalent to the membership of f into the *Orlicz space* $L \log L$.

Another example is provided by the theory of Sobolev spaces. It is classically well known that if f is a weakly differentiable function on $\mathbb{R}^n$ whose weak derivatives belong locally to the space L^p for some $p > n$, then the function f is differentiable in the usual sense at almost every point, and that a similar conclusion is not true anymore if we only have the information that the weak derivatives of f belong locally to the borderline space L^n. This nonsharp result is often not sufficient, and if one wants to find a sharp condition on the weak gradient of a function f in order to ensure the differentiability of f almost everywhere, one has to go again beyond the class of Lebesgue spaces. It was shown in [55] that the *Lorentz space* $L^{n,1}$, consisting of all functions g for which

$$\int_0^\infty |\{x \in \mathbb{R}^n : |g(x)| > t\}|^{\frac{1}{n}} \, dt < \infty,$$

is actually the appropriate one in this situation.

In this thesis we focus on the general class of *Banach function spaces*. This class of function spaces provides a common roof for Lebesgue, Lorentz and Orlicz spaces mentioned above, as well as for many other less standard families of function spaces. We put a particular emphasis on applications of the Banach function spaces theory in the theory of Sobolev spaces and in harmonic analysis.

2 Banach function spaces – a brief introduction

In this section we introduce the class of Banach function spaces, which is going to play a primary role in the thesis. We start with definitions of particular families of Banach function spaces, such as Lebesgue, Orlicz and Lorentz spaces, and we then turn our attention to the class of Banach function spaces in its full generality. For simplicity, we consider here only Banach function spaces defined on (Lebesgue) measurable sets in $\mathbb{R}^n$ since we are going to work in this setting most of the time. However, we emphasize that Banach function spaces can in fact be defined on any totally σ-finite measure space, and we refer the reader to the book [6] for a detailed treatment of Banach function spaces in this more general context.

Let us fix $n \in \mathbb{N}$ and a measurable set $E \subseteq \mathbb{R}^n$. We denote by $\mathcal{M}(E)$ the set of all measurable functions on $\mathbb{R}^n$ having their values in $[-\infty, \infty]$. If F is a measurable subset of E, then $|F|$ denotes the Lebesgue measure of F.

Given $p \in [1, \infty]$, the *Lebesgue space* $L^p(E)$ consists of all functions $f \in \mathcal{M}(E)$ for which $\|f\|_{L^p(E)} < \infty$, where

$$\|f\|_{L^p(E)} = \begin{cases} \left(\int_E |f|^p\right)^{\frac{1}{p}}, & p \in [1, \infty); \\ \operatorname{esssup_E} |f|, & p = \infty. \end{cases}$$

Extensions of Lebesgue spaces in several different directions are nowadays available in the literature. Perhaps the most common one is provided by the notion of *Orlicz spaces*. Each Orlicz space corresponds to a *Young function*. Before stating the precise properties we shall require from a Young function, we should note that the definitions differ slightly in the literature, and sometimes stronger assumptions than those we employ here are required. We say that a function $A : [0, \infty) \to [0, \infty]$ is a Young function if it is convex, nontrivial, left-continuous and vanishes at 0. The Orlicz space corresponding to the Young function A is denoted by $L^A(E)$ and contains all functions $f \in \mathcal{M}(E)$ for which

$$\|f\|_{L^A(E)} = \inf\left\{\lambda > 0 : \int_E A\left(\frac{|f|}{\lambda}\right) \leq 1\right\} < \infty.$$

The particular choice of $A(t) = t^p$ for $p \in [1, \infty)$ yields the Lebesgue space $L^p(E)$, while the Lebesgue space $L^\infty(E)$ is induced by the Young function $A(t) = \infty\chi_{(1,\infty)}$. Let us also provide two more delicate examples of Orlicz spaces. To state them we restrict ourselves to sets E of finite measure. The first example is the Orlicz space $L^p(\log L)^\alpha(E)$ associated with a Young function equivalent to $t^p(\log t)^\alpha$ near infinity, where either $p \in (1, \infty)$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$. The latter one concerns the Orlicz space $\exp L^\beta(E)$ built upon a Young function equivalent to e^{t^β} near infinity, where $\beta > 0$. An extensive treatment of the theory of Orlicz spaces can be found, e.g., in the book [48].

A generalization of Lebesgue spaces in a different direction is provided by the notion of *Lorentz spaces*. As in the case of Orlicz spaces, the norm (or quasinorm) in a Lorentz space depends only on the size of a function. More precisely, "the size of a function f" stands for the measure of the level sets of $|f|$, and there are two common ways how to express it. One can either make use of the *distribution function* f_*, defined by

$$f_*(\lambda) = |\{x \in E : |f(x)| > \lambda\}|, \quad \lambda \geq 0,$$

or of its generalized inverse, called the *non-increasing rearrangement* of f and denoted by f^*. Namely, the function f^* is given by

$$f^*(t) = \inf\{\lambda \geq 0 : f_*(\lambda) \leq t\}, \quad t \geq 0,$$

and it is the unique nonnegative right-continuous non-increasing function on $[0, \infty)$ having the same measure of level sets as $|f|$.

Given $p \in [1, \infty]$ and $q \in [1, \infty]$, the Lorentz space $L^{p,q}(E)$ consists of all functions $f \in \mathcal{M}(E)$ for which

$$\|f\|_{L^{p,q}(E)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,\infty)} < \infty.$$

Alternatively, the quantity $\|f\|_{L^{p,q}(E)}$ can be expressed in terms of the distribution function of f as

$$\|f\|_{L^{p,q}(E)} = p^{\frac{1}{q}} \|t^{1-\frac{1}{q}} (f_*(t))^{\frac{1}{p}}\|_{L^q(0,\infty)}.$$

Nowadays, there exist also several generalizations of Lorentz spaces. For instance, the *Lorentz-Zygmund space* $L^{p,q;\alpha}(E)$ consists of all functions $f \in \mathcal{M}(E)$ for which

$$\|f\|_{L^{p,q;\alpha}(E)} = \|t^{\frac{1}{p}-\frac{1}{q}} (1 + |\log t|)^\alpha f^*(t)\|_{L^q(0,\infty)} < \infty,$$

where $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$. The choice $\alpha = 0$ yields the Lorentz space $L^{p,q}(E)$. Lorentz-Zygmund spaces overlap somewhat also with Orlicz spaces. Namely, assuming that E is a set of finite measure, we have $L^{p,p;\alpha}(E) = L^p(\log L)^{p\alpha}(E)$ if $p \in (1, \infty)$ and $\alpha \in \mathbb{R}$, or if $p = 1$ and $\alpha \geq 0$, and $L^{\infty,\infty,-\beta}(E) = \exp L^{\frac{1}{\beta}}(E)$ if $\beta > 0$, up to equivalent norms. More details about Lorentz-Zygmund spaces can be found, e.g., in [5, 19, 41].

A yet further generalization of Lorentz spaces, which covers also all Lorentz-Zygmund spaces, is provided by the notion of *classical Lorentz spaces*. Given $p \in [1, \infty]$ and a weight (that is, a nonnegative measurable function) w, the classical Lorentz space $\Lambda^p_w(E)$ consists of all $f \in \mathcal{M}(E)$ for which $\|f\|_{\Lambda^p_w(E)} < \infty$, where

$$\|f\|_{\Lambda^p_w(E)} = \begin{cases} \left(\int_0^\infty (f^*(t))^p w(t)\, dt\right)^{\frac{1}{p}}, & p \in [1, \infty); \\ \sup_{t>0} f^*(t) w(t), & p = \infty. \end{cases}$$

These spaces were introduced by Lorentz [32] and their intensive study has continued up to the present. We refer the interested reader to the book [45] for the references and further information on classical Lorentz spaces.

All the function spaces we mentioned so far were "rearrangement invariant" in the sense that the (quasi) norm of these spaces depended only on the measure of level sets of a function (or, equivalently, on the non-increasing rearrangement of a function). An important example of function spaces that do not have this property are *weighted Lebesgue spaces*. For any $p \in (1, \infty)$ and any weight w, the weighted Lebesgue space $L^p_w(E)$ consists of all functions f in $\mathcal{M}(E)$ for which

$$\|f\|_{L^p_w(E)} = \left(\int_E |f|^p w\right)^{\frac{1}{p}} < \infty.$$

Although the function spaces we introduced differ in several aspects, they still share a lot of properties. An inspection of common properties of these (and several other) families of function spaces then leads to a notion of a *Banach function space*, which provides a common roof for most of the function spaces mentioned above.

We say that a functional $\|\cdot\|_{X(E)} : \mathcal{M}(E) \to [0,\infty]$ is a *Banach function norm* if, for all functions f, $g \in \mathcal{M}(E)$, for all sequences $(f_k)_{k=1}^{\infty}$ in $\mathcal{M}(E)$ and for all constants $a \in \mathbb{R}$, the following properties hold:

(P1) $\|f\|_{X(E)} = 0$ if and only if $f = 0$ a.e.; $\|af\|_{X(E)} = |a|\|f\|_{X(E)}$;
$\|f+g\|_{X(E)} \le \|f\|_{X(E)} + \|g\|_{X(E)}$;

(P2) $|f| \le |g|$ a.e. implies $\|f\|_{X(E)} \le \|g\|_{X(E)}$;

(P3) $|f_k| \nearrow |f|$ a.e. implies $\|f_k\|_{X(E)} \nearrow \|f\|_{X(E)}$;

(P4) if $F \subseteq E$ with $|F| < \infty$ then $\|\chi_F\|_{X(E)} < \infty$;

(P5) if $F \subseteq E$ with $|F| < \infty$ then $\int_F |f(x)|\,dx \le C_F \|f\|_{X(E)}$ for some constant C_F depending on F but independent of f.

The collection of all $f \in \mathcal{M}(E)$ for which $\|f\|_{X(E)} < \infty$ is denoted by $X(E)$ and is called a Banach function space.

If a Banach function norm $\|\cdot\|_{X(E)}$ satisfies also the property

(P6) $f^* = g^*$ implies $\|f\|_{X(E)} = \|g\|_{X(E)}$,

then it is called a *rearrangement-invariant Banach function norm* and the corresponding Banach function space is called a *rearrangement-invariant Banach function space*.

Examples of Banach function spaces include Lebesgue and Orlicz spaces mentioned earlier in this section. The Lorentz space $L^{p,q}(E)$ is a Banach function space (up to equivalent norms) if and only if $p \in (1,\infty)$ and $q \in [1,\infty]$, or $p = q = 1$, or $p = q = \infty$. The description of those classical Lorentz spaces that are equivalent to a Banach function space is a bit complicated, so we do not include it here and just refer to the papers [51, 10]. The particular case of Lorentz-Zygmund spaces is treated in [19]. All these spaces are rearrangement invariant. The weighted Lebesgue space $L^p_w(E)$ is a Banach function space if and only if both w and $w^{-\frac{1}{p-1}}$ are integrable over subsets of E of finite measure; it is not rearrangement invariant unless the weight w is constant a.e.

As we have seen, the class of Banach function spaces contains most of the function spaces we introduced in this section, but it does not include all of them. If we wanted to get a yet more general class that would cover all of our examples, we would need to relax some of the conditions (P1) – (P5). However, admitting such an extension would mean losing several important properties of Banach function spaces. Let us now present an example of this phenomenon.

A basic property of Lebesgue spaces is that they fulfil the Hölder inequality

$$\int_E |fg| \le \|f\|_{L^p(E)} \|g\|_{L^{p'}(E)}, \quad f,g \in \mathcal{M}(E),$$

where the indices p, p' are connected by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. This inequality is sharp in the sense that

$$\|g\|_{L^{p'}(E)} = \sup_{\|f\|_{L^p(E)} \leq 1} \int_E |fg|, \quad g \in \mathcal{M}(E).$$

An analogue of the Hölder inequality holds also for any Banach function space. Indeed, for each Banach function space $X(E)$, the functional $\|\cdot\|_{X'(E)}$ defined by

$$\|g\|_{X'(E)} = \sup_{\|f\|_{X(E)} \leq 1} \int_E |fg|, \quad g \in \mathcal{M}(E), \tag{2}$$

is a Banach function norm, and the general Hölder inequality

$$\int_E |fg| \leq \|f\|_{X(E)} \|g\|_{X'(E)}, \quad f, g \in \mathcal{M}(E),$$

is then an obvious consequence of (2). We recall that the Banach function space $X'(E)$ is called the *associate space* of $X(E)$. The relation of "being the associate space" is symmetric, that is, the associate space to $X'(E)$ is the original space $X(E)$.

If we wanted to extend the class of Banach function spaces so that it included, for instance, the Lorentz space $L^{1,\infty}(E)$, we would face several problems in the attempt to generalize the Hölder inequality to this setting. First of all, the "associate space" to $L^{1,\infty}(E)$, defined by the expression (2) with $X(E)$ replaced by $L^{1,\infty}(E)$, contains only the zero function. Moreover, the space $L^{1,\infty}(E)$ cannot be recovered as the associate space to the trivial space $(L^{1,\infty}(E))'$.

To conclude this section we would like to express our hope that the reader now shares with us the impression that *the class of Banach function spaces is a class of function spaces generalizing those of Lebesgue, which is sufficiently wide but still preserves several important properties of Lebesgue spaces.*

3 Banach function spaces meet Sobolev spaces

These days, Banach function spaces meet Sobolev spaces on a regular basis. Let us now have a quick look at the earlier stages of this relationship, when the meetings of these two kinds of spaces were not so common. There are definitely many important moments that helped to build this relationship and we have by no means the ambition to cover them all. Instead, we focus just on a few classical instances and then jump to those that directly inspired the research presented in this thesis.

Assume that Ω is an open subset of $\mathbb{R}^n$. Given $p \in [1, \infty]$, the classical Sobolev space $W^{1,p}(\Omega)$ consists of those weakly differentiable functions u on Ω which fulfil $u \in L^p(\Omega)$ and $|\nabla u| \in L^p(\Omega)$, where ∇u denotes the weak gradient of u and $|\nabla u|$ its Euclidean length. The space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \||\nabla u|\|_{L^p(\Omega)}.$$

Let us assume for the time being that Ω is a regular domain in $\mathbb{R}^n$, for instance, a bounded domain having a Lipschitz boundary. The classical Sobolev embedding tells us that for $p \in [1, n)$,

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \tag{3}$$

where $p^* = \frac{np}{n-p}$. Further, if $p > n$ then we have $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. In both cases, the range spaces are the best possible among all Lebesgue spaces. The situation, however, turns out to be more complicated in the borderline case $p = n$. The Sobolev space $W^{1,n}(\Omega)$ is continuously embedded into all Lebesgue spaces $L^q(\Omega)$ with $q < \infty$, but not into $L^\infty(\Omega)$. This means that no definite best possible Lebesgue space range for $W^{1,n}(\Omega)$ can be provided. Nevertheless, a sharp range for $W^{1,n}(\Omega)$ can still be found if a suitable refinement of the Lebesgue scale is considered.

One possible scale that can be successfully used to solve the above problem is the scale of Orlicz spaces. Namely, the space $\exp L^{\frac{n}{n-1}}(\Omega)$ (see Section 2 for the definition) is the smallest Orlicz space that contains the Sobolev space $W^{1,n}(\Omega)$. The embedding

$$W^{1,n}(\Omega) \hookrightarrow \exp L^{\frac{n}{n-1}}(\Omega) \tag{4}$$

was proved by Trudinger [56] (see also Yudovich [58], Peetre [42] and Pokhozhaev [46]), and its optimality within the scale of Orlicz spaces is due to Hempel, Morris and Trudinger [24]. Let us also note that the embedding (3), which was mentioned to be optimal among all Lebesgue spaces, can be improved neither in the class of Orlicz spaces (see [12]).

Embeddings (3) and (4) may now seem quite satisfactory since they are sharp in a certain sense, but, interestingly, they still do not reflect the integrability properties of functions from Sobolev spaces in the best possible way. The improved results involve spaces of Lorentz and Lorentz-Zygmund type (see Section 2 for definitions of these spaces).

The result due to Peetre [42] (see also O'Neil [40] and Hunt [26]) tells us that

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*,p}(\Omega), \quad 1 < p < n, \tag{5}$$

providing a nontrivial improvement of the embedding (3) since $L^{p^*,p}(\Omega) \subsetneq L^{p^*}(\Omega)$. The refined version of (4) has the form

$$W^{1,n}(\Omega) \hookrightarrow L^{\infty,n;-1}(\Omega) \tag{6}$$

and was proved by Hansson [23] and Brézis and Wainger [9] (and can be also derived from capacitary estimates by Maz'ya - see [36]). The range spaces in (5) and (6) are the best possible within the class of rearrangement-invariant Banach function spaces, see [15] and [18].

The relation between Sobolev spaces and Banach function spaces we discussed so far was restricted to the situation when a Banach function space plays the role of the range space in a Sobolev embedding. However, an even more intimate relationship between these two classes of function spaces is possible, and has found several applications. Namely, one can consider the Sobolev-type space of those weakly differentiable functions whose weak derivatives belong to a certain Banach function space. For instance, the study of the so called Orlicz-Sobolev spaces was motivated by variational problems and by partial differential equations with nonlinearities that are not necessarily of polynomial type, and the early contributions on this topic include the papers [16, 17, 22, 2]. Let us also recall the result due to Stein [55] presented in Section 1 which shows the use of a Lorentz-Sobolev space in connection with differentiability of Sobolev functions.

To date there exists a number of contributions to the study of various properties of Sobolev spaces built upon Banach function spaces. Let us just focus on a few of them that directly inspired the research presented in this thesis.

Edmunds, Kerman and Pick [18] described optimal embeddings of Sobolev-type spaces built upon rearrangement-invariant quasinorms and containing functions vanishing on the boundary of a bounded Euclidean domain. A result in the spirit of [18], working with rearrangement-invariant norms instead of quasinorms and valid for Sobolev-type spaces on Lipschitz domains, was obtained by Kerman and Pick [27]. These two papers basically solved the problem of optimality of Sobolev-type embeddings on regular Euclidean domains in the context of rearrangement-invariant Banach function spaces.

The question that remained open concerned Sobolev-type embeddings on irregular Euclidean domains and on domains in $\mathbb{R}^n$ equipped with measures different from the Lebesgue one. The domain of particular importance was the Gauss space, namely, $\mathbb{R}^n$ equipped with the Gauss measure

$$d\,\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}\,dx.$$

A phenomenon that holds for Sobolev embeddings on the Gauss space, making them quite different from those on Euclidean domains, is that these embeddings do not depend on the dimension n. This important property of Gaussian Sobolev embeddings allows to generalize them to the infinite-dimensional setting, which finds several applications in the study of quantum fields.

The celebrated result due to Gross [22] shows that whenever u is a weakly differentiable function belonging to $L^2(\mathbb{R}^n, \gamma_n)$ together with all its weak derivatives, then u does actually belong to the (smaller) Orlicz space $L^2 \log L(\mathbb{R}^n, \gamma_n)$, and the constant in the corresponding logarithmic Sobolev inequality is independent of n. Many papers then followed, studying Gaussian Sobolev embeddings in various settings. The paper that was of a particular significance for this thesis was the one written by Cianchi and Pick [14]. In this paper, a complete characterization of optimal first-order Gaussian Sobolev embeddings in the general context of rearrangement-invariant Banach function spaces was provided. The question that immediately arised from this paper was how to generalize this result to higher-order embeddings. The answer to this question is not straightforward since the symmetrization technique used to prove the first-order result cannot be applied to derive higher-order Sobolev embeddings. Also, the interpolation method which was succesfully used in [27] to prove higher-order Sobolev embeddings on regular Euclidean domains does not work in the Gauss space setting. We provide the solution of this problem in Paper I of this thesis.

The question which is closely related to the results we have discussed is that of compactness of Sobolev-type embeddings in the context of rearrangement-invariant Banach function spaces. Kerman and Pick [28] gave an answer to this question, provided that Sobolev-type spaces over regular Euclidean domains are considered. The paper [53] then addressed this problem in quite general setting, and the results of this paper included a characterization of compactness of higher-order Sobolev-type embeddings on the Gauss space. It is worth noting that an important tool used to prove compactness of Sobolev embeddings in [53] is a characterization of optimal Sobolev embeddings given in Paper I of this thesis. (A preliminary version of [53] appeared already as a diploma thesis of the author; this is the reason why the paper [53] is not contained in the present thesis.)

Another problem we address in this thesis is that of finding a characterization of those Sobolev-type spaces which are commutative Banach algebras with respect to a pointwise multiplication a.e. In the setting of classical Sobolev spaces over regular Euclidean domains, such a characterization is well known and can be found, e.g., in [1]. An extension of this result to the setting of Sobolev spaces built upon Orlicz spaces, but still on regular Euclidean domains, was obtained by Cianchi [13]. The interesting phenomenon behind these results is the following: a Sobolev space is a Banach algebra if and only if it is continuously embedded into L^∞.

In contrast, it was proved by Maz'ya and Netrusov [37] that there exists a (very irregular) domain $\Omega \subseteq \mathbb{R}^2$ for which the second-order Sobolev space $W^{2,2}(\Omega)$ intersected with $L^\infty(\Omega)$ is not a Banach algebra. In fact, the domain Ω from their counterexample can be slightly modified in order to ensure that the Sobolev space $W^{2,2}(\Omega)$ is actually embedded into $L^\infty(\Omega)$ but is still not a Banach algebra. We provide the construction in a note to Paper II of this thesis.

There are two questions that naturally arise in connection with the papers [13] and [37]. Firstly, is it true that every Sobolev space built upon a general rearrangement-invariant Banach function space over a regular Euclidean domain is a Banach algebra if and only if it is continuously embedded into L^∞? And if so, is such an equivalence just a privilege of regular domains, or it continues to hold also for domains with some irregularity? We address these two questions in Paper II of the present thesis.

3.1 Paper I: Higher-order Sobolev embeddings and isoperimetric inequalities

In this paper, motivated by the problem of finding a characterization of optimal higher-order Gaussian Sobolev embeddings, we prove that iteration of optimal first-order Sobolev embeddings leads to optimal higher-order counterparts, provided that the optimality is understood within the context of rearrangement-invariant Banach function spaces. As an important consequence of this result, we show that optimal higher-order Sobolev embeddings follow from isoperimetric inequalities. Let us note that the connection between Sobolev embeddings and isoperimetric inequalities has been known in the first-order setting for more than fifty years (see [34, 35, 20]), however, it was widely believed not to be true in the higher-order case.

Although an iteration of first-order Sobolev embeddings is a well known and natural technique for deriving higher-order embeddings, its implementation in order to obtain sharp results is not straightforward. Indeed, even in the basic case when standard families of Sobolev norms are considered, iteration of optimal first-order embeddings need not lead to optimal higher-order counterparts. A similar loss of information throughout the iteration process can occur also in the

Orlicz setting. On the other hand, we show that there is no loss of optimality when one iterates first-order embeddings that are optimal within the class of rearrangement-invariant Banach function spaces.

As an application of the iteration technique, we obtain a reduction of higher-order Sobolev embeddings to much simpler one-dimensional inequalities involving certain kernel integral operator. Our results hold for Sobolev-type spaces of functions defined on underlying domains in $\mathbb{R}^n$, equipped with fairly general measures. This, in particular, includes Euclidean John domains, Maz'ya classes of Euclidean domains and product probability spaces, of which the Gauss space is a classical instance.

3.2 Paper II: Banach algebras of weakly differentiable functions

In this paper we study the question when a general Sobolev space built upon a rearrangement-invariant Banach function space is a Banach algebra with respect to a pointwise multiplication a.e. We first consider Sobolev-type spaces over a fairly general class of regular Euclidean domains, called John domains, and we show that in this setting, a Sobolev space is a Banach algebra if and only if it is continuously embedded into L^∞. We then turn our attention to Sobolev spaces on domains having a more general isoperimetric behaviour, showing that such an equivalence still continues to hold in a relaxed sense, when families of domains are considered instead of single domains. Moreover, in each of the families of (irregular) domains one single domain can be found, for which the above mentioned equivalence holds in the usual (nonrelaxed) sense. A typical example of such a domain is a domain with a cusp.

We complement the paper by a note showing an example of a domain Ω in $\mathbb{R}^2$ for which the Sobolev space $W^{2,2}(\Omega)$ is embedded into $L^\infty(\Omega)$ but is not a Banach algebra.

4 Banach function spaces meet harmonic analysis

One of the operators that appear frequently in harmonic analysis is the Hardy-Littlewood maximal operator, defined by (1). Its importance stems from the fact that it can be used to estimate several more complicated operators, such as singular integral operators. It also plays a significant role in the study of differentiability properties of functions. In this section we would like to recall some of the important moments from the story "how Banach function spaces met the Hardy-Littlewood maximal operator".

It is well known that the Hardy-Littlewood maximal operator M is bounded on $L^p(\mathbb{R}^n)$ for every $p \in (1, \infty]$, but it is not bounded on $L^1(\mathbb{R}^n)$. If we intend to find a substitute for the L^1-boundedness, we can proceed in two different directions. We can either preserve $L^1(\mathbb{R}^n)$ as the domain space and look for a function space $X(\mathbb{R}^n)$ such that Mf belongs to $X(\mathbb{R}^n)$ for every $f \in L^1(\mathbb{R}^n)$; alternatively, we can try to get $L^1(\mathbb{R}^n)$ as the range space and search for a function space $Y(\mathbb{R}^n)$ with the property that $Mf \in L^1(\mathbb{R}^n)$ for every $f \in Y(\mathbb{R}^n)$.

In the former case, the space $X(\mathbb{R}^n) = L^{1,\infty}(\mathbb{R}^n)$ gives a solution to the problem (we emphasize that $L^{1,\infty}(\mathbb{R}^n)$ is not a Banach function space - in fact, the problem cannot be solved within the class of Banach function spaces). Let us note that the boundedness of the Hardy-Littlewood maximal operator M from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$ has an important consequence, namely, the Lebesgue differentiation theorem. It tells us that if f is a locally integrable function on $\mathbb{R}^n$ then

$$f(x) = \lim_{r\to 0_+} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} f(y)\,dy \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where $Q(x,r)$ denotes the cube centered in x and with sidelength r.

The latter way how to find a substitute for the L^1-boundedness of the Hardy-Littlewood maximal operator is slightly more complicated since, as we already mentioned in Section 1, the only function f for which $Mf \in L^1(\mathbb{R}^n)$ is the function $f = 0$ a.e. Thus, in order to obtain a reasonable result, we need to relax the requirement of the integrability of Mf. A suitable substitute is the local integrability - we recall the result by Stein [54], which tells us that whenever f is a measurable function supported in a ball B, then Mf is integrable on B if and only if f belongs to the Orlicz space $L\log L(B)$.

A precise description of integrability properties of the Hardy-Littlewood maximal function is provided by the following couple of rearrangement inequalities:

$$\frac{C_1}{t}\int_0^t f^*(s)\,ds \le (Mf)^*(t) \le \frac{C_2}{t}\int_0^t f^*(s)\,ds, \quad t>0. \tag{7}$$

The latter of these two inequalities was proved in the one-dimensional case by Riesz [49] and in the n-dimensional setting by Wiener [57]. The former inequality was established much later by Herz [25].

Inequalities (7) can be used to characterize boundedness of the Hardy-Littlewood maximal operator between rearrangement-invariant Banach function spaces but are of no use if the question of boundedness of M on function spaces that are not rearrangement invariant comes into consideration. In particular, (7) does not tell us how the operator M behaves on weighted Lebesgue spaces. We shall discuss this problem later in this section. Before doing this, let us introduce generalizations of the Hardy-Littlewood maximal operator which arise naturally not only in connection with this problem, but also in connection with several other problems.

Given a Banach function space $X(\mathbb{R}^n)$, we define the X-average of a measurable function f over a cube $Q \subseteq \mathbb{R}^n$ by

$$\|f\|_{X,Q} = \|\tau_{\ell(Q)} f \chi_Q\|_{X(\mathbb{R}^n)},$$

where τ_δ denotes, for $\delta > 0$, the dilation operator $\tau_\delta f(x) = f(\delta x)$, and $\ell(Q)$ stands for the sidelength of the cube Q. The maximal operator M_X is then given by

$$M_X f(x) = \sup_{Q\ni x} \|f\|_{X,Q}, \quad x \in \mathbb{R}^n.$$

It is worth noticing that the standard Hardy-Littlewood maximal operator is recovered by the choice $X(\mathbb{R}^n) = L^1(\mathbb{R}^n)$. Further, if $p \in (1,\infty)$ then

$$M_{L^p} f = (M(|f|^p))^{\frac{1}{p}} = \sup_{Q\ni x} \left(\frac{1}{|Q|}\int_Q |f|^p\right)^{\frac{1}{p}}, \quad f \in \mathcal{M}(\mathbb{R}^n).$$

The maximal operator built upon the Lorentz space $L^{p,1}(\mathbb{R}^n)$ came into play in the paper by Stein [55], mentioned in Section 1. He proved that if the weak derivatives of a weakly differentiable function f on $\mathbb{R}^n$ belong to the space $L^{n,1}(\mathbb{R}^n)$ then the function f is differentiable in the usual sense at almost every point. An important step of the proof was a maximal inequality showing that, for $p \in [1, \infty)$, the maximal operator $M_{L^{p,1}}$ is bounded from $L^{p,1}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$. It was also noted that even a more general version of this result is true, namely, that the maximal operator $M_{L^{p,q}}$ is bounded from $L^{p,q}(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ whenever $1 \le q \le p$.

A reversible weak-type inequality for the maximal operator built upon the classical Lorentz space $\Lambda^1_\varphi(\mathbb{R}^n)$ (see Section 2 for the definition of this space) was derived by Leckband [29]. A motivation for the study of this maximal operator arose in the earlier work of Leckband and Neugebauer [30], where they showed that the N-th iteration of the Hardy-Littlewood maximal operator is equivalent to the operator $M_{\Lambda^1_\varphi}$ with $\varphi(t) = (\log \frac{e}{t})^{N-1}$. It is worth noting that the reversible weak-type inequality from [29] is closely related to a version of the Riesz-Wiener-Herz inequality (7) for the maximal operator built upon the classical Lorentz space $\Lambda^1_\varphi(\mathbb{R}^n)$.

The maximal operator built upon Orlicz spaces was studied by Bagby and Parsons [3]. They showed that the Orlicz-type maximal operator satisfies an analogy of the Riesz-Wiener-Herz inequality (7). As a consequence, for any pair $L^A(\mathbb{R}^n)$, $L^B(\mathbb{R}^n)$ of Orlicz spaces they were able to find a third Orlicz space $L^C(\mathbb{R}^n)$ such that $M_{L^A}f$ belongs to $L^B(\mathbb{R}^n)$ if and only if f belongs to $L^C(\mathbb{R}^n)$. This provided an extension of the earlier mentioned result from [54], where the important special case when $L^A = L^B = L^1$ (locally) was addressed.

Several further results generalizing inequality (7) appeared later in the literature. Bastero, Milman and Ruiz [4] showed that, given $p \in (1, \infty)$, the Herz-type inequality (an analogue of the first inequality in (7)) holds for the maximal operator $M_{L^{p,q}}$ if and only if $1 < p \le q \le \infty$, while the Riesz-Wiener-type inequality (an analogue of the second inequality in (7)) is fulfilled if and only if $1 \le q \le p$. A simple sufficient condition for the validity of the Riesz-Wiener-type inequality for very general maximal operators (including, in particular, maximal operators built upon rearrangement-invariant Banach function spaces) was proposed by Lerner [31]. The approach introduced in [31] led to alternative proofs of the Riesz-Wiener-type inequality for Orlicz and Lorentz spaces, and was also used by Mastyło and Pérez [33] to prove the Riesz-Wiener-type inequality for further families of rearrangement-invariant Banach function spaces, including, in particular, the classical Lorentz spaces $\Lambda^1_\varphi(\mathbb{R}^n)$ with nonincreasing φ (called also "Lorentz endpoint spaces").

Closely related to maximal inequalities for the operator M_X is the problem of validity of the Lebesgue differentiation theorem in the context of rearrangement-invariant Banach function spaces. This problem arose from the recent work of Cavaliere and Cianchi [11], who studied a generalization of Taylor expansion in the L^p-sense for Sobolev functions into the setting of rearrangement-invariant Banach function norms. An open problem mentioned in that paper concerned finding necessary and sufficient conditions on a rearrangement-invariant Banach function space $X(\mathbb{R}^n)$ in order to ensure that

$$\lim_{r \to 0_+} \|f - f(x)\|_{X,Q(x,r)} = 0 \quad \text{for a.e. } x \in \mathbb{R}^n \tag{8}$$

holds for every function $f \in \mathcal{M}(\mathbb{R}^n)$. We address this problem in Paper III of the present thesis. We also would like to mention that when our paper was almost in final form, it was pointed out to us by A. Gogatishvili that the validity of the Lebesgue differentiation theorem in the context of rearrangement-invariant Banach function spaces has also been investigated in [7, 8, 47, 50]. The analysis of those papers is however limited to the case of functions of one variable.

Let us now turn our attention back to the standard Hardy-Littlewood maximal operator M. As we have mentioned, the question of boundedness of the operator M between rearrangement-invariant Banach function spaces can be answered using inequality (7), but the problem of boundedness of M between function spaces that are not rearrangement invariant is more complicated. This problem is particularly interesting in the context of weighted Lebesgue spaces, and has become the object of several papers. Although lots of interesting results have been obtained during the years, a satisfactory solution of the problem has not been found yet.

Boundedness of the Hardy-Littlewood maximal operator M from the weighted Lebesgue space $L^p_v(\mathbb{R}^n)$ into the weighted Lebesgue space with another weight $L^p_w(\mathbb{R}^n)$ is equivalent to the validity of inequality

$$\int_{\mathbb{R}^n} w(Mf)^p \le C \int_{\mathbb{R}^n} v|f|^p \tag{9}$$

for every function $f \in \mathcal{M}(\mathbb{R}^n)$ and some positive constant C. In the special case when the two weights coincide, inequality (9) was characterized by Muckenhoupt [38]. He showed that the correct necessary and sufficient condition is the A_p-condition

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} < \infty. \tag{10}$$

Here, and in what follows, the notation $\sup_Q$ means that the supremum is taken over all cubes Q in $\mathbb{R}^n$.

The problem of characterizing inequality (9) is much more complicated in the two-weighted setting. Before discussing this problem in detail, let us reformulate it in terms of the pair of weights (w, σ), where $\sigma = v^{-\frac{1}{p-1}}$, since this formulation seems to be more convenient for us. We are going to assume, for simplicity, that $0 < v < \infty$ a.e. Then, setting $g = fv^{\frac{1}{p-1}}$, inequality (9) turns into the form

$$\int_{\mathbb{R}^n} w(M(g\sigma))^p \le C \int_{\mathbb{R}^n} \sigma|g|^p. \tag{11}$$

It is well known that the two-weighted version of the A_p-condition (10), namely,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q \sigma\right)^{p-1} < \infty, \tag{12}$$

is necessary for (11), but it is not sufficient any more (see, e.g., [21, Chapter 4, Example 1.15]). A solution to the two-weighted problem was given by Sawyer [52], who showed that (11) holds if and only if there is a positive constant C such that

$$\int_Q w(M(\chi_Q\sigma))^p \le C \int_Q \sigma \tag{13}$$

for every cube Q. This characterizing condition, however, still involves the operator M itself, and hence does not give a quite satisfactory answer to the abovementioned problem.

Another approach to the two-weighted problem (11) consists in finding sufficient conditions for (11) that are close in form to the A_p-condition (12). These conditions are called "bump conditions" in the literature. They are more explicit than (13), and thus more appropriate for the use in applications.

To introduce the bump theory, let us first observe that the A_p-condition (12) can be written in the form

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{L^{p'},Q} < \infty. \tag{14}$$

Neugebauer [39] showed that if the norms in (14) are replaced by stronger Lebesgue norms, namely, if

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^{pr},Q}\|\sigma^{\frac{1}{p'}}\|_{L^{p'r},Q} < \infty \tag{15}$$

holds for some $r > 1$, then the two-weighted maximal inequality (11) is fulfilled.

Pérez [43] found a way how to weaken the sufficient condition (15). He noticed that in order to obtain (11) one just needs to "bump" in a suitable way the $L^{p'}$-norm of $\sigma^{\frac{1}{p'}}$ in (14). He also showed that more general norms than just those of Lebesgue can be used in this connection. Namely, if $X(\mathbb{R}^n)$ is a Banach function space such that the maximal operator $M_{X'}$ corresponding to the associate space $X'(\mathbb{R}^n)$ of $X(\mathbb{R}^n)$ is bounded on $L^p(\mathbb{R}^n)$ (see (2) for the definition of the associate space), then the bump condition

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{X,Q} < \infty \tag{16}$$

was proved in [43] to be sufficient for (11).

A basic example of a Banach function space for which this result can be applied is the Lebesgue space $L^q(\mathbb{R}^n)$ with $q > p'$. The strength of the result lies, however, in Banach function spaces that are "closer to $L^{p'}(\mathbb{R}^n)$", such as, for instance, an Orlicz space which locally coincides with $L^{p'}(\log L)^\gamma$, where $\gamma > p' - 1$.

The requirement of boundedness of $M_{X'}$ on $L^{p'}(\mathbb{R}^n)$ can be weakened if we allow it to depend on σ. Namely, the following implication holds: if X is a Banach function space such that (16) is fulfilled and there is a positive constant C for which

$$\int_Q (M_{X'}(\sigma^{\frac{1}{p}}\chi_Q))^p \le C\int_Q \sigma \tag{17}$$

for every cube Q, then (11) holds. This was proved by Pérez and Rela [44] as a consequence of the Sawyer characterization (13) of the two-weighted maximal inequality. We note that the result in [44] is restricted only to Orlicz spaces, however, it is easy to observe that the proof given there works equally well for an arbitrary Banach function space over $\mathbb{R}^n$. Moreover, the paper [44] gives even a quantitative version of this result which is shown to hold, at least for Orlicz spaces, not only in the Euclidean setting, but also in the more general context of spaces of homogeneous type.

A question which is not discussed in any of these papers, but would be of interest, is the necessity of these bump conditions for (11). We discuss this question in Paper IV of this thesis.

4.1 Paper III: Norms supporting the Lebesgue differentiation theorem

In this paper we characterize those rearrangement-invariant Banach function spaces $X(\mathbb{R}^n)$ that support the Lebesgue differentiation theorem (8). We show that local separability of the space in question is necessary for such a theorem to be true. Further, we prove that local separability combined with a Riesz-Wiener-type rearrangement inequality for the maximal operator M_X, or with a local boundedness of M_X from $X(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$, or with an easily verifiable condition in terms of concavity of certain functional depending on the norm in $X(\mathbb{R}^n)$, is actually necessary and sufficient for (8). As an application, we characterize those Orlicz, Lorentz and other customary spaces for which the Lebesgue differentiation theorem holds.

We would like to remark that, since the origin of Paper III was motivated by the paper [11], we decided to employ the notation of the X-average used in that paper. This notation is different from the one we have introduced earlier in this section.

4.2 Paper IV: On the necessity of bump conditions for the two-weighted maximal inequality

In this paper we are concerned with necessity of the bump conditions that appeared in [39, 43, 44] for the two-weighted L^p-boundedness of the Hardy-Littlewood maximal operator. Although in several specific situations, these conditions are indeed necessary for (9), we show that this is not the case in general. The question whether one can characterize the two-weighted maximal inequality in terms of suitable Muckenhoupt-type bump conditions thus remains open.

List of papers included in the thesis

[I] A. Cianchi, L. Pick, and L. Slavíková. Higher-order Sobolev embeddings and isoperimetric inequalities. *Adv. Math.*, 273: 568 – 650, 2015.

[II] A. Cianchi, L. Pick, and L. Slavíková. Banach algebras of weakly differentiable functions. Preprint, 2015.

[III] P. Cavaliere, A. Cianchi, L. Pick, and L. Slavíková. Norms supporting the Lebesgue differentiation theorem. Preprint, 2015.

[IV] L. Slavíková. On the necessity of bump conditions for the two-weighted maximal inequality. Preprint, 2015.

Paper [II] is complemented by the note

[II'] L. Slavíková. A Sobolev space embedded to L^∞ does not need to be a Banach algebra.

Bibliography

[1] R. A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] R. A. Adams. On the Orlicz-Sobolev imbedding theorem. *J. Funct. Anal.*, 24(3):241–257, 1977.

[3] R. J. Bagby and J. D. Parsons. Orlicz spaces and rearranged maximal functions. *Math. Nachr.*, 132:15–27, 1987.

[4] J. Bastero, M. Milman, and F. J. Ruiz. Rearrangement of Hardy-Littlewood maximal functions in Lorentz spaces. *Proc. Amer. Math. Soc.*, 128(1):65–74, 2000.

[5] C. Bennett and K. Rudnick. On Lorentz-Zygmund spaces. *Dissertationes Math. (Rozprawy Mat.)*, 175:1–72, 1980.

[6] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics.* Academic Press, Inc., Boston, MA, 1988.

[7] M. Sh. Braverman and A. A. Sedaev. On a characteristic of symmetric spaces. *Funktsional. Anal. i Prilozhen.*, 15(2):65–66, 1981.

[8] M. Sh. Braverman and A. A. Sedaev. L. Kh. Poritskaya's characterization of symmetric spaces. In *Qualitative and approximate methods for investigating operator equations (Russian)*, pages 21–33, 167. Yaroslav. Gos. Univ., Yaroslavl', 1982.

[9] H. Brézis and S. Wainger. A note on limiting cases of Sobolev embeddings and convolution inequalities. *Comm. Partial Differential Equations*, 5(7):773–789, 1980.

[10] M. J. Carro, A. García del Amo, and J. Soria. Weak-type weights and normable Lorentz spaces. *Proc. Amer. Math. Soc.*, 124(3):849–857, 1996.

[11] P. Cavaliere and A. Cianchi. Classical and approximate Taylor expansions of weakly differentiable functions. *Ann. Acad. Sci. Fenn. Math.*, 39(2):527–544, 2014.

[12] A. Cianchi. A sharp embedding theorem for Orlicz-Sobolev spaces. *Indiana Univ. Math. J.*, 45(1):39–65, 1996.

[13] A. Cianchi. Orlicz-Sobolev algebras. *Potential Anal.*, 28(4):379–388, 2008.

[14] A. Cianchi and L. Pick. Optimal Gaussian Sobolev embeddings. *J. Funct. Anal.*, 256(11):3588–3642, 2009.

[15] M. Cwikel and E. Pustylnik. Sobolev type embeddings in the limiting case. *J. Fourier Anal. Appl.*, 4(4-5):433–446, 1998.

[16] T. Donaldson. Nonlinear elliptic boundary value problems in Orlicz-Sobolev spaces. *J. Differential Equations*, 10:507–528, 1971.

[17] T. K. Donaldson and N. S. Trudinger. Orlicz-Sobolev spaces and imbedding theorems. *J. Funct. Anal.*, 8:52–75, 1971.

[18] D. E. Edmunds, R. Kerman, and L. Pick. Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms. *J. Funct. Anal.*, 170(2):307–355, 2000.

[19] W. D. Evans, B. Opic, and L. Pick. Interpolation of operators on scales of generalized Lorentz-Zygmund spaces. *Math. Nachr.*, 182:127–181, 1996.

[20] H. Federer and W. H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[21] J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[22] L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97(4):1061–1083, 1975.

[23] K. Hansson. Imbedding theorems of Sobolev type in potential theory. *Math. Scand.*, 45(1):77–102, 1979.

[24] J. A. Hempel, G. R. Morris, and N. S. Trudinger. On the sharpness of a limiting case of the Sobolev imbedding theorem. *Bull. Austral. Math. Soc.*, 3:369–373, 1970.

[25] C. Herz. The Hardy-Littlewood maximal theorem. *Symposium on Harmonic Analysis, University of Warwick*, 1968.

[26] R. A. Hunt. On $L(p, q)$ spaces. *Enseignement Math. (2)*, 12:249–276, 1966.

[27] R. Kerman and L. Pick. Optimal Sobolev imbeddings. *Forum Math.*, 18(4):535–570, 2006.

[28] R. Kerman and L. Pick. Compactness of Sobolev imbeddings involving rearrangement-invariant norms. *Studia Math.*, 186(2):127–160, 2008.

[29] M. A. Leckband. A note on maximal operators and reversible weak type inequalities. *Proc. Amer. Math. Soc.*, 92(1):19–26, 1984.

[30] M. A. Leckband and C. J. Neugebauer. Weighted iterates and variants of the Hardy-Littlewood maximal operator. *Trans. Amer. Math. Soc.*, 279(1):51–61, 1983.

[31] A. K. Lerner. On an estimate of Calderón-Zygmund operators by dyadic positive operators. *J. Anal. Math.*, 121:141–161, 2013.

[32] G. G. Lorentz. On the theory of spaces Λ. *Pacific J. Math.*, 1:411–429, 1951.

[33] M. Mastyło and C. Pérez. The Hardy-Littlewood maximal type operators between Banach function spaces. *Indiana Univ. Math. J.*, 61(3):883–900, 2012.

[34] V. G. Maz'ya. Classes of domains and imbedding theorems for function spaces. *Soviet Math. Dokl.*, 1:882–885, 1960.

[35] V. G. Maz'ya. p-conductivity and theorems on imbedding certain functional spaces into a C-space. *Dokl. Akad. Nauk SSSR*, 140:299–302, 1961.

[36] V. G. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, augmented edition, 2011.

[37] V. G. Maz'ya and Yu. V. Netrusov. Some counterexamples for the theory of Sobolev spaces on bad domains. *Potential Anal.*, 4(1):47–65, 1995.

[38] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[39] C. J. Neugebauer. Inserting A_p-weights. *Proc. Amer. Math. Soc.*, 87(4):644–648, 1983.

[40] R. O'Neil. Convolution operators and $L(p, q)$ spaces. *Duke Math. J.*, 30:129–142, 1963.

[41] B. Opic and L. Pick. On generalized Lorentz-Zygmund spaces. *Math. Inequal. Appl.*, 2(3):391–467, 1999.

[42] J. Peetre. Espaces d'interpolation et théorème de Soboleff. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):279–317, 1966.

[43] C. Pérez. On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights. *Proc. London Math. Soc. (3)*, 71(1):135–157, 1995.

[44] C. Pérez and E. Rela. A new quantitative two weight theorem for the Hardy-Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 143(2):641–655, 2015.

[45] L. Pick, A. Kufner, O. John, and S. Fučík. *Function spaces. Vol. 1*, volume 14 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, extended edition, 2013.

[46] S. I. Pokhozhaev. On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Dokl. Akad. Nauk SSSR*, 165:36–39, 1965.

[47] L. H. Poritskaya. Lebesgue-Banach points of functions in symmetric spaces. *Mat. Zametki*, 23(4):581–592, 1978.

[48] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1991.

[49] F. Riesz. Sur un Theoreme de Maximum de Mm. Hardy et Littlewood. *J. London Math. Soc.*, S1-7(1):10–13, 1932.

[50] D. V. Salehov and E. M. Semenov. On Lebesgue-Banach points. *Uspehi Mat. Nauk*, 20(6 (126)):151–156, 1965.

[51] E. Sawyer. Boundedness of classical operators on classical Lorentz spaces. *Studia Math.*, 96(2):145–158, 1990.

[52] E. T. Sawyer. A characterization of a two-weight norm inequality for maximal operators. *Studia Math.*, 75(1):1–11, 1982.

[53] L. Slavíková. Compactness of higher-order Sobolev embeddings. *Publ. Mat.*, 59(2):373–448, 2015.

[54] E. M. Stein. Note on the class $L \log L$. *Studia Math.*, 32:305–310, 1969.

[55] E. M. Stein. Editor's note: the differentiability of functions in $\mathbf{R}^n$. *Ann. of Math. (2)*, 113(2):383–385, 1981.

[56] N. S. Trudinger. On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.*, 17:473–483, 1967.

[57] N. Wiener. The ergodic theorem. *Duke Math. J.*, 5(1):1–18, 1939.

[58] V. I. Yudovich. Some estimates connected with integral operators and with solutions of elliptic equations. *Dokl. Akad. Nauk SSSR*, 138:805–808, 1961.

Paper I

A. Cianchi, L. Pick, and L. Slavíková. Higher-order Sobolev embeddings and isoperimetric inequalities. *Adv. Math.*, 273: 568 – 650, 2015.
doi:10.1016/j.aim.2014.12.027

Advances in Mathematics 273 (2015) 568–650

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Higher-order Sobolev embeddings and isoperimetric inequalities

Andrea Cianchi [a,*], Luboš Pick [b], Lenka Slavíková [b]

[a] *Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Piazza Ghiberti 27, 50122 Firenze, Italy*
[b] *Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic*

A R T I C L E   I N F O

Article history:
Received 27 February 2014
Accepted 16 December 2014
Available online 22 January 2015
Communicated by Erwin Lutwak

MSC:
46E35
46E30

Keywords:
Isoperimetric function
Higher-order Sobolev embeddings
Rearrangement-invariant spaces
John domains
Maz'ya domains
Product probability measures
Gaussian Sobolev inequalities
Hardy operator

A B S T R A C T

Optimal higher-order Sobolev type embeddings are shown to follow via isoperimetric inequalities. This establishes a higher-order analogue of a well-known link between first-order Sobolev embeddings and isoperimetric inequalities. Sobolev type inequalities of any order, involving arbitrary rearrangement-invariant norms, on open sets in $\mathbb{R}^n$, possibly endowed with a measure density, are reduced to much simpler one-dimensional inequalities for suitable integral operators depending on the isoperimetric function of the relevant sets. As a consequence, the optimal target space in the relevant Sobolev embeddings can be determined both in standard and in non-standard classes of function spaces and underlying measure spaces. In particular, our results are applied to any-order Sobolev embeddings in regular (John) domains of the Euclidean space, in Maz'ya classes of (possibly irregular) Euclidean domains described in terms of their isoperimetric function, and in families of product probability spaces, of which the Gauss space is a classical instance.

* Corresponding author.
E-mail addresses: cianchi@unifi.it (A. Cianchi), pick@karlin.mff.cuni.cz (L. Pick), slavikova@karlin.mff.cuni.cz (L. Slavíková).

http://dx.doi.org/10.1016/j.aim.2014.12.027

Contents

1. Introduction

Sobolev inequalities and isoperimetric inequalities had traditionally been investigated along independent lines of research, which had led to the cornerstone results by Sobolev [81,82], Gagliardo [43] and Nirenberg [73] on the one hand, and by De Giorgi [34] on the other hand, until their intimate connection was discovered some half a century ago. Such breakthrough goes back to the work of Maz'ya [68,69], who proved that quite general Sobolev inequalities are equivalent to either isoperimetric or isocapacitary inequalities. Independently, Federer and Fleming [41] also exploited De Giorgi's isoperimetric theorem to exhibit the best constant in the special case of the Sobolev inequality for functions whose gradient is integrable with power one in $\mathbb{R}^n$. These advances paved the way to an extensive research, along diverse directions, on the interplay between isoperimetric and Sobolev inequalities, and to a number of remarkable applications, such as the classics by Moser [72], Talenti [87], Aubin [2], Brézis and Lieb [13]. The contributions to this field now constitute the corpus of a vast literature, which includes the papers [1,3,5,9,10,14,15,19,21,23,26,29,30,32,37,39,44,47,48,51,52,54–56,61–63,71,77,86,88,91] and the monographs [16,18,20,45,49,70,80]. Needless to say, this list of references is by no means exhaustive.

The strength of the approach to Sobolev embeddings via isoperimetric inequalities stems from the fact that not only it applies to a broad range of situations, but also typically yields sharp results. The available results, however, essentially deal with first-order Sobolev inequalities, apart from few exceptions on quite specific issues concerning the higher-order case. Indeed, isoperimetric inequalities are usually considered ineffectual in proving optimal higher-order Sobolev embeddings. Customary techniques that are crucial in the derivation of first-order Sobolev inequalities from isoperimetric inequalities, such as symmetrization, or just truncation, cannot be adapted to the proof of higher-order Sobolev inequalities. A major drawback is that these operations do not preserve higher-order (weak) differentiability. A new approach to the sharp Sobolev inequality in $\mathbb{R}^n$, based on mass transportation techniques, has been introduced in [33], and has later been developed in various papers to attack other Sobolev type inequalities, but still in the first-order case. On the other hand, methods which can be employed to handle higher-order Sobolev inequalities, such as representation formulas, Fourier transforms, atomic decomposition, are not flexible enough to produce sharp conclusions in full generality. A paradigmatic instance in this connection is provided by the standard Sobolev embedding in $\mathbb{R}^n$ to which we alluded above, whose original proof via representation formulas [81,82] does not include the borderline case when derivatives are just integrable with power one. This case was restored in [43] and [73] through a completely different technique that rests upon one-dimensional integration combined with a clever use of Hölder's inequality.

One main purpose of the present paper is to show that, this notwithstanding, isoperimetric inequalities do imply optimal higher-order Sobolev embeddings in quite general frameworks. Sobolev embeddings for functions defined on underlying domains in $\mathbb{R}^n$, equipped with fairly general measures, are included in our discussion. Also, Sobolev-type norms built upon any rearrangement-invariant Banach function norm are considered. The use of isoperimetric inequalities is shown to allow for a unified approach to the relevant embeddings, which is based on the reduction to considerably simpler one-dimensional inequalities. Such reduction principle is crucial in a characterization of the best possible target for arbitrary-order Sobolev embeddings, in the class of all rearrangement-invariant Banach function spaces. As a consequence, the optimal target space in arbitrary-order Sobolev embeddings involving various customary and non-standard underlying domains and norms can be exhibited. In fact, establishing optimal higher-order Gaussian Sobolev embeddings, namely Sobolev embeddings in $\mathbb{R}^n$ endowed with the Gauss measure, was our original motivation for the present research. Failure of standard strategies in the solution of this problem led us to develop the general picture which is now the subject of this paper.

A key step in our proofs amounts to the development of a sharp iteration method involving subsequent applications of optimal Sobolev embeddings. We consider this method of independent interest for its possible use in different problems, where regularity properties of functions endowed with higher-order derivatives are in question.

2. An overview

We shall deal with Sobolev inequalities in an open connected set – briefly, a domain – Ω in $\mathbb{R}^n$, $n \geq 1$, equipped with a finite measure ν which is absolutely continuous with respect to the Lebesgue measure, with density ω. Namely,

$$d\nu(x) = \omega(x)\, dx, \tag{2.1}$$

where ω is a Borel function such that $\omega(x) > 0$ a.e. in Ω. Throughout the paper, we assume, for simplicity of notation, that ν is normalized in such a way that $\nu(\Omega) = 1$.

The basic case when ν is the Lebesgue measure will be referred to as Euclidean. Sobolev embeddings of arbitrary order for functions defined in Ω, with unconstrained values on $\partial\Omega$, will be considered. However, the even simpler case of functions vanishing (in the suitable sense) on $\partial\Omega$ together with their derivatives up to the order $m-1$ could be included in our discussion.

The isoperimetric inequality relative to (Ω, ν) tells us that

$$P_\nu(E, \Omega) \geq I_{\Omega,\nu}\big(\nu(E)\big), \tag{2.2}$$

where E is any measurable subset of Ω, and $P_\nu(E, \Omega)$ stands for its perimeter in Ω with respect to ν. Moreover, $I_{\Omega,\nu}$ denotes the largest non-decreasing function in $[0, \frac{1}{2}]$ for which (2.2) holds, called the isoperimetric function (or isoperimetric profile) of (Ω, ν), which was introduced in [68].

In the Euclidean case, (Ω, ν) will be simply denoted by Ω, and $I_{\Omega,\nu}$ by I_Ω. The isoperimetric function $I_{\Omega,\nu}$ is known only in few special instances, e.g. when Ω is a Euclidean ball [70], or agrees with the space $\mathbb{R}^n$ equipped with the Gauss measure [12]. However, the asymptotic behavior of $I_{\Omega,\nu}$ at 0 – the piece of information relevant in our applications – can be evaluated for various classes of domains, such as: Euclidean bounded domains whose boundary is locally a graph of a Lipschitz function [70], or, more generally, has a prescribed modulus of continuity [22,58]; Euclidean John domains, and even s-John domains; the space $\mathbb{R}^n$ equipped with the Gauss measure [12], or with product probability measures which generalize it [3,4]. The literature on isoperimetric inequalities is very rich. Let us limit ourselves to mentioning that, besides those quoted above, recent contributions on isoperimetric problems in (domains in) $\mathbb{R}^n$ endowed with a measure ν include [17,35,42,79].

Given a Banach function space $X(\Omega, \nu)$ of measurable functions on Ω, and a positive integer $m \in \mathbb{N}$, the m-th order Sobolev type space built upon $X(\Omega, \nu)$ is the normed linear space $V^m X(\Omega, \nu)$ of all functions on Ω whose m-th order weak derivatives belong to $X(\Omega, \nu)$, equipped with a natural norm induced by $X(\Omega, \nu)$.

A Sobolev embedding amounts to the boundedness of the identity operator from the Sobolev space $V^m X(\Omega, \nu)$ into another function space $Y(\Omega, \nu)$ and will be denoted by

$$V^m X(\Omega, \nu) \to Y(\Omega, \nu). \tag{2.3}$$

When $m = 1$, we refer to (2.3) as a first-order embedding; otherwise, we call it a higher-order embedding.

Necessary and sufficient conditions for the validity of first-order Euclidean Sobolev embeddings with $X(\Omega) = L^1(\Omega)$ and $Y(\Omega) = L^q(\Omega)$ for some $q \geq 1$ can be given through the isoperimetric function I_Ω. Sufficient conditions for first-order Sobolev embeddings when $X(\Omega) = L^p(\Omega)$ for some $p > 1$ and $Y(\Omega) = L^q(\Omega)$, for some $q \geq 1$ can also be provided in terms of I_Ω. These results were established in [68,69], and are exposed in detail in [70, Section 6.4.3].

More recently, first-order Sobolev embeddings of the general form (2.3) (with $m = 1$), where $X(\Omega, \nu)$ and $Y(\Omega, \nu)$ are Banach function spaces whose norm depends only on the measure of level sets of functions, called rearrangement-invariant spaces in the literature, have been shown to follow from one-dimensional inequalities for suitable Hardy type operators which depend on the isoperimetric function $I_{\Omega,\nu}$, and involve the representation function norms $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ of $X(\Omega, \nu)$ and $Y(\Omega, \nu)$, respectively.

Although a reverse implication need not hold in very pathological settings (e.g. in Euclidean domains of Nikodým type [70, Remark 6.5.2]), first-order Sobolev inequalities are known to be equivalent to the associated one-dimensional Hardy inequalities in most situations of interest in applications. This is the case, for instance, when Ω is a regular Euclidean domain – specifically, a John domain in $\mathbb{R}^n$, $n \geq 2$ (see Section 6 for a definition). The class of John domains includes other more classical families of domains, such as Lipschitz domains, and domains with the cone property. The John domains arise in connection with the study of holomorphic dynamical systems and quasiconformal mappings. John domains are known to support a first-order Sobolev inequality with the same exponents as in the standard Sobolev inequality [11,48,54]. In fact, being a John domain is a necessary condition for such a Sobolev inequality to hold in the class of two-dimensional simply connected open sets, and in quite general classes of higher-dimensional domains [14]. The isoperimetric function I_Ω of any John domain is known to satisfy

$$I_\Omega(s) \approx s^{\frac{1}{n'}} \tag{2.4}$$

near 0, where $n' = \frac{n}{n-1}$. Here, and in what follows, the notation $\approx$ means that the two sides are bounded by each other up to multiplicative constants independent of appropriate quantities. For instance, in (2.4) such constants depend only on Ω.

As a consequence of (2.4), one can show that the first-order Sobolev embedding

$$V^1X(\Omega) \to Y(\Omega) \tag{2.5}$$

holds if and only if the Hardy type inequality

$$\left\| \int_t^1 f(s)s^{-1+\frac{1}{n}}\, ds \right\|_{Y(0,1)} \leq C\|f\|_{X(0,1)} \tag{2.6}$$

holds for some constant C, and for every nonnegative $f \in X(0,1)$. Results of this kind, showing that Sobolev embeddings follow from (and are possibly equivalent to) one-dimensional inequalities will be referred to as reduction principles or reduction theorems. The equivalence of (2.5) and (2.6) is a key tool in determining the optimal target $Y(\Omega)$ for $V^1X(\Omega)$ in (2.5) within families of rearrangement-invariant function spaces, such as Lebesgue, Lorentz, and Orlicz spaces, provided that such an optimal target space does exist [23,25,37]. An even more standard version of this reduction result, which holds for functions vanishing on $\partial\Omega$, and is called Pólya–Szegö symmetrization principle, is a crucial step in exhibiting the sharp constant in the classical Sobolev inequalities to which we alluded above [2,13,72,87].

A version of this picture for higher-order Sobolev inequalities is exhibited in the present paper. We show that any m-th order Sobolev embedding involving arbitrary rearrangement-invariant norms can be reduced to a suitable one-dimensional inequality for an integral operator, with a kernel depending on $I_{\Omega,\nu}$ and m.

Just to give an idea of the conclusions which follow from our results, let us mention that, if, for instance, Ω is a Euclidean John domain in $\mathbb{R}^n$, $n \geq 2$, then a full higher-order analogue of the equivalence of (2.5) and (2.6) holds. Namely, the m-th order Sobolev embedding

$$V^m X(\Omega) \to Y(\Omega)$$

holds if and only if the Hardy type inequality

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{n}}\, ds \right\|_{Y(0,1)} \leq C \|f\|_{X(0,1)} \tag{2.7}$$

holds for some constant C, and for every nonnegative $f \in X(0,1)$ (Theorem 6.1, Section 6).

Our approach to reduction principles for higher-order Sobolev embeddings relies on the iteration of first-order results. Loosely speaking, iteration is understood in the sense that, given a rearrangement-invariant space and $m \in \mathbb{N}$, a first-order optimal Sobolev embedding is applied to show that the $(m-1)$-th order derivatives of functions from the relevant Sobolev space belong to a suitable rearrangement-invariant space. Another first-order optimal Sobolev embedding is then applied to show that the $(m-2)$-th order derivatives belong to another rearrangement-invariant space, and so on. Eventually, m optimal first-order Sobolev embeddings are exploited to deduce that the functions themselves belong to a certain space.

Let us warn that, although this strategy is quite natural in principle, its implementation is not straightforward. Indeed, even in the basic setting when Ω is a Euclidean domain with a smooth boundary, and standard families of norms are considered, iteration of optimal first-order embeddings need not lead to optimal higher-order counterparts.

To see this, recall, for instance, that, if Ω is a regular domain in $\mathbb{R}^2$, then

$$V^2 L^1(\Omega) \to L^\infty(\Omega). \tag{2.8}$$

On the other hand, iterating twice the classical first-order Sobolev embedding only tells us that

$$V^2 L^1(\Omega) \to V^1 L^2(\Omega) \to L^q(\Omega) \tag{2.9}$$

for every $q < \infty$, and neither of the iterated embeddings can be improved in the framework of Lebesgue spaces. This shows that subsequent applications of optimal first-order Sobolev embeddings in the class of Lebesgue spaces do not necessarily yield optimal higher-order counterparts.

One might relate the loss of optimality in the chain of embeddings (2.9) to the lack of an optimal Lebesgue target space for the first-order Sobolev embedding of $V^1L^2(\Omega)$ when $n = 2$. However, non-optimal targets may appear after iteration even in situations where optimal first-order target spaces do exist. Consider, for example, Euclidean Sobolev embeddings involving Orlicz spaces. The optimal target in Sobolev embeddings of any order always exists in this class of spaces, and can be explicitly determined [23,27], see also [50]. In particular, Orlicz spaces naturally arise in the borderline case of the Sobolev embedding theorem. Indeed, if Ω is a regular domain in $\mathbb{R}^n$ and $1 \le m < n$, then

$$V^m L^{\frac{n}{m}}(\Omega) \to \exp L^{\frac{n}{n-m}}(\Omega) \tag{2.10}$$

[78,84,90]; see also [89] for $m = 1$. Here, $\exp L^\alpha(\Omega)$, with $\alpha > 0$, denotes the Orlicz space associated with the Young function given by $e^{t^\alpha} - 1$ for $t \ge 0$. Observe that the target space in (2.10) is actually optimal in the class of all Orlicz spaces [23,25]. Now, assume, for instance, that $n \ge 3$ and $m = 2$. Then (2.10) reduces to

$$V^2 L^{\frac{n}{2}}(\Omega) \to \exp L^{\frac{n}{n-2}}(\Omega).$$

Via the iteration of optimal first-order embeddings, one gets

$$V^2 L^{\frac{n}{2}}(\Omega) \to V^1 L^n(\Omega) \to \exp L^{\frac{n}{n-1}}(\Omega) \supsetneq \exp L^{\frac{n}{n-2}}(\Omega).$$

Thus, subsequent applications of optimal Sobolev embeddings even in the class of Orlicz spaces, where optimal target spaces always exist, need not result in optimal higher-order Sobolev embeddings.

The underlying idea behind the method that we shall introduce is that such a loss of optimality of the target space under iteration does not occur, provided that first-order (in fact, any-order) Sobolev embeddings whose targets are optimal among all rearrangement-invariant spaces are iterated. We thus proceed via a two-step argument, which can be outlined as follows. Firstly, given any function norm $\|\cdot\|_{X(0,1)}$ and the isoperimetric function $I_{\Omega,\nu}$ of (Ω, ν), the optimal target among all rearrangement-invariant function norms for the first-order Sobolev space $V^1X(\Omega, \nu)$ is characterized; secondly, first-order Sobolev embeddings with an optimal target are iterated to derive optimal targets in arbitrary-order Sobolev embeddings.

In order to grasp this procedure in a simple situation, observe that, when applied in the proof of embedding (2.8), it amounts to strengthening the chain in (2.9) by

$$V^2 L^1(\Omega) \to V^1 L^{2,1}(\Omega) \to L^\infty(\Omega), \tag{2.11}$$

where $L^{2,1}(\Omega)$ denotes a Lorentz space (strictly contained in $L^2(\Omega)$). We refer to [53,74,76] for standard Sobolev embeddings in Lorentz spaces. Note that both targets in the embeddings in (2.11) are actually optimal among all rearrangement-invariant spaces.

As mentioned above, our reduction principle asserts that the Sobolev embedding (2.3) follows from a suitable one-dimensional inequality for an integral operator depending on $I_{\Omega,\nu}$, m, $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$. Interestingly, in contrast with the first-order case, the relevant integral operator is not just of Hardy type, but involves a genuine kernel. The latter takes back the form of a basic (weighted) Hardy operator only if, loosely speaking, the isoperimetric function $I_{\Omega,\nu}(s)$ does not decay too fast to 0 when s tends to 0. This is the case, for instance, of (2.7). A major consequence of the reduction principle is a characterization of a target space $Y(\Omega,\nu)$ in embedding (2.3), depending on $X(\Omega,\nu)$, m, and $I_{\Omega,\nu}$, which turns out to be optimal among all rearrangement-invariant spaces whenever Sobolev embeddings and associated one-dimensional inequalities in the reduction principle are actually equivalent. This latter property depends on the geometry of (Ω,ν), and is fulfilled in most customary situations, to some of which a substantial part of this paper is devoted.

Besides regular Euclidean domains, namely the John domains which we have already briefly discussed, the implementations of our results that will be presented concern Maz'ya classes of (possibly irregular) Euclidean domains, and product probability spaces, of which the Gauss space and the Boltzmann spaces are distinguished instances.

The Maz'ya classes are defined as families of domains whose isoperimetric function is bounded from below by some fixed power. Sobolev embeddings in all domains from a class of this type take the same form, and a worst, in a sense, domain from the relevant class can be singled out to demonstrate the sharpness of the results.

The product probability spaces in $\mathbb{R}^n$ that are taken into account were analyzed in [3,4], and share common features with the Gauss space, namely $\mathbb{R}^n$ endowed with the probability measure $d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$. In particular, the Boltzmann spaces can be handled via our approach.

For the reader's convenience, we list at the end of the paper the main symbols employed throughout, with a reference to the equation where they are introduced.

3. Spaces of measurable functions

In this section, we briefly recall some basic facts from the theory of rearrangement-invariant spaces. For more details, a standard reference is [7].

Let (Ω,ν) be as in Section 2. Recall that we are assuming $\nu(\Omega) = 1$. The measure of any measurable set $E \subset \Omega$ is thus given by

$$\nu(E) = \int_E \omega(x)\, dx.$$

We set

$$\mathcal{M}(\Omega,\nu)=\{u:\Omega\to[-\infty,\infty]:\ u \text{ is } \nu\text{-measurable in } \Omega\}, \tag{3.1}$$
$$\mathcal{M}_+(\Omega,\nu)=\{u\in\mathcal{M}(\Omega,\nu):u\geq 0\}, \tag{3.2}$$

and

$$\mathcal{M}_0(\Omega,\nu)=\{u\in\mathcal{M}(\Omega,\nu):u \text{ is finite a.e. in } \Omega\}. \tag{3.3}$$

The *decreasing rearrangement* $u^*:[0,1]\to[0,\infty]$ of a function $u\in\mathcal{M}(\Omega,\nu)$ is defined as

$$u^*(s)=\inf\{t\geq 0:\nu(\{x\in\Omega:|u(x)|>t\})\leq s\}\quad \text{for } s\in[0,1]. \tag{3.4}$$

The operation $u\mapsto u^*$ is monotone in the sense that

$$|u|\leq|v| \quad \text{a.e. in } \Omega \quad \text{implies} \quad u^*\leq v^* \quad \text{in } [0,1].$$

We also define $u^{**}:(0,1]\to[0,\infty]$ as

$$u^{**}(s)=\frac{1}{s}\int_0^s u^*(r)\,dr \quad \text{for } s\in(0,1]. \tag{3.5}$$

Note that u^{**} is also non-increasing, and $u^*\leq u^{**}$ in $(0,1]$. Moreover,

$$\int_0^s (u+v)^*(r)\,dr\leq\int_0^s u^*(r)\,dr+\int_0^s v^*(r)\,dr \quad \text{for } s\in[0,1], \tag{3.6}$$

for every $u,v\in\mathcal{M}_+(\Omega,\nu)$.

A basic property of rearrangements is the *Hardy–Littlewood inequality*, which tells us that, if $u,v\in\mathcal{M}(\Omega,\nu)$, then

$$\int_\Omega |u(x)v(x)|d\nu(x)\leq\int_0^1 u^*(s)v^*(s)ds. \tag{3.7}$$

A special case of (3.7) states that for every $u\in\mathcal{M}(\Omega,\nu)$ and every measurable set $E\subset\Omega$,

$$\int_E |u(x)|d\nu(x)\leq\int_0^{\nu(E)} u^*(s)\,ds.$$

We say that a functional

$$\| \cdot \|_{X(0,1)} : \mathcal{M}_+(0,1) \to [0,\infty] \tag{3.8}$$

is a *function norm*, if, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}_+(0,1)$, and every $\lambda \geq 0$, the following properties hold:

(P1) $\|f\|_{X(0,1)} = 0$ if and only if $f = 0$ a.e.; $\|\lambda f\|_{X(0,1)} = \lambda\|f\|_{X(0,1)}$; $\|f+g\|_{X(0,1)} \leq \|f\|_{X(0,1)} + \|g\|_{X(0,1)}$;
(P2) $f \leq g$ a.e. implies $\|f\|_{X(0,1)} \leq \|g\|_{X(0,1)}$;
(P3) $f_j \nearrow f$ a.e. implies $\|f_j\|_{X(0,1)} \nearrow \|f\|_{X(0,1)}$;
(P4) $\|1\|_{X(0,1)} < \infty$;
(P5) $\int_0^1 f(x)\,dx \leq C\|f\|_{X(0,1)}$ for some constant C independent of f.

If, in addition,

(P6) $\|f\|_{X(0,1)} = \|g\|_{X(0,1)}$ whenever $f^* = g^*$,

we say that $\| \cdot \|_{X(0,1)}$ is a *rearrangement-invariant function norm.*

With any rearrangement-invariant function norm $\| \cdot \|_{X(0,1)}$, it is associated another functional on $\mathcal{M}_+(0,1)$, denoted by $\| \cdot \|_{X'(0,1)}$, and defined, for $g \in \mathcal{M}_+(0,1)$, as

$$\|g\|_{X'(0,1)} = \sup_{\substack{f\geq 0 \\ \|f\|_{X(0,1)}\leq 1}} \int_0^1 f(s)g(s)\,ds. \tag{3.9}$$

It turns out that $\| \cdot \|_{X'(0,1)}$ is also a rearrangement-invariant function norm, which is called the *associate function norm* of $\| \cdot \|_{X(0,1)}$. Moreover, for every rearrangement-invariant function norm $\| \cdot \|_{X(0,1)}$ and every function $f \in \mathcal{M}_+(0,1)$, we have

$$\|f\|_{X(0,1)} = \sup_{\substack{g\geq 0 \\ \|g\|_{X'(0,1)}\leq 1}} \int_0^1 f(s)g(s)\,ds. \tag{3.10}$$

We also introduce yet another functional on $\mathcal{M}_+(0,1)$, the *down associate function norm* of $\| \cdot \|_{X(0,1)}$. It is denoted by $\| \cdot \|_{X'_d(0,1)}$, and defined, for $g \in \mathcal{M}_+(0,1)$, as

$$\|g\|_{X'_d(0,1)} = \sup_{\|f\|_{X(0,1)}\leq 1} \int_0^1 f^*(t)g(t)\,dt. \tag{3.11}$$

Clearly, one has that $\|g\|_{X'_d(0,1)} \leq \|g\|_{X'(0,1)}$ for every $g \in \mathcal{M}_+(0,1)$, and $\|g\|_{X'_d(0,1)} = \|g\|_{X'(0,1)}$ if g is non-increasing.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, the space $X(\Omega,\nu)$ is defined as the collection of all functions $u \in \mathcal{M}(\Omega,\nu)$ such that the expression

$$\|u\|_{X(\Omega,\nu)} = \|u^*\|_{X(0,1)} \tag{3.12}$$

is finite. Such expression defines a norm on $X(\Omega,\nu)$, and the latter is a Banach space endowed with this norm, called a rearrangement-invariant space. Moreover, $X(\Omega,\nu) \subset \mathcal{M}_0(\Omega,\nu)$ for any rearrangement-invariant space $X(\Omega,\nu)$. The space $X(0,1)$ is called the *representation space* of $X(\Omega,\nu)$.

We also denote

$$X_{\mathrm{loc}}(\Omega,\nu) = \big\{u \in \mathcal{M}(\Omega,\nu) : u\chi_G \in X(\Omega,\nu) \text{ for every compact set } G \subset \Omega\big\}. \tag{3.13}$$

Here, χ_G denotes the characteristic function of G.

The rearrangement-invariant space $X'(\Omega,\nu)$ built upon the function norm $\|\cdot\|_{X'(0,1)}$ is called the *associate space* of $X(\Omega,\nu)$. It turns out that $X''(\Omega,\nu) = X(\Omega,\nu)$. Furthermore, the *Hölder inequality*

$$\int_\Omega \big|u(x)v(x)\big|\, d\nu(x) \le \|u\|_{X(\Omega,\nu)}\|v\|_{X'(\Omega,\nu)}$$

holds for every $u \in X(\Omega,\nu)$ and $v \in X'(\Omega,\nu)$.

For any rearrangement-invariant spaces $X(\Omega,\nu)$ and $Y(\Omega,\nu)$, we have that

$$X(\Omega,\nu) \to Y(\Omega,\nu) \quad \text{if and only if} \quad Y'(\Omega,\nu) \to X'(\Omega,\nu), \tag{3.14}$$

with the same embedding norms [7, Chapter 1, Proposition 2.10].

Given any $\lambda > 0$, the *dilation operator* E_λ, defined at $f \in \mathcal{M}(0,1)$ by

$$(E_\lambda f)(s) = \begin{cases} f(\lambda^{-1}s) & \text{if } 0 < s \le \lambda \\ 0 & \text{if } \lambda < s < 1, \end{cases} \tag{3.15}$$

is bounded on any rearrangement-invariant space $X(0,1)$, with norm not exceeding $\max\{1, \frac{1}{\lambda}\}$.

Hardy's lemma tells us that if $f_1, f_2 \in \mathcal{M}_+(0,1)$ satisfy

$$\int_0^s f_1(r)dr \le \int_0^s f_2(r)dr \quad \text{for every } s \in (0,1),$$

then

$$\int_0^1 f_1(r)h(r)dr \le \int_0^1 f_2(r)h(r)dr$$

for every non-increasing function $h : (0,1) \to [0,\infty]$. A consequence of this result is the *Hardy–Littlewood–Pólya principle* which asserts that if the functions $u, v \in \mathcal{M}(\Omega,\nu)$ satisfy

$$\int_0^s u^*(r)\,dr \le \int_0^s v^*(r)\,dr \quad \text{for } s \in (0,1),$$

then

$$\|u\|_{X(\Omega,\nu)} \le \|v\|_{X(\Omega,\nu)}$$

for every rearrangement-invariant space $X(\Omega,\nu)$.

Let $X(\Omega,\nu)$ and $Y(\Omega,\nu)$ be rearrangement-invariant spaces. By [7, Chapter 1, Theorem 1.8],

$$X(\Omega,\nu) \subset Y(\Omega,\nu) \quad \text{if and only if} \quad X(\Omega,\nu) \to Y(\Omega,\nu).$$

For every rearrangement-invariant space $X(\Omega,\nu)$, one has that

$$L^\infty(\Omega,\nu) \to X(\Omega,\nu) \to L^1(\Omega,\nu). \tag{3.16}$$

An embedding of the form

$$X_{\mathrm{loc}}(\Omega,\nu) \to Y_{\mathrm{loc}}(\Omega,\mu),$$

where μ is a measure enjoying the same properties as ν, means that, for every compact set $G \subset \Omega$, there exists a constant C such that

$$\|u\chi_G\|_{Y(\Omega,\mu)} \le C\|u\chi_G\|_{X(\Omega,\nu)},$$

for every $u \in X_{\mathrm{loc}}(\Omega,\nu)$.

Throughout, we use the convention that $\frac{1}{\infty} = 0$, and $0 \cdot \infty = 0$.

A basic example of a function norm is the standard *Lebesgue norm* $\|\cdot\|_{L^p(0,1)}$, for $p \in [1,\infty]$, upon which the Lebesgue spaces $L^p(\Omega,\nu)$ are built.

The *Lorentz spaces* yield an extension of the Lebesgue spaces. Assume that $1 \le p, q \le \infty$. We define the functionals $\|\cdot\|_{L^{p,q}(0,1)}$ and $\|\cdot\|_{L^{(p,q)}(0,1)}$ as

$$\begin{aligned} \|f\|_{L^{p,q}(0,1)} &= \left\| s^{\frac{1}{p}-\frac{1}{q}} f^*(s) \right\|_{L^q(0,1)} \quad \text{and} \\ \|f\|_{L^{(p,q)}(0,1)} &= \left\| s^{\frac{1}{p}-\frac{1}{q}} f^{**}(s) \right\|_{L^q(0,1)}, \end{aligned} \tag{3.17}$$

respectively, for $f \in \mathcal{M}_+(0,1)$. One can show that

$$L^{p,q}(\Omega,\nu) = L^{(p,q)}(\Omega,\nu) \quad \text{if } 1 < p \le \infty, \tag{3.18}$$

with equivalent norms. If one of the conditions

$$\begin{cases} 1 < p < \infty, & 1 \le q \le \infty, \\ p = q = 1, \\ p = q = \infty, \end{cases} \tag{3.19}$$

is satisfied, then $\|\cdot\|_{L^{p,q}(0,1)}$ is equivalent to a rearrangement-invariant function norm. The corresponding rearrangement-invariant space $L^{p,q}(\Omega,\nu)$ is called a *Lorentz space*.

Let us recall that $L^{p,p}(\Omega,\nu) = L^p(\Omega,\nu)$ for every $p \in [1,\infty]$ and that $1 \le q \le r \le \infty$ implies $L^{p,q}(\Omega,\nu) \to L^{p,r}(\Omega,\nu)$ with equality if and only if $q = r$.

Assume now that $1 \le p, q \le \infty$, and a third parameter $\alpha \in \mathbb{R}$ is called into play. We define the functionals $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ and $\|\cdot\|_{L^{(p,q;\alpha)}(0,1)}$ as

$$\begin{cases} \|f\|_{L^{p,q;\alpha}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\frac{2}{s}\right) f^*(s) \right\|_{L^q(0,1)}, \\ \|f\|_{L^{(p,q;\alpha)}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\frac{2}{s}\right) f^{**}(s) \right\|_{L^q(0,1)}, \end{cases} \tag{3.20}$$

respectively, for $f \in \mathcal{M}_+(0,1)$. If one of the following conditions

$$\begin{cases} 1 < p < \infty, & 1 \le q \le \infty, & \alpha \in \mathbb{R}; \\ p = 1, & q = 1, & \alpha \ge 0; \\ p = \infty, & q = \infty, & \alpha \le 0; \\ p = \infty, & 1 \le q < \infty, & \alpha + \frac{1}{q} < 0, \end{cases} \tag{3.21}$$

is satisfied, then $\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ is equivalent to a rearrangement-invariant function norm, called a *Lorentz–Zygmund function norm*. The corresponding rearrangement-invariant space $L^{p,q;\alpha}(\Omega,\nu)$ is a *Lorentz–Zygmund space*. At a few occasions, we shall need also the so-called *generalized Lorentz–Zygmund space* $L^{p,q;\alpha,\beta}(\Omega,\nu)$, where $p, q \in [1,\infty]$ and $\alpha, \beta \in \mathbb{R}$. It is the space built upon the functional given by

$$\|f\|_{L^{p,q;\alpha,\beta}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\frac{2}{s}\right) \log^{\beta}\left(1 + \log\left(\frac{2}{s}\right)\right) f^*(s) \right\|_{L^q(0,1)} \tag{3.22}$$

for $f \in \mathcal{M}_+(0,1)$. The values of p, q, α and β, for which $\|\cdot\|_{L^{p,q;\alpha,\beta}(0,1)}$ is actually equivalent to a rearrangement-invariant function norm, are characterized in [40]. For more details on (generalized) Lorentz–Zygmund spaces, see e.g. [6,40,75]. Assume that one of the conditions in (3.21) is satisfied. Then the associate space $(L^{p,q;\alpha})'(\Omega,\nu)$ of the Lorentz–Zygmund space $L^{p,q;\alpha}(\Omega,\nu)$ satisfies (up to equivalent norms)

$$\big(L^{p,q;\alpha}\big)'(\Omega,\nu) = \begin{cases} L^{p',q';-\alpha}(\Omega,\nu) & \text{if } 1 < p < \infty,\ 1 \le q \le \infty,\ \alpha \in \mathbb{R}; \\ L^{\infty,\infty;-\alpha}(\Omega,\nu) & \text{if } p = 1,\ q = 1,\ \alpha \ge 0; \\ L^{1,1;-\alpha}(\Omega,\nu) & \text{if } p = \infty,\ q = \infty,\ \alpha \le 0; \\ L^{(1,q';-\alpha-1)}(\Omega,\nu) & \text{if } p = \infty,\ 1 \le q < \infty,\ \alpha + \frac{1}{q} < 0 \end{cases} \tag{3.23}$$

[75, Theorems 6.11 and 6.12]. Moreover,

$$L^{(p,q;\alpha)}(\Omega,\nu) = \begin{cases} L^{p,q;\alpha}(\Omega,\nu) & \text{if } 1 < p \le \infty; \\ L^{1,1;\alpha+1}(\Omega,\nu) & \text{if } p = q = 1,\ \alpha > -1, \end{cases} \tag{3.24}$$

and

$$L^{p}(\Omega,\nu) \to L^{(1,q)}(\Omega,\nu) \quad \text{for every } 1 < p \le \infty,\ 1 \le q \le \infty$$

[75, Theorem 3.16 (i), (ii)].

A generalization of the Lebesgue spaces in a different direction is provided by the Orlicz spaces. Let $A : [0,\infty) \to [0,\infty]$ be a Young function, namely a convex (non-trivial), left-continuous function vanishing at 0. Any such function takes the form

$$A(t) = \int_0^t a(\tau)d\tau \quad \text{for } t \ge 0, \tag{3.25}$$

for some non-decreasing, left-continuous function $a : [0,\infty) \to [0,\infty]$ which is neither identically equal to 0, nor to ∞. The Orlicz space $L^A(\Omega,\nu)$ is the rearrangement-invariant space associated with the *Luxemburg function norm* defined as

$$\|f\|_{L^A(0,1)} = \inf\left\{\lambda > 0 : \int_0^1 A\left(\frac{f(s)}{\lambda}\right)ds \le 1\right\} \tag{3.26}$$

for $f \in \mathcal{M}_+(0,1)$. In particular, $L^A(\Omega,\nu) = L^p(\Omega,\nu)$ if $A(t) = t^p$ for some $p \in [1,\infty)$, and $L^A(\Omega,\nu) = L^\infty(\Omega,\nu)$ if $A(t) = \infty\chi_{(1,\infty)}(t)$.

A Young function A is said to dominate another Young function B near infinity if positive constants c and t_0 exist such that

$$B(t) \le A(ct) \quad \text{for } t \ge t_0.$$

The functions A and B are called equivalent near infinity if they dominate each other near infinity. One has that

$$L^A(\Omega,\nu) \to L^B(\Omega,\nu) \quad \text{if and only if} \quad A \text{ dominates } B \text{ near infinity.} \tag{3.27}$$

We denote by $L^p \log^\alpha L(\Omega,\nu)$ the Orlicz space associated with a Young function equivalent to $t^p(\log t)^\alpha$ near infinity, where either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \ge 0$. The notation $\exp L^{\beta}(\Omega,\nu)$ will be used for the Orlicz space built upon a Young function equivalent to e^{t^β} near infinity, where $\beta > 0$. Also, $\exp\exp L^{\beta}(\Omega,\nu)$ stands for the Orlicz space associated with a Young function equivalent to $e^{e^{t^\beta}}$ near infinity.

The classes of Orlicz and (generalized) Lorentz–Zygmund spaces overlap, up to equivalent norms. For instance, if $1 \le p < \infty$ and $\alpha \in \mathbb{R}$, then

$$L^{p,p;\alpha}(\Omega,\nu) = L^p \log^{p\alpha} L(\Omega,\nu).$$

Moreover, if $\beta > 0$, then

$$L^{\infty,\infty;-\beta}(\Omega,\nu) = \exp L^{\frac{1}{\beta}}(\Omega,\nu)$$

and [40, Lemma 2.2]

$$L^{\infty,\infty;0,-\beta}(\Omega,\nu) = \exp\exp L^{\frac{1}{\beta}}(\Omega,\nu).$$

A common extension of the Orlicz and Lorentz spaces is provided by a family of Orlicz–Lorentz spaces defined as follows. Given $p \in (1,\infty)$, $q \in [1,\infty)$ and a Young function D such that

$$\int^{\infty} \frac{D(t)}{t^{1+p}}\, dt < \infty,$$

we denote by $L(p,q,D)(\Omega,\nu)$ the Orlicz–Lorentz space associated with the rearrangement-invariant function norm defined, for $f \in \mathcal{M}_+(0,1)$, as

$$\|f\|_{L(p,q,D)(0,1)} = \left\|s^{-\frac{1}{p}} f^*\left(s^{\frac{1}{q}}\right)\right\|_{L^D(0,1)}. \tag{3.28}$$

The fact that $\|\cdot\|_{L(p,q,D)(0,1)}$ is actually a function norm follows via easy modifications in the proof of [25, Proposition 2.1]. Observe that the class of the spaces $L(p,q,D)(\Omega,\nu)$ actually includes (up to equivalent norms) Orlicz spaces and various instances of Lorentz and Lorentz–Zygmund spaces.

4. Spaces of Sobolev type and the isoperimetric function

Let (Ω,ν) be as in Section 2. Define the perimeter of a measurable set E in (Ω,ν)

$$P_\nu(E,\Omega) = \int_{\Omega\cap\partial^M E} \omega(x)\, d\mathcal{H}^{n-1}(x), \tag{4.1}$$

where $\partial^M E$ denotes the essential boundary of E, in the sense of geometric measure theory [70,92]. The isoperimetric function $I_{\Omega,\nu} : [0,1] \to [0,\infty]$ of (Ω,ν) is then given by

$$I_{\Omega,\nu}(s) = \inf\left\{P_\nu(E,\Omega) : E \subset \Omega, s \leq \nu(E) \leq \frac{1}{2}\right\} \quad \text{if } s \in \left[0, \frac{1}{2}\right], \tag{4.2}$$

and $I_{\Omega,\nu}(s) = I_{\Omega,\nu}(1-s)$ if $s \in (\frac{1}{2}, 1]$. The isoperimetric inequality (2.2) in (Ω, ν) is a straightforward consequence of this definition and of the fact that $P_\nu(E,\Omega) = P_\nu(\Omega \setminus E, \Omega)$ for every set $E \subset \Omega$.

Let us observe that, actually, $I_{\Omega,\nu}(s) < \infty$ for $s \in [0, \frac{1}{2})$. To verify this fact, fix any $x_0 \in \Omega$, and let $R > 0$ be such that $\nu(\Omega \cap B_R(x_0)) = \frac{1}{2}$. Here, $B_R(x_0)$ denotes the ball, centered at x_0, with radius R. By the polar-coordinates formula for integrals,

$$\frac{1}{2} = \int\limits_{\Omega\cap B_R(x_0)} \omega(x)\,dx = \int\limits_0^R \int\limits_{\Omega\cap\partial B_\rho(x_0)} \omega(x)\,d\mathcal{H}^{n-1}(x)\,d\rho = \int\limits_0^R P_\nu\big(\Omega\cap B_\rho(x_0), \Omega\big)\,d\rho, \tag{4.3}$$

whence $P_\nu(\Omega \cap B_\rho(x_0), \Omega) < \infty$ for a.e. $\rho \in (0, R)$. The finiteness of $I_{\Omega,\nu}$ in $[0, \frac{1}{2})$ now follows by its very definition.

The next result shows that the best possible behavior of an isoperimetric function at 0 is that given by (2.4), in the sense that $I_{\Omega,\nu}(s)$ cannot decay more slowly than $s^{\frac{1}{n'}}$ as $s \to 0$, whatever (Ω, ν) is.

Proposition 4.1. *There exists a positive constant $C = C(\Omega, \nu)$ such that*

$$I_{\Omega,\nu}(s) \leq C s^{\frac{1}{n'}} \quad \textit{near } 0. \tag{4.4}$$

Proof. Let x_0 be any Lebesgue point of ω, namely a point such that

$$\lim_{r\to 0^+} \frac{1}{|B_r(x_0)|} \int\limits_{B_r(x_0)} \omega(x)\,dx \tag{4.5}$$

exists and is finite. Here, $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{R}^n$. By (4.5), there exists $r_0 > 0$ and $C > 0$ such that

$$\int\limits_{B_r(x_0)} \omega(x)\,dx \leq Cr^n \quad \text{if } 0 < r < r_0. \tag{4.6}$$

By an analogous chain as in (4.3),

$$\int\limits_{B_r(x_0)} \omega(x)\,dx = \int\limits_0^r P_\nu\big(B_\rho(x_0), \Omega\big)\,d\rho \geq \frac{r}{2} \inf\left\{P_\nu\big(B_\rho(x_0), \Omega\big) : \frac{r}{2} \leq \rho \leq r\right\} \tag{4.7}$$

if $0 < r < r_0$. From (4.6) and (4.7) we deduce that there exists a constant C such that

$$C\big|B_r(x_0)\big|^{\frac{1}{n'}} \geq \inf\left\{P_\nu\big(B_\rho(x_0), \Omega\big) : \frac{r}{2} \leq \rho \leq r\right\}$$
$$= \inf\left\{P_\nu\big(B_\rho(x_0), \Omega\big) : \frac{1}{2^n}\big|B_r(x_0)\big| \leq \big|B_\rho(x_0)\big| \leq \big|B_r(x_0)\big|\right\} \quad \text{if } 0 < r < r_0.$$

Thus, there exists a constant C such that

$$Cs^{\frac{1}{n'}} \geq \inf\left\{P_\nu(E, \Omega) : s \leq |E| \leq \frac{1}{2}\right\},$$

provided that s is sufficiently small, and hence (4.4) follows. □

Let $m \in \mathbb{N}$ and let $X(\Omega, \nu)$ be a rearrangement-invariant space. We define the m-th order Sobolev space $V^m X(\Omega, \nu)$ as

$$V^m X(\Omega, \nu) = \big\{u : u \text{ is } m\text{-times weakly differentiable in } \Omega, \text{ and } \big|\nabla^m u\big| \in X(\Omega, \nu)\big\}. \tag{4.8}$$

Here, $\nabla^m u$ denotes the vector of all m-th order weak derivatives of u. We shall also set $\nabla^0 u = u$. Let us notice that in the definition of $V^m X(\Omega, \nu)$ it is only required that the derivatives of the highest order m of u belong to $X(\Omega, \nu)$. This assumption does not entail, in general, that also u and its derivatives up to the order $m-1$ belong to $X(\Omega, \nu)$, or even to $L^1(\Omega, \nu)$. Thus, it may happen that $V^m X(\Omega, \nu) \not\subseteq V^k X(\Omega, \nu)$ for $m > k$. Such inclusion indeed fails, for instance, when $(\Omega, \nu) = (\mathbb{R}^n, \gamma_n)$, the Gauss space, and $\|\cdot\|_{X(0,1)} = \|\cdot\|_{L^\infty(0,1)}$ (or $\|\cdot\|_{X(0,1)} = \|\cdot\|_{\exp L^\beta(0,1)}$ for some $\beta > 0$). Examples of Euclidean domains for which $V^m X(\Omega) \not\subseteq L^1(\Omega)$ are those of Nikodým type, see, e.g., [70, Sections 5.2 and 5.4].

However, if $I_{\Omega,\nu}(s)$ does not decay at 0 faster than linearly, namely if there exists a positive constant C such that

$$I_{\Omega,\nu}(s) \geq Cs \quad \text{for } s \in \left[0, \frac{1}{2}\right], \tag{4.9}$$

then any function $u \in V^m X(\Omega, \nu)$ does at least belong to $L^1(\Omega, \nu)$, together with all its derivatives up to the order $m-1$. This is a consequence of the next result. Such result in the case when ν is the Lebesgue measure is established in [70, Theorem 5.2.3]; the general case rests upon an analogous argument. We provide a proof for completeness.

Proposition 4.2 *(Condition for $V^1L^1(\Omega, \nu) \subset L^1(\Omega, \nu)$). Assume that (4.9) holds. Then $V^1L^1(\Omega, \nu) \subset L^1(\Omega, \nu)$, and*

$$\frac{C}{2}\left\|u - \int_\Omega u\, d\nu\right\|_{L^1(\Omega,\nu)} \leq \|\nabla u\|_{L^1(\Omega,\nu)} \tag{4.10}$$

for every $u \in V^1L^1(\Omega, \nu)$, where C is the same constant as in (4.9).

Proof. Let $\operatorname{med}(u)$ denote the median of a function $u \in \mathcal{M}(\Omega,\nu)$, given by

$$\operatorname{med}(u) = \sup\left\{t \in \mathbb{R} : \nu\big(\{x \in \Omega :\, u(x) > t\}\big) > \frac{1}{2}\right\}. \tag{4.11}$$

We begin by showing that

$$C\big\|u - \operatorname{med}(u)\big\|_{L^1(\Omega,\nu)} \le \|\nabla u\|_{L^1(\Omega,\nu)} \tag{4.12}$$

for every $u \in V^1L^1(\Omega,\nu)$. On replacing, if necessary, u by $u - \operatorname{med}(u)$, we may assume, without loss of generality, that $\operatorname{med}(u) = 0$. Let us set $u_+ = \frac{1}{2}(|u| + u)$ and $u_- = \frac{1}{2}(|u| - u)$, the positive and the negative parts of u, respectively. Thus,

$$\nu\big(\{u_\pm > t\}\big) \le \frac{1}{2} \quad \text{for } t > 0. \tag{4.13}$$

By (2.2) and (4.9),

$$P_\nu\big(\{u_\pm > t\}, \Omega\big) \ge I_{\Omega,\nu}\big(\nu\big(\{u_\pm > t\}\big)\big) \ge C\nu\big(\{u_\pm > t\}\big).$$

Therefore, owing to (4.13), and to the coarea formula, we have that

$$\begin{aligned} C\|u_\pm\|_{L^1(\Omega,\nu)} &= C\int_0^\infty \nu\big(\{u_\pm > t\}\big)\,dt \le \int_0^\infty P_\nu\big(\{u_\pm > t\}, \Omega\big)\,dt \\ &= \int_0^\infty \int_{\partial^M\{u_\pm > t\}\cap\Omega} \omega(x)d\mathcal{H}^{n-1}(x)\,dt = \int_\Omega |\nabla u_\pm| d\nu. \end{aligned}$$

Hence, (4.12) follows. In particular, (4.12) tells us that $V^1L^1(\Omega,\nu) \subset L^1(\Omega,\nu)$. Inequality (4.10) is a consequence of (4.12) and of the fact that

$$\left\|u - \int_\Omega u\,d\nu\right\|_{L^1(\Omega,\nu)} \le 2\big\|u - \operatorname{med}(u)\big\|_{L^1(\Omega,\nu)}$$

for every $u \in L^1(\Omega,\nu)$. □

Corollary 4.3. *Assume that (4.9) holds. Let $m \ge 1$. Let $X(\Omega,\nu)$ be any rearrangement-invariant space. Then $V^mX(\Omega,\nu) \subset V^kL^1(\Omega,\nu)$ for every $k = 0,\dots,m-1$.*

Proof. By property (P5) of rearrangement-invariant spaces, $V^mX(\Omega,\nu) \to V^mL^1(\Omega,\nu)$. Thus, the conclusion follows from an iterated use of Proposition 4.2. □

Under (4.9), an assumption which will always be kept in force hereafter, $V^m X(\Omega,\nu)$ is easily seen to be a normed linear space, equipped with the norm

$$\|u\|_{V^m X(\Omega,\nu)} = \sum_{k=0}^{m-1} \big\|\nabla^k u\big\|_{L^1(\Omega,\nu)} + \big\|\nabla^m u\big\|_{X(\Omega,\nu)}. \tag{4.14}$$

Standard arguments show that $V^m X(\Omega,\nu)$ is complete, and hence a Banach space, under the additional assumption that

$$L^1_{\mathrm{loc}}(\Omega,\nu) \to L^1_{\mathrm{loc}}(\Omega).$$

We also define the subspace $V^m_\perp X(\Omega,\nu)$ of $V^m X(\Omega,\nu)$ as

$$V^m_\perp X(\Omega,\nu) = \left\{u \in V^m X(\Omega,\nu) : \int_\Omega \nabla^k u \, d\nu = 0, \text{ for } k = 0,\dots,m-1\right\}. \tag{4.15}$$

The Sobolev embedding (2.3) turns out to be equivalent to a Poincaré type inequality for functions in $V^m_\perp X(\Omega,\nu)$.

Proposition 4.4 *(Equivalence of Sobolev and Poincaré inequalities). Assume that (Ω,ν) fulfills (4.9) and that $m \geq 1$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then*

$$V^m X(\Omega,\nu) \to Y(\Omega,\nu) \tag{4.16}$$

if and only if there exists a constant C such that

$$\|u\|_{Y(\Omega,\nu)} \leq C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{4.17}$$

for every $u \in V^m_\perp X(\Omega,\nu)$.

Proof. Assume that (4.16) holds. Thus, there exists a constant C such that

$$\|u\|_{Y(\Omega,\nu)} \leq C\left(\sum_{k=0}^{m-1} \big\|\nabla^k u\big\|_{L^1(\Omega,\nu)} + \big\|\nabla^m u\big\|_{X(\Omega,\nu)}\right) \tag{4.18}$$

for every $u \in V^m X(\Omega,\nu)$. Iterating inequality (4.10) implies that there exist constants $C_1,\dots,C_m$ such that

$$\|u\|_{L^1(\Omega,\nu)} \leq C_1\|\nabla u\|_{L^1(\Omega,\nu)} \leq C_2\big\|\nabla^2 u\big\|_{L^1(\Omega,\nu)} \leq \cdots \leq C_m\big\|\nabla^m u\big\|_{L^1(\Omega,\nu)} \tag{4.19}$$

for every $u \in V^m_\perp X(\Omega,\nu)$. By property (P5) of rearrangement-invariant function norms, there exists a constant C, independent of u, such that $\|\nabla^m u\|_{L^1(\Omega,\nu)} \leq C\|\nabla^m u\|_{X(\Omega,\nu)}$. Thus, (4.17) follows from (4.18) and (4.19).

Suppose next that (4.17) holds. Given $k \in \mathbb{N}$, denote by $\mathcal{P}^k$ the space of polynomials whose degree does not exceed k. Observe that $\mathcal{P}^k \subset L^1(\Omega,\nu)$ for every $k \in \mathbb{N}$. Indeed, $\nabla^h P = 0$ for every $P \in \mathcal{P}^k$, provided that $h > k$, and hence $\mathcal{P}^k \subset V^h X(\Omega,\nu)$ for any rearrangement-invariant space $X(\Omega,\nu)$. The inclusion $\mathcal{P}^k \subset L^1(\Omega,\nu)$ thus follows via Corollary 4.3. Next, it is not difficult to verify that, for each $u \in V^m X(\Omega,\nu)$, there exists a (unique) polynomial $P_u \in \mathcal{P}^{m-1}$ such that $u - P_u \in V_\perp^m X(\Omega,\nu)$. Moreover, the coefficients of P_u are linear combinations of the components of $\int_\Omega \nabla^k u\, d\nu$, for $k = 0,\dots,m-1$, with coefficients depending on n, m and (Ω,ν). Now, we claim that

$$\mathcal{P}^m \subset Y(\Omega,\nu). \tag{4.20}$$

This inclusion is trivial in the case when Ω is bounded, owing to axioms (P2) and (P4) of the definition of rearrangement-invariant function norms, since any polynomial is bounded in Ω. To verify (4.20) in the general case, consider, for each $i = 1,\dots,n$, the polynomial $Q(x) = x_i^m \in \mathcal{P}^m$. Let $P_Q \in \mathcal{P}^{m-1}$ be the polynomial associated with Q as above, such that $Q - P_Q \in V_\perp^m X(\Omega,\nu)$. Note that the polynomial P_Q also depends only on x_i. From (4.17) applied with $u = Q - P_Q$ we deduce that $Q - P_Q \in Y(\Omega,\nu)$. This inclusion and the inequality $|Q - P_Q| \geq C|x_i|^m$, which holds, for a suitable positive constant C, if $|x_i|$ is sufficiently large, tell us, via axiom (P2) of the definition of rearrangement-invariant function norms, that $|x_i|^m \in Y(\Omega,\nu)$ as well. Thus, $|x|^m \in Y(\Omega,\nu)$, and by axiom (P2) again, any polynomial of degree not exceeding m also belongs to $Y(\Omega,\nu)$. Hence, (4.20) follows. Thus, given any $u \in V^m X(\Omega,\nu)$, we have that

$$\begin{aligned}
\|u\|_{Y(\Omega,\nu)} &\leq \|u - P_u\|_{Y(\Omega,\nu)} + \|P_u\|_{Y(\Omega,\nu)} \\
&\leq C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} + \sum_{k=0}^{m-1} C \int_\Omega \big|\nabla^k u\big|\, d\nu \sum_{\alpha_1+\dots+\alpha_n=k} \big\||x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n}\big\|_{Y(\Omega,\nu)} \\
&\leq C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} + C' \sum_{k=0}^{m-1} \big\|\nabla^k u\big\|_{L^1(\Omega,\nu)},
\end{aligned}$$

for some constants C and C' independent of u. Hence, embedding (4.16) follows. □

Let us incidentally mention that more customary Sobolev type spaces $W^m X(\Omega,\nu)$ can be defined as

$$\begin{aligned}
W^m X(\Omega,\nu) = \big\{u :\ & u \text{ is } m\text{-times weakly differentiable in } \Omega, \\
& \big|\nabla^k u\big| \in X(\Omega,\nu) \text{ for } k = 0,\dots,m\big\},
\end{aligned} \tag{4.21}$$

and equipped with the norm

$$\|u\|_{W^m X(\Omega,\nu)} = \sum_{k=0}^{m} \big\|\nabla^k u\big\|_{X(\Omega,\nu)}. \tag{4.22}$$

The space $W^m X(\Omega, \nu)$ is a normed linear space, and it is a Banach space if

$$X_{\mathrm{loc}}(\Omega, \nu) \to L^1_{\mathrm{loc}}(\Omega).$$

By the second embedding in (3.16),

$$W^m X(\Omega, \nu) \to V^m X(\Omega, \nu) \tag{4.23}$$

for every (Ω, ν) fulfilling (4.9), but, in general, $W^m X(\Omega, \nu) \subsetneqq V^m X(\Omega, \nu)$. For instance, if $(\Omega, \nu) = (\mathbb{R}^n, \gamma_n)$, the Gauss space, and $\|\cdot\|_{X(0,1)} = \|\cdot\|_{L^\infty(0,1)}$ (or $\|\cdot\|_{X(0,1)} = \|\cdot\|_{\exp L^\beta(0,1)}$ for some $\beta > 0$), then $V^m X(\Omega, \nu) \neq W^m X(\Omega, \nu)$. However, the spaces $W^m X(\Omega, \nu)$ and $V^m X(\Omega, \nu)$ agree if condition (4.9) is slightly strengthened to

$$\int_0 \frac{ds}{I_{\Omega,\nu}(s)} < \infty. \tag{4.24}$$

Note that (4.24) indeed implies (4.9), since $\frac{1}{I_{\Omega,\nu}}$ is a non-increasing function.

Proposition 4.5 *(Condition for $W^m X(\Omega, \nu) = V^m X(\Omega, \nu)$). Let (Ω, ν) be as above, and let $m \in \mathbb{N}$. Assume that (4.24) holds. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then*

$$W^m X(\Omega, \nu) = V^m X(\Omega, \nu), \tag{4.25}$$

up to equivalent norms.

A proof of this proposition relies upon one of our main results, and can be found at the end of Section 9.

5. Main results

The present section contains the main results of this paper, which link embeddings and Poincaré inequalities for Sobolev-type spaces of arbitrary order to isoperimetric inequalities. The relevant results depend only on a lower bound for the isoperimetric function $I_{\Omega,\nu}$ of (Ω, ν) in terms of some other non-decreasing function $I : [0, 1] \to [0, \infty)$; precisely, on the existence of a positive constant c such that

$$I_{\Omega,\nu}(s) \geq cI(cs) \quad \text{for } s \in \left[0, \frac{1}{2}\right]. \tag{5.1}$$

As mentioned in Proposition 4.2 and the preceding remarks, it is reasonable to suppose that the function $I_{\Omega,\nu}$ satisfies the estimate (4.9). In the light of this fact, in what follows we shall assume that

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0. \tag{5.2}$$

Theorem 5.1 *(Reduction principle). Assume that (Ω, ν) fulfills (5.1) for some non-decreasing function I satisfying (5.2). Let $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. If there exists a constant C_1 such that*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{5.3}$$

for every nonnegative $f \in X(0,1)$, then

$$V^m X(\Omega, \nu) \to Y(\Omega, \nu), \tag{5.4}$$

and there exists a constant C_2 such that

$$\|u\|_{Y(\Omega,\nu)} \le C_2 \big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.5}$$

for every $u \in V^m_\perp X(\Omega, \nu)$.

Remark 5.2. It turns out that inequality (5.3) holds for every nonnegative $f \in X(0,1)$ if and only if it just holds for every nonnegative and non-increasing $f \in X(0,1)$. This fact will be proved in Corollary 9.8, Section 9, and can be of use in concrete applications of Theorem 5.1. Indeed, the available criteria for the validity of one-dimensional inequalities for integral operators take, in general, different forms according to whether trial functions are arbitrary, or just monotone.

As already stressed in Sections 1 and 2, the first-order case ($m = 1$) of Theorem 5.1 is already well known; the novelty here amounts to the higher-order case when $m > 1$. To be more precise, when $m = 1$, a version of Theorem 5.1 in the standard Euclidean case, for functions vanishing on $\partial\Omega$, is by now classical, and has been exploited in the proof of Sobolev inequalities with sharp constants, including [2,13,72,87]. An argument showing that (5.3) with $m = 1$ implies (5.4) and (5.5), for functions with arbitrary boundary values, for Orlicz norms, on regular Euclidean domains, or, more generally, on domains in Maz'ya classes, is presented [23, Proof of Theorem 2 and Remark 2]. A proof for arbitrary rearrangement-invariant norms, in Gauss space, is given in [32]. The same proof translates verbatim to general measure spaces (Ω, ν) as in Theorem 5.1 – see e.g. [66].

A major feature of Theorem 5.1 is the difference occurring in (5.3) between the first-order case ($m = 1$) and the higher-order case ($m > 1$). Indeed, the integral operator appearing in (5.3) when $m = 1$ is just a weighted Hardy-type operator, namely a primitive of f times a weight, whereas, in the higher-order case, a genuine kernel, with a more complicated structure, comes into play. In fact, this seems to be the first known instance where such a kernel operator is needed in a reduction result for Sobolev-type embeddings. Of course, this makes the proof of inequalities of the form (5.3) more challenging, although several contributions on one-dimensional inequalities for kernel operators are fortunately available in the literature (see e.g. the survey papers [57,67,83], and the monographs [36,38]).

Remark 5.3. As we shall see, the Sobolev embedding (5.4) (or the Poincaré inequality (5.5)) and inequality (5.3), with a function I equivalent to the isoperimetric function $I_{\Omega,\nu}$ on some neighborhood of zero, are actually equivalent in customary families of measure spaces (Ω, ν), and hence, Theorem 5.4 below will enable us to determine the optimal rearrangement-invariant target spaces in Sobolev embeddings for these measure spaces. Incidentally, let us mention that when $m = 1$, this is the case whenever the geometry of (Ω, ν) allows the construction of a family of trial functions u in (5.4) or (5.5) characterized by the following properties: the level sets of u are isoperimetric (or almost isoperimetric) in (Ω, ν); $|\nabla u|$ is constant (or almost constant) on the boundary of the level sets of u. If $m > 1$, then the latter requirement has to be complemented by requiring that the derivatives of u up to the order m restricted to the boundary of the level sets satisfy certain conditions depending on I. The relevant conditions have, however, a technical nature, and it is not worth to state them explicitly. In fact, heuristically speaking, properties (5.3), (5.5) and (5.4) turn out to be equivalent for every $m \geq 1$ on the same measure spaces (Ω, ν) as they are equivalent for $m = 1$. Such equivalence certainly holds in any customary, non-pathological situation, including the three frameworks to which our results will be applied, namely John domains, Euclidean domains from Maz'ya classes, and product probability spaces in $\mathbb{R}^n$ extending the Gauss space.

Now we are in a position to characterize the space which, in the situation discussed in Remark 5.3, is the optimal rearrangement-invariant target space in the Sobolev embedding (5.4). Such an optimal space is the one built upon the rearrangement-invariant function norm $\|\cdot\|_{X_{m,I}(0,1)}$, whose associate norm is defined as

$$\|f\|_{X'_{m,I}(0,1)} = \left\| \frac{1}{I(t)} \int_0^t \left(\int_s^t \frac{dr}{I(r)} \right)^{m-1} f^*(s)ds \right\|_{X'(0,1)} \tag{5.6}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 5.4 *(Optimal target). Assume that (Ω, ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 5.1. Then the functional $\|\cdot\|_{X'_{m,I}(0,1)}$, given by (5.6), is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,I}(0,1)}$ satisfies*

$$V^m X(\Omega,\nu) \to X_{m,I}(\Omega,\nu), \tag{5.7}$$

and there exists a constant C such that

$$\|u\|_{X_{m,I}(\Omega,\nu)} \le C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.8}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

Moreover, if (Ω,ν) is such that (5.4), or equivalently (5.5), implies (5.3), and hence (5.3), (5.4) and (5.5) are equivalent, then the function norm $\|\cdot\|_{X_{m,I}(0,1)}$ is optimal in (5.7) and (5.8) among all rearrangement-invariant norms.

An important special case of Theorems 5.1 and 5.4 is enucleated in the following corollary.

Corollary 5.5 *(Sobolev embeddings into L^∞). Assume that (Ω,ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 5.1. If*

$$\left\|\frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1}\right\|_{X'(0,1)} < \infty, \tag{5.9}$$

then

$$V^m X(\Omega,\nu) \to L^\infty(\Omega,\nu), \tag{5.10}$$

and there exists a constant C such that

$$\|u\|_{L^\infty(\Omega,\nu)} \le C\big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.11}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

Moreover, if (Ω,ν) is such that (5.4), or equivalently (5.5), implies (5.3), and hence (5.3), (5.4) and (5.5) are equivalent, then (5.9) is necessary for (5.10) or (5.11) to hold.

Remark 5.6. If (Ω,ν) is such that (5.4), or equivalently (5.5), implies (5.3), and hence (5.3), (5.4) and (5.5) are equivalent, then (5.10) cannot hold, whatever $\|\cdot\|_{X(0,1)}$ is, if I decays so fast at 0 that

$$\int_0 \frac{dr}{I(r)} = \infty.$$

Our last main result concerns the preservation of optimality in targets among all rearrangement-invariant spaces under iteration of Sobolev embeddings of arbitrary order.

Theorem 5.7 *(Iteration principle). Assume that (Ω, ν), I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 5.1. Let $k, h \in \mathbb{N}$. Then*

$$(X_{k,I})_{h,I}(\Omega, \nu) = X_{k+h,I}(\Omega, \nu),$$

up to equivalent norms.

We now focus on the case when

$$\int_0^s \frac{dr}{I(r)} \approx \frac{s}{I(s)} \quad \text{for } s \in (0,1). \tag{5.12}$$

If the function I satisfies (5.12), then the results of Theorems 5.1, 5.4 and 5.7 can be somewhat simplified. This is the content of the next three corollaries. Let us preliminarily observe that, since the right-hand side of (5.12) does not exceed its left-hand side for any non-decreasing function I, only the estimate in the reverse direction is relevant in (5.12).

Corollary 5.8 *(Reduction principle under (5.12)). Let (Ω, ν), m, I, $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be as in Theorem 5.1. Assume, in addition, that I fulfills (5.12). If there exists a constant C_1 such that*

$$\left\| \int_t^1 f(s) \frac{s^{m-1}}{I(s)^m}\, ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{5.13}$$

for every nonnegative $f \in X(0,1)$, then

$$V^m X(\Omega, \nu) \to Y(\Omega, \nu), \tag{5.14}$$

and there exists a constant C_2 such that

$$\|u\|_{Y(\Omega,\nu)} \le C_2 \big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.15}$$

for every $u \in V_\perp^m X(\Omega, \nu)$.

Let us notice that a remark parallel to Remark 5.2 applies on the equivalence of the validity of (5.13) for any f, or for any non-increasing f (see Proposition 8.6, Section 8).

The next corollary tells us that, under the extra condition (5.12), the optimal rearrangement-invariant target space takes a simplified form. Namely, it can be equivalently defined via the rearrangement-invariant function norm $\|\cdot\|_{X^\sharp_{m,I}(0,1)}$ obeying

$$\|f\|_{(X^{\sharp}_{m,I})'(0,1)} = \left\| \frac{t^{m-1}}{I(t)^m} \int\limits_0^t f^*(s)\, ds \right\|_{X'(0,1)} \tag{5.16}$$

for every $f \in \mathcal{M}_+(0,1)$.

Corollary 5.9 *(Optimal target under (5.12)). Assume that (Ω,ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Corollary 5.8. Then the functional $\|\cdot\|_{(X^{\sharp}_{m,I})'(0,1)}$, given by (5.16), is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X^{\sharp}_{m,I}(0,1)}$ satisfies*

$$V^m X(\Omega,\nu) \to X^{\sharp}_{m,I}(\Omega,\nu), \tag{5.17}$$

and there exists a constant C such that

$$\|u\|_{X^{\sharp}_{m,I}(\Omega,\nu)} \le C \big\|\nabla^m u\big\|_{X(\Omega,\nu)} \tag{5.18}$$

for every $u \in V^m_{\perp} X(\Omega,\nu)$.

Moreover, if (Ω,ν) is such that the validity of (5.14), or equivalently (5.15), implies (5.13), and hence (5.13), (5.14) and (5.15) are equivalent, then the function norm $\|\cdot\|_{X^{\sharp}_{m,I}(0,1)}$ is optimal in (5.17) and (5.18) among all rearrangement-invariant norms.

We conclude this section with a stability result for the iterated embeddings under the additional condition (5.12).

Corollary 5.10 *(Iteration principle under (5.12)). Assume that (Ω,ν), I and $\|\cdot\|_{X(0,1)}$ are as in Corollary 5.8. Let $k,h \in \mathbb{N}$. Then*

$$\big(X^{\sharp}_{k,I}\big)^{\sharp}_{h,I}(\Omega,\nu) = X^{\sharp}_{k+h,I}(\Omega,\nu),$$

up to equivalent norms.

6. Euclidean–Sobolev embeddings

The main results of this section are reduction theorems and their consequences for Euclidean Sobolev embeddings, of arbitrary order m, on John domains, and on domains from Maz'ya classes.

We begin with the reduction theorem for John domains. Recall that a bounded open set Ω in $\mathbb{R}^n$ is called a *John domain* if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parameterized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}\big(\varpi(r), \partial\Omega\big) \ge cr \quad \text{for } r \in [0,l].$$

Theorem 6.1 *(Reduction principle for John domains). Let $n \in \mathbb{N}$, $n \geq 2$, and let $m \in \mathbb{N}$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following assertions are equivalent.*

(i) *The Hardy type inequality*

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{n}}\, ds \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)} \tag{6.1}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The Sobolev embedding*

$$V^m X(\Omega) \to Y(\Omega) \tag{6.2}$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\Omega)} \leq C_2 \big\|\nabla^m u\big\|_{X(\Omega)} \tag{6.3}$$

holds for some constant C_2 and every $u \in V_\perp^m X(\Omega)$.

Forerunners of Theorem 6.1 are known. The first order case ($m = 1$) on Lipschitz domains was obtained in [37]. In the case when $m = 2$, and functions vanishing on $\partial\Omega$ are considered, the equivalence of (6.1) and (6.3) was proved in [26], as a consequence of a non-standard rearrangement inequality for second-order derivatives (see also [24] for a related one-dimensional second-order rearrangement inequality). The equivalence of (6.1) and (6.2), when $m \leq n - 1$ and Ω is a Lipschitz domain, was established in [53] by a method relying upon interpolation techniques. Such a method does not carry over to the more general setting of Theorem 6.1, since it requires that Ω be an extension domain.

Let us also warn that results reducing higher-order Sobolev embeddings to one-dimensional inequalities can be obtained via more standard methods, such as, for instance, representation formulas of convolution type combined with O'Neil rearrangement estimates for convolutions, or plain iteration of certain first-order pointwise rearrangement estimates [64]. However, these approaches lead to optimal Sobolev embeddings only under additional technical assumptions on the involved rearrangement-invariant function norms $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$ and $m \in \mathbb{N}$, we define $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ as the rearrangement-invariant function norm, whose associate function norm is given by

$$\|f\|_{X'_{m,\mathrm{John}}(0,1)} = \left\| s^{-1+\frac{m}{n}} \int_0^s f^*(r)dr \right\|_{X'(0,1)} \tag{6.4}$$

for $f \in \mathcal{M}_+(0,1)$. The function norm $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ is optimal, as a target, for Sobolev embeddings of $V^m X(\Omega)$.

Theorem 6.2 *(Optimal target for John domains). Let n, m, Ω and $\|\cdot\|_{X(0,1)}$ be as in Theorem 6.1. Then the functional $\|\cdot\|_{X'_{m,\mathrm{John}}(0,1)}$, given by (6.4), is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ satisfies*

$$V^m X(\Omega) \to X_{m,\mathrm{John}}(\Omega), \tag{6.5}$$

and

$$\|u\|_{X_{m,\mathrm{John}}(\Omega)} \le C\big\|\nabla^m u\big\|_{X(\Omega)} \tag{6.6}$$

for some constant C and every $u \in V^m_\perp X(\Omega)$.

Moreover, the function norm $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ is optimal in (6.5) and (6.6) among all rearrangement-invariant norms.

The iteration principle for optimal target norms in Sobolev embeddings on John domains reads as follows.

Theorem 6.3 *(Iteration principle for John domains). Let $n \in \mathbb{N}$, Ω and $\|\cdot\|_{X(0,1)}$ be as in Theorem 6.1. Let $k, h \in \mathbb{N}$. Then*

$$(X_{k,\mathrm{John}})_{h,\mathrm{John}}(\Omega) = X_{k+h,\mathrm{John}}(\Omega),$$

up to equivalent norms.

Let us now focus on Maz'ya classes of domains. Given $\alpha \in [\frac{1}{n'}, 1]$, we denote by $\mathcal{J}_\alpha$ the *Maz'ya class* of all Euclidean domains Ω satisfying (5.1), with $I(s) = s^\alpha$ for $s \in [0,1]$, namely

$$\mathcal{J}_\alpha = \left\{\Omega : I_\Omega(s) \ge C s^\alpha \text{ for some constant } C > 0 \text{ and for } s \in \left[0, \frac{1}{2}\right]\right\}. \tag{6.7}$$

Thanks to (2.4), any John domain belongs to the class $\mathcal{J}_{\frac{1}{n'}}$.

The reduction theorem in the class $\mathcal{J}_\alpha$ takes the following form.

Theorem 6.4 *(Reduction principle for Maz'ya classes). Let $n \in \mathbb{N}$, $n \ge 2$, $m \in \mathbb{N}$ and $\alpha \in [\frac{1}{n'}, 1]$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Assume that either $\alpha \in [\frac{1}{n'}, 1)$ and there exists a constant C_1 such that*

$$\left\| \int_t^1 f(s) s^{-1+m(1-\alpha)}\, ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{6.8}$$

for every nonnegative $f \in X(0,1)$*, or* $\alpha = 1$ *and there exists a constant* C_1 *such that*

$$\left\| \int_t^1 f(s)\frac{1}{s}\left(\log\frac{s}{t}\right)^{m-1} ds \right\|_{Y(0,1)} \le C_1\|f\|_{X(0,1)} \tag{6.9}$$

for every nonnegative $f \in X(0,1)$*. Then the Sobolev embedding*

$$V^m X(\Omega) \to Y(\Omega) \tag{6.10}$$

holds for every $\Omega \in \mathcal{J}_\alpha$ *and, equivalently, the Poincaré inequality*

$$\|u\|_{Y(\Omega)} \le C_2\|\nabla^m u\|_{X(\Omega)} \tag{6.11}$$

holds for every $\Omega \in \mathcal{J}_\alpha$*, for some constant* C_2*, depending on* Ω, m, X *and* Y*, and every* $u \in V_\perp^m X(\Omega)$*.*

Conversely, if the Sobolev embedding (6.10), or, equivalently, the Poincaré inequality (6.11), holds for every $\Omega \in \mathcal{J}_\alpha$*, then either inequality (6.8), or (6.9) holds, according to whether* $\alpha \in [\frac{1}{n'}, 1)$ *or* $\alpha = 1$*.*

A major consequence of Theorem 6.4 is the identification of the optimal rearrangement-invariant target space $Y(\Omega)$ associated with a given domain $X(\Omega)$ in embedding (6.10), as Ω is allowed to range among all domains in the class $\mathcal{J}_\alpha$. This is the content of the next result. The rearrangement-invariant function norm yielding such an optimal space will be denoted by $\|\cdot\|_{X_{m,\alpha}(0,1)}$. Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, $m \in \mathbb{N}$, and $\alpha \in [\frac{1}{n'}, 1]$, it is characterized through its associate function norm defined by

$$\|f\|_{X'_{m,\alpha}(0,1)} = \begin{cases} \|s^{-1+m(1-\alpha)}\int_0^s f^*(r)dr\|_{X'(0,1)} & \text{if } \alpha \in [\frac{1}{n'}, 1), \\ \|\frac{1}{s}\int_0^s (\log\frac{s}{r})^{m-1} f^*(r)dr\|_{X'(0,1)} & \text{if } \alpha = 1, \end{cases} \tag{6.12}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 6.5 *(Optimal target for Maz'ya classes). Let* $n \in \mathbb{N}$*,* $n \ge 2$*,* $m \in \mathbb{N}$*,* α *and* $\|\cdot\|_{X(0,1)}$ *be as in Theorem 6.4. Then the functional* $\|\cdot\|_{X'_{m,\alpha}(0,1)}$*, given by (6.12), is a rearrangement-invariant function norm, whose associate norm* $\|\cdot\|_{X_{m,\alpha}(0,1)}$ *satisfies*

$$V^m X(\Omega) \to X_{m,\alpha}(\Omega) \tag{6.13}$$

for every $\Omega \in \mathcal{J}_\alpha$*, and*

$$\|u\|_{X_{m,\alpha}(\Omega)} \le C\|\nabla^m u\|_{X(\Omega)} \tag{6.14}$$

for every $\Omega \in \mathcal{J}_\alpha$, for some constant C, depending on Ω, m, X and Y, and every $u \in V_\perp^m X(\Omega)$.

Moreover, the function norm $\| \cdot \|_{X_{m,\alpha}(0,1)}$ is optimal in (6.13) and (6.14) among all rearrangement-invariant norms, as Ω ranges in $\mathcal{J}_\alpha$.

Theorem 6.5 is a straightforward consequence of Theorem 6.4, and either Corollary 5.9 or Theorem 5.4, according to whether $\alpha \in [\frac{1}{n'}, 1)$ or $\alpha = 1$.

The stability of the process of finding optimal rearrangement-invariant targets in Euclidean Sobolev embeddings on Maz'ya domains under iteration is the object of the last main result of the present section. This is the key ingredient which bridges the first-order case of Theorems 6.4 and 6.5 to their higher-order versions.

Theorem 6.6 *(Iteration principle for Maz'ya classes). Let $n \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1]$ and $\|\cdot\|_{X(0,1)}$ be as in Theorem 6.4. Let $k, h \in \mathbb{N}$. Assume that $\Omega \in \mathcal{J}_\alpha$. Then,*

$$(X_{k,\alpha})_{h,\alpha}(\Omega) = X_{k+h,\alpha}(\Omega),$$

up to equivalent norms.

Theorem 6.6 follows from a specialization of Corollary 5.10 ($\alpha \in [\frac{1}{n'}, 1)$), or Theorem 5.7 ($\alpha = 1$).

Remark 6.7. Note that there is one important difference between the reduction and the optimal-target theorems concerning John domains on the one hand, and their counterparts for general Maz'ya domains on the other hand. Namely, the equivalence in Theorem 6.1 and the optimality result in Theorem 6.2 are valid *for each single John domain*, whereas the necessity of condition (6.8) or (6.9) for (6.10) (and (6.11)) in Theorem 6.4 as well as the optimality of the target space in Theorem 6.5 are valid in the *class of all $\Omega \in \mathcal{J}_\alpha$*. This is inevitable, since, of course, each class $\mathcal{J}_\alpha$ contains all regular domains, and for such domains Sobolev embeddings with stronger target norms hold.

The remaining part of this section is devoted to applications of Theorems 6.4–6.6 to customary function norms. Consider first the case when Lebesgue or Lorentz norms are concerned. Our conclusions take a different form, according to whether $\alpha \in [\frac{1}{n'}, 1)$, or $\alpha = 1$.

We begin by assuming that $\alpha \in [\frac{1}{n'}, 1)$. Note that results for regular (i.e. John) domains are covered by the choice $\alpha = \frac{1}{n'}$.

Sobolev embeddings involving usual Lebesgue norms are contained in the following theorem.

Theorem 6.8. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_\alpha$ for some $\alpha \in [\frac{1}{n'}, 1)$. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then*

$$V^m L^p(\Omega) \to \begin{cases} L^{\frac{p}{1-mp(1-\alpha)}}(\Omega) & \text{if } m(1-\alpha) < 1 \text{ and } 1 \le p < \frac{1}{m(1-\alpha)}, \\ L^r(\Omega) & \text{for any } r \in [1,\infty), \text{ if } m(1-\alpha) < 1 \text{ and } p = \frac{1}{m(1-\alpha)}, \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{6.15}$$

Moreover, in the first and the third cases, the target spaces in (6.15) are optimal among all Lebesgue spaces, as Ω ranges in $\mathcal{J}_\alpha$.

Although the target spaces in (6.15) cannot be improved in the class of Lebesgue spaces, the first two embeddings in (6.15) can be strengthened if more general rearrangement-invariant spaces are employed. Such a strengthening can be obtained as a special case of a Sobolev embedding for Lorentz spaces which reads as follows.

Theorem 6.9. *Let $n \in \mathbb{N}$, $n \ge 2$, and let $\Omega \in \mathcal{J}_\alpha$ for some $\alpha \in [\frac{1}{n'}, 1)$. Let $m \in \mathbb{N}$ and $p, q \in [1,\infty]$. Assume that one of the conditions in (3.19) holds. Then*

$$V^m L^{p,q}(\Omega) \to \begin{cases} L^{\frac{p}{1-mp(1-\alpha)},q}(\Omega) & \text{if } m(1-\alpha) < 1 \text{ and } 1 \le p < \frac{1}{m(1-\alpha)}, \\ L^{\infty,q;-1}(\Omega) & \text{if } m(1-\alpha) < 1,\ p = \frac{1}{m(1-\alpha)} \text{ and } q > 1, \\ L^\infty(\Omega) & \text{otherwise.} \end{cases} \tag{6.16}$$

Moreover, the target spaces in (6.16) are optimal among all rearrangement-invariant spaces, as Ω ranges in $\mathcal{J}_\alpha$.

The particular choice of parameters $p = q$, $1 \le p < \frac{1}{m(1-\alpha)}$ in Theorem 6.9 shows that

$$V^m L^p(\Omega) \to L^{\frac{p}{1-mp(1-\alpha)},p}(\Omega).$$

This is a non-trivial strengthening of the first embedding in (6.15), since $L^{\frac{p}{1-mp(1-\alpha)},p}(\Omega) \subsetneqq L^{\frac{p}{1-mp(1-\alpha)}}$. Likewise, the choice $m(1-\alpha) < 1$ and $p = q = \frac{1}{m(1-\alpha)}$ shows that also the second embedding in (6.15) can be essentially improved by

$$V^m L^p(\Omega) \to L^{\infty,p;-1}(\Omega).$$

Assume now that $\alpha = 1$. The embedding theorem in Lebesgue spaces takes the following form.

Theorem 6.10. *Let $n \in \mathbb{N}$, $n \ge 2$, and let $\Omega \in \mathcal{J}_1$. Let $m \in \mathbb{N}$ and $p \in [1,\infty]$. Then*

$$V^m L^p(\Omega) \to \begin{cases} L^p(\Omega) & \text{if } 1 \le p < \infty, \\ L^r(\Omega) & \text{for any } r \in [1,\infty), \text{ if } p = \infty. \end{cases} \tag{6.17}$$

Moreover, in the former case of (6.17), the target space is optimal among all Lebesgue spaces, as Ω ranges in $\mathcal{J}_1$.

Optimal embeddings for Lorentz–Sobolev spaces are provided in the next theorem.

Theorem 6.11. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_1$. Let $m \in \mathbb{N}$ and $p, q \in [1, \infty]$. Assume that one of the conditions in (3.19) holds. Then*

$$V^m L^{p,q}(\Omega) \to \begin{cases} L^{p,q}(\Omega) & \text{if } 1 \leq p < \infty, \\ \exp L^{\frac{1}{m}}(\Omega) & \text{if } p = q = \infty. \end{cases} \tag{6.18}$$

The target spaces are optimal in (6.18) among all rearrangement-invariant spaces, as Ω ranges in $\mathcal{J}_1$.

Our last application in this section concerns Orlicz–Sobolev spaces. Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$, and let A be a Young function. If $m < \frac{1}{1-\alpha}$, we may assume, without loss of generality, that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt < \infty. \tag{6.19}$$

Indeed, by (3.27), the function A can be modified near 0, if necessary, in such a way that (6.19) is fulfilled, on leaving the space $V^m L^A(\Omega)$ unchanged (up to equivalent norms).

If $m < \frac{1}{1-\alpha}$ and the integral

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt \tag{6.20}$$

diverges, we define the function $H_{m,\alpha} : [0, \infty) \to [0, \infty)$ as

$$H_{m,\alpha}(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt\right)^{1-m(1-\alpha)} \quad \text{for } s \geq 0,$$

and the Young function $A_{m,\alpha}$ as

$$A_{m,\alpha}(t) = A\big(H_{m,\alpha}^{-1}(t)\big) \quad \text{for } t \geq 0. \tag{6.21}$$

Theorem 6.12. *Assume that $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$ and $\Omega \in \mathcal{J}_\alpha$. Let A be a Young function fulfilling (6.19). Then*

$$V^m L^A(\Omega) \to \begin{cases} L^{A_{m,\alpha}}(\Omega) & \text{if } m < \frac{1}{1-\alpha}, \text{ and the integral (6.20) diverges,} \\ L^\infty(\Omega) & \text{if either } m \geq \frac{1}{1-\alpha}, \text{ or } m < \frac{1}{1-\alpha} \\ & \text{and the integral (6.20) converges.} \end{cases} \tag{6.22}$$

Moreover, the target spaces in (6.22) are optimal among all Orlicz spaces, as Ω ranges in $\mathcal{J}_\alpha$.

Theorem 6.12 follows from Theorem 6.4, via [28, Theorem 4].

The first case of embedding (6.22) can be enhanced, on replacing the optimal Orlicz target spaces with the optimal rearrangement-invariant target spaces. The latter turn out to belong to the family of Orlicz–Lorentz spaces defined in Section 3.

Assume that $m < \frac{1}{1-\alpha}$, and the integral (6.20) diverges. Let a be the left-continuous function appearing in (3.25), and let B be the Young function given by

$$B(t) = \int_0^t b(\tau)d\tau \quad \text{for } t \geq 0,$$

where b is the non-decreasing, left-continuous function in $[0, \infty)$ obeying

$$b^{-1}(s) = \left(\int_{a^{-1}(s)}^{\infty} \left(\int_0^{\tau} \left(\frac{1}{a(t)} \right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt \right)^{-\frac{1}{m(1-\alpha)}} \frac{d\tau}{a(\tau)^{\frac{1}{1-m(1-\alpha)}}} \right)^{\frac{m(1-\alpha)}{m(1-\alpha)-1}} \quad \text{for } s \geq 0.$$

Here, a^{-1} and b^{-1} denote the (generalized) left-continuous inverses of a and b, respectively.

Recall from Section 3 that $L(\frac{1}{m(1-\alpha)}, 1, B)(\Omega)$ is the Orlicz–Lorentz space built upon the function norm given by

$$\|f\|_{L(\frac{1}{m(1-\alpha)},1,B)(0,1)} = \left\| s^{-m(1-\alpha)} f^*(s) \right\|_{L^B(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 6.13. *Assume that $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$ and $\Omega \in \mathcal{J}_\alpha$. Let A be a Young function fulfilling (6.19). Assume that $m < \frac{1}{1-\alpha}$, and the integral in (6.20) diverges. Then*

$$V^m L^A(\Omega) \to L\left(\frac{1}{m(1-\alpha)}, 1, B \right)(\Omega), \tag{6.23}$$

and the target space in (6.23) is optimal among all rearrangement-invariant spaces, as Ω ranges in $\mathcal{J}_\alpha$.

Embedding (6.23) is a consequence of Theorem 6.4, and of [25, inequality (3.1)].

Example 6.14. Consider the case when

$$A(t) \approx t^p (\log t)^\beta \quad \text{near infinity, where either } p > 1 \text{ and } \beta \in \mathbb{R}, \text{ or } p = 1 \text{ and } \beta \geq 0.$$

Hence, $L^A(\Omega) = L^p \log^\beta L(\Omega)$. An application of Theorem 6.12 tells us that

$$V^m L^p \log^\beta L(\Omega) \to \begin{cases} L^{\frac{p}{1-pm(1-\alpha)}} \log^{\frac{\beta}{1-pm(1-\alpha)}} L(\Omega) \quad \text{if } mp(1-\alpha) < 1, \\ \exp L^{\frac{1}{1-(1+\beta)m(1-\alpha)}}(\Omega) \\ \quad \text{if } mp(1-\alpha) = 1 \text{ and } \beta < \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ \exp\exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega) \\ \quad \text{if } mp(1-\alpha) = 1 \text{ and } \beta = \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ L^\infty(\Omega) \quad \text{if either } mp(1-\alpha) > 1, \\ \quad \text{or } mp(1-\alpha) = 1 \text{ and } \beta > \frac{1-m(1-\alpha)}{m(1-\alpha)}. \end{cases} \tag{6.24}$$

Moreover, the target spaces in (6.24) are optimal among all Orlicz spaces, as Ω ranges in $\mathcal{J}_\alpha$.

The first three embeddings in (6.24) can be improved on allowing more general rearrangement-invariant target spaces. Indeed, we have that

$$V^m L^p \log^\beta L(\Omega) \to \begin{cases} L^{\frac{p}{1-pm(1-\alpha)}, p; \frac{\beta}{p}}(\Omega) & \text{if } mp(1-\alpha) < 1, \\ L^{\infty, \frac{1}{m(1-\alpha)}; m(1-\alpha)\beta - 1}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta < \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ L^{\infty, \frac{1}{m(1-\alpha)}; -m(1-\alpha), -1}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta = \frac{1-m(1-\alpha)}{m(1-\alpha)}, \end{cases} \tag{6.25}$$

the targets being optimal among all rearrangement-invariant spaces in (6.25) as Ω ranges among all domains in $\mathcal{J}_\alpha$. This is a consequence of Theorem 6.13, and of the fact that the Orlicz–Lorentz spaces $L(\frac{1}{m(1-\alpha)}, 1, B)(\Omega)$ associated with the present choices of the function A agree (up to equivalent norms) with the (generalized) Lorentz–Zygmund spaces appearing on the right-hand side of (6.25).

7. Sobolev embeddings in product probability spaces

The class of product probability measures in $\mathbb{R}^n$, $n \geq 1$, which we consider in this section, arises in connection with the study of generalized hypercontractivity theory and integrability properties of the associated heat semigroups. The isoperimetric problem in the corresponding probability spaces was studied in [4] – see also [3,8,9,31,59,60].

Assume that $\Phi : [0,\infty) \to [0,\infty)$ is a strictly increasing convex function, twice continuously differentiable in $(0,\infty)$, such that $\sqrt{\Phi}$ is concave and $\Phi(0) = 0$. Let μ_Φ be the probability measure on $\mathbb{R}$ given by

$$d\mu_\Phi(x) = c_\Phi e^{-\Phi(|x|)}\, dx, \tag{7.1}$$

where c_Φ is a constant chosen in such a way that $\mu_\Phi(\mathbb{R}) = 1$. The product measure $\mu_{\Phi,n}$ on $\mathbb{R}^n$, $n \geq 1$, generated by μ_Φ, is then defined as

$$\mu_{\Phi,n} = \underbrace{\mu_\Phi \times \cdots \times \mu_\Phi}_{n\text{-times}}. \tag{7.2}$$

Clearly, $\mu_{\Phi,1} = \mu_\Phi$, and $(\mathbb{R}^n, \mu_{\Phi,n})$ is a probability space for every $n \in \mathbb{N}$.

The main example of a measure μ_Φ is obtained by taking

$$\Phi(t) = \frac{1}{2}t^2. \tag{7.3}$$

This choice yields $\mu_{\Phi,n} = \gamma_n$, the Gauss measure which obeys

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx. \tag{7.4}$$

More generally, given any $\beta \in [1,2]$, the Boltzmann measure $\gamma_{n,\beta}$ in $\mathbb{R}^n$, associated with

$$\Phi(t) = \frac{1}{\beta}t^\beta, \tag{7.5}$$

satisfies the above assumptions.

Let $H : \mathbb{R} \to (0,1)$ be defined as

$$H(t) = \int_t^\infty c_\Phi e^{-\Phi(|r|)}\, dr \quad \text{for } t \in \mathbb{R}, \tag{7.6}$$

and let $F_\Phi : [0,1] \to [0,\infty)$ be given by

$$F_\Phi(s) = c_\Phi e^{-\Phi(|H^{-1}(s)|)} \quad \text{for } s \in (0,1), \quad \text{and} \quad F_\Phi(0) = F_\Phi(1) = 0. \tag{7.7}$$

Since μ_Φ is a probability measure and $\mu_{\Phi,n}$ is defined by (7.2), it is easily seen that, for each $i = 1, \dots, n$,

$$\mu_{\Phi,n}\big(\{(x_1, \dots, x_n) : x_i > t\}\big) = H(t) \quad \text{for } t \in \mathbb{R}, \tag{7.8}$$

and

$$P_{\mu_{\Phi,n}}\big(\{(x_1, \dots, x_n) : x_i > t\}, \mathbb{R}^n\big) = c_\Phi e^{-\Phi(|t|)} = -H'(t) \quad \text{for } t \in \mathbb{R}.$$

Hence, $F_\Phi(s)$ agrees with the perimeter of any half-space of the form $\{x_i > t\}$, whose measure is s.

Next, define $L_\Phi : [0,1] \to [0,\infty)$ as

$$L_\Phi(s) = s\Phi'\left(\Phi^{-1}\left(\log\left(\frac{2}{s}\right)\right)\right) \quad \text{for } s \in (0,1], \quad \text{and} \quad L_\Phi(0) = 0. \tag{7.9}$$

Then the isoperimetric function of $(\mathbb{R}^n, \mu_{\Phi,n})$ satisfies

$$I_{\mathbb{R}^n,\mu_{\Phi,n}}(s) \approx F_\Phi(s) \approx L_\Phi(s) \quad \text{for } s \in \left[0, \frac{1}{2}\right] \tag{7.10}$$

(see [4, Proposition 13 and Theorem 15]; note that the second equivalence in (7.10) also relies upon Lemma 11.1(ii) of Section 11). Furthermore, half-spaces, whose boundary is orthogonal to a coordinate axis, are "approximate solutions" to the isoperimetric problem in $(\mathbb{R}^n, \mu_{\Phi,n})$ in the sense that there exist constants C_1 and C_2, depending on n, such that, for every $s \in (0,1)$, any such half-space V with measure s satisfies

$$C_1 P_{\mu_{\Phi,n}}\big(V, \mathbb{R}^n\big) \le I_{\mathbb{R}^n,\mu_{\Phi,n}}(s) \le C_2 P_{\mu_{\Phi,n}}\big(V, \mathbb{R}^n\big).$$

In the special case when $\mu_{\Phi,n} = \gamma_n$, the Gauss measure, Eq. (7.10) yields

$$I_{\mathbb{R}^n,\gamma_n}(s) \approx s\left(\log \frac{2}{s}\right)^{\frac{1}{2}} \quad \text{for } s \in \left(0, \frac{1}{2}\right].$$

Moreover, any half-space is, in fact, an exact minimizer in the isoperimetric inequality [12,85].

Our reduction theorem for Sobolev embeddings in product probability spaces reads as follows.

Theorem 7.1 *(Reduction principle for product probability spaces). Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, let $\mu_{\Phi,n}$ be the probability measure defined by (7.2), and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The inequality*

$$\left\| \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}}\right)^m \int_t^1 \frac{f(s)}{s}\left(\log \frac{s}{t}\right)^{m-1} ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{7.11}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The embedding*

$$V^m X\big(\mathbb{R}^n, \mu_{\Phi,n}\big) \to Y\big(\mathbb{R}^n, \mu_{\Phi,n}\big) \tag{7.12}$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\mu_{\Phi,n})} \le C_2 \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\mu_{\Phi,n})} \tag{7.13}$$

holds for some constant C_2 and every $u \in V_\perp^m X(\mathbb{R}^n, \mu_{\Phi,n})$.

Let us notice that inequality (7.11) is not just a specialization of (5.3), but even a further simplification of such specialization.

Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm, and let $n, m \in \mathbb{N}$. The rearrangement-invariant function norm $\|\cdot\|_{X_{m,\Phi}(0,1)}$ which yields the optimal rearrangement-invariant target space $Y(\mathbb{R}^n, \mu_{\Phi,n})$ in embedding (7.12) is defined as follows. Consider the rearrangement-invariant function norm $\|\cdot\|_{\widetilde{X}_m(0,1)}$ whose associate norm fulfills

$$\|g\|_{\widetilde{X}'_m(0,1)} = \left\| \frac{1}{s} \int_0^s \left(\log \frac{s}{r}\right)^{m-1} g^*(r)\, dr \right\|_{X'(0,1)} \tag{7.14}$$

for $g \in \mathcal{M}_+(0,1)$. Then $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is given by

$$\|f\|_{X_{m,\Phi}(0,1)} = \left\| \left(\frac{\log \frac{2}{s}}{\Phi^{-1}(\log \frac{2}{s})} \right)^m f^*(s) \right\|_{\widetilde{X}_m(0,1)} \tag{7.15}$$

for $f \in \mathcal{M}_+(0,1)$.

Remark 7.2. Note that if $\Phi(t) = t$, and $m \in \mathbb{N}$, we have that

$$X_{m,\Phi}(0,1) = \widetilde{X}_m(0,1)$$

for every rearrangement-invariant norm $\|\cdot\|_{X(0,1)}$.

Theorem 7.3 *(Optimal target for product probability spaces). Let n, m, $\mu_{\Phi,n}$ and $\|\cdot\|_{X(0,1)}$ be as in Theorem 7.1. Then the functional $\|\cdot\|_{X_{m,\Phi}(0,1)}$, given by (7.15), is a rearrangement-invariant function norm satisfying*

$$V^m X(\mathbb{R}^n, \mu_{\Phi,n}) \to X_{m,\Phi}(\mathbb{R}^n, \mu_{\Phi,n}), \tag{7.16}$$

and there exists a constant C such that

$$\|u\|_{X_{m,\Phi}(\mathbb{R}^n,\mu_{\Phi,n})} \le C \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_{\Phi,n})}, \tag{7.17}$$

for every $u \in V^m_\perp X(\mathbb{R}^n, \mu_{\Phi,n})$.

Moreover, the function norm $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is optimal in (7.16) and in (7.17) among all rearrangement-invariant norms.

Remark 7.4. Let us emphasize that inequality (7.11) implies embedding (7.12) with a norm independent of n, and the Poincaré inequality (7.13) with constant C_2 independent of n. The norm of the optimal embedding (7.16), and the constant C in the corresponding Poincaré inequality (7.17) are independent of n as well.

For a broad class of rearrangement-invariant function norms $\|\cdot\|_{X(0,1)}$ the expression of the associated optimal Sobolev target norm $\|\cdot\|_{X_{m,\Phi}(0,1)}$ can be substantially simplified, as observed in the next proposition.

Proposition 7.5. *Let $m \in \mathbb{N}$ and let Φ be as in (7.1). Suppose that $\|\cdot\|_{X(0,1)}$ is a rearrangement-invariant function norm such that the operator*

$$f \mapsto f^{**}$$

is bounded on $X'(0,1)$. Then

$$\|f\|_{X_{m,\Phi}(0,1)} \approx \left\| \left(\frac{\log \frac{2}{s}}{\Phi^{-1}(\log \frac{2}{s})} \right)^m f^*(s) \right\|_{X(0,1)}$$

up to multiplicative constants independent of $f \in \mathcal{M}_+(0,1)$.

The rearrangement-invariant spaces on which the operator "$**$" is bounded are fully characterized in terms of their upper Boyd index. In particular, the assumptions of Proposition 7.5 are satisfied if and only if the upper Boyd index of $X'(0,1)$ is strictly smaller that 1 [7, Theorem 5.15].

The iteration principle for Sobolev embeddings on product probability measure spaces, on which Theorem 7.1 rests, reads as follows.

Theorem 7.6 *(Iteration principle for product probability spaces). Let n, $\mu_{\Phi,n}$ and $\|\cdot\|_{X(0,1)}$ be as in Theorem 7.1, and let $k, h \in \mathbb{N}$. Then,*

$$(X_{k,\Phi})_{h,\Phi}\big(\mathbb{R}^n, \mu_{\Phi,n}\big) = X_{k+h,\Phi}\big(\mathbb{R}^n, \mu_{\Phi,n}\big),$$

up to equivalent norms.

Specialization of Theorems 7.1, 7.3 and 7.6 to the case of (7.3) easily leads to the following results for Gaussian Sobolev embeddings of any order.

Theorem 7.7 *(Reduction principle in Gauss space). Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The inequality*

$$\left\| \frac{1}{(\log \frac{2}{s})^{\frac{m}{2}}} \int_s^1 \frac{f(r)}{r} \left(\log \frac{r}{s}\right)^{m-1} dr \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The embedding*

$$V^m X\big(\mathbb{R}^n, \gamma_n\big) \to Y\big(\mathbb{R}^n, \gamma_n\big)$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\gamma_n)} \le C_2 \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\gamma_n)}$$

holds for some constant C_2, and for every $u \in V_\perp^m X(\mathbb{R}^n,\gamma_n)$.

Given $n, m \in \mathbb{N}$, and a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, define the rearrangement-invariant function norm $\|\cdot\|_{X_{m,G}(0,1)}$ by

$$\|f\|_{X_{m,G}(0,1)} = \left\| \left(\log \frac{2}{s}\right)^{\frac{m}{2}} f^*(s) \right\|_{\widetilde{X}_m(0,1)} \tag{7.18}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 7.8 *(Optimal target in Gauss space). Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the functional $\|\cdot\|_{X_{m,G}(0,1)}$, given by (7.18), is a rearrangement-invariant function norm satisfying*

$$V^m X\big(\mathbb{R}^n,\gamma_n\big) \to X_{m,G}\big(\mathbb{R}^n,\gamma_n\big) \tag{7.19}$$

and

$$\|u\|_{X_{m,G}(\mathbb{R}^n,\gamma_n)} \le C\big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\gamma_n)} \tag{7.20}$$

for some constant C and every $u \in V_\perp^m X(\mathbb{R}^n,\gamma_n)$.

Moreover, the function norm $\|\cdot\|_{X_{m,G}(0,1)}$ is optimal in (7.19) and (7.20) among all rearrangement-invariant norms.

Observe that, even for $m = 1$, Theorems 7.7 and 7.8 provide us with a characterization of Gaussian Sobolev embeddings which somewhat simplifies earlier results in a similar direction [32,65].

Theorem 7.9 *(Iteration principle in Gauss space). Let $n, k, h \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then,*

$$(X_{k,G})_{h,G}\big(\mathbb{R}^n,\gamma_n\big) = X_{k+h,G}\big(\mathbb{R}^n,\gamma_n\big),$$

up to equivalent norms.

Of course, versions of Theorems 7.7–7.9, with the Gauss measure replaced with the Boltzmann measure, given by the choice (7.5), can similarly be deduced from Theorems 7.1, 7.3 and 7.6. The reduction principle and the optimal target space then take the following form.

Theorem 7.10 *(Reduction principle in Boltzmann spaces). Assume that $n, m \in \mathbb{N}$, and $\beta \in [1,2]$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The inequality*

$$\left\| \frac{1}{(\log \frac{2}{s})^{\frac{m(\beta-1)}{\beta}}} \int_s^1 \frac{f(r)}{r} \left(\log \frac{r}{s}\right)^{m-1} dr \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The embedding*

$$V^m X\big(\mathbb{R}^n, \gamma_{n,\beta}\big) \to Y\big(\mathbb{R}^n, \gamma_{n,\beta}\big)$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\gamma_{n,\beta})} \leq C_2 \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\gamma_{n,\beta})}$$

holds for some constant C_2 and for every $u \in V_\perp^m X(\mathbb{R}^n, \gamma_{n,\beta})$.

Given $n, m \in \mathbb{N}$, $\beta \in [1,2]$, and a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, define the rearrangement-invariant function norm $\|\cdot\|_{X_{m,B,\beta}(0,1)}$ by

$$\|f\|_{X_{m,B,\beta}(0,1)} = \left\| \left(\log \frac{2}{s}\right)^{\frac{m(\beta-1)}{\beta}} f^*(s) \right\|_{\widetilde{X}_m(0,1)} \tag{7.21}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 7.11 *(Optimal target in Boltzmann spaces). Let $n, m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the functional $\|\cdot\|_{X_{m,B,\beta}(0,1)}$, given by (7.21), is a rearrangement-invariant function norm satisfying*

$$V^m X\big(\mathbb{R}^n, \gamma_{n,\beta}\big) \to X_{m,B,\beta}\big(\mathbb{R}^n, \gamma_{n,\beta}\big) \tag{7.22}$$

and

$$\|u\|_{X_{m,B,\beta}(\mathbb{R}^n,\gamma_{n,\beta})} \leq C \big\|\nabla^m u\big\|_{X(\mathbb{R}^n,\gamma_{n,\beta})} \tag{7.23}$$

for some constant C and every $u \in V_\perp^m X(\mathbb{R}^n, \gamma_{n,\beta})$.

Moreover, the function norm $\|\cdot\|_{X_{m,B,\beta}(0,1)}$ is optimal in (7.22) and (7.23) among all rearrangement-invariant norms.

We present an application of the results of this section to the particular case when $\mu_{\Phi,n}$ is a Boltzmann measure, and the norms are of Lorentz–Zygmund type.

Theorem 7.12. *Let $n, m \in \mathbb{N}$, let $\beta \in [1,2]$ and let $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in (3.21) is satisfied. Then*

$$V^m L^{p,q;\alpha}(\mathbb{R}^n, \gamma_{n,\beta}) \to \begin{cases} L^{p,q;\alpha+\frac{m(\beta-1)}{\beta}}(\mathbb{R}^n, \gamma_{n,\beta}) & \text{if } p < \infty; \\ L^{\infty,q;\alpha-\frac{m}{\beta}}(\mathbb{R}^n, \gamma_{n,\beta}) & \text{if } p = \infty. \end{cases}$$

Moreover, in both cases, the target space is optimal among all rearrangement-invariant spaces.

When $\beta = 2$, Theorem 7.12 yields the following sharp Sobolev type embeddings in Gauss space.

Theorem 7.13. *Let $n, m \in \mathbb{N}$, and let $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in (3.21) is satisfied. Then*

$$V^m L^{p,q;\alpha}(\mathbb{R}^n, \gamma_n) \to \begin{cases} L^{p,q;\alpha+\frac{m}{2}}(\mathbb{R}^n, \gamma_n) & \text{if } p < \infty; \\ L^{\infty,q;\alpha-\frac{m}{2}}(\mathbb{R}^n, \gamma_n) & \text{if } p = \infty. \end{cases}$$

Moreover, in both cases, the target space is optimal among all rearrangement-invariant spaces.

A further specialization of the indices p, q, α appearing in Theorem 7.13 leads to the following basic embeddings. In particular, when $m = 1$ we recover a classical Gaussian Sobolev embedding [46].

Corollary 7.14. *Let $n, m \in \mathbb{N}$.*

(i) *Assume that $p \in [1,\infty)$. Then*

$$V^m L^p(\mathbb{R}^n, \gamma_n) \to L^p \log^{\frac{mp}{2}} L(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

(ii) *Assume that $\gamma > 0$. Then*

$$V^m \exp L^\gamma(\mathbb{R}^n, \gamma_n) \to \exp L^{\frac{2\gamma}{2+m\gamma}}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

(iii)

$$V^m L^\infty(\mathbb{R}^n, \gamma_n) \to \exp L^{\frac{2}{m}}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

Note that the target space in the second embedding of Theorem 7.13, and in the embeddings (ii) and (iii) of Corollary 7.14 increases in m. This is related to the fact that $V^m L^{\infty,q;\alpha}(\mathbb{R}^n, \gamma_n) \not\subseteq V^k L^{\infty,q;\alpha}(\mathbb{R}^n, \gamma_n)$ if $m > k$.

8. Optimal target function norms

In this section we collect some basic properties about certain one-dimensional operators playing a role in the proofs of our main results.

Let $T : \mathcal{M}_+(0,1) \to \mathcal{M}_+(0,1)$ be a sublinear operator, namely an operator such that

$$T(\lambda f) = \lambda T f, \quad \text{and} \quad T(f+g) \leq C(Tf + Tg),$$

for some positive constant C, and for every $\lambda \geq 0$ and $f, g \in \mathcal{M}_+(0,1)$.

Given two rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$, we say that T is bounded from $X(0,1)$ into $Y(0,1)$, and write

$$T : X(0,1) \to Y(0,1), \tag{8.1}$$

if the quantity

$$\|T\| = \sup\bigl\{\|Tf\|_{Y(0,1)};\, f \in X(0,1) \cap \mathcal{M}_+(0,1), \|f\|_{X(0,1)} \leq 1\bigr\}$$

is finite. Such a quantity will be called the norm of T. The space $Y(0,1)$ will be called optimal, within a certain class, in (8.1) if, whenever $Z(0,1)$ is another rearrangement-invariant space, from the same class, such that $T : X(0,1) \to Z(0,1)$, we have that $Y(0,1) \to Z(0,1)$. Equivalently, the function norm $\|\cdot\|_{Y(0,1)}$ will be said to be optimal in (8.1) in the relevant class.

Two operators T and T' from $\mathcal{M}_+(0,1)$ into $\mathcal{M}_+(0,1)$ will be called mutually associate if

$$\int_0^1 Tf(s)g(s)\,ds = \int_0^1 f(s)T'g(s)\,ds$$

for every $f, g \in \mathcal{M}_+(0,1)$.

Lemma 8.1. *Let T and T' be mutually associate operators, and let $X(0,1)$ and $Y(0,1)$ be rearrangement-invariant spaces. Then,*

$$T : X(0,1) \to Y(0,1) \quad \textit{if and only if} \quad T' : Y'(0,1) \to X'(0,1),$$

and

$$\|T\| = \|T'\|.$$

Proof. The conclusion is a consequence of the following chain:

$$\|T\| = \sup_{\substack{f\geq 0\\ \|f\|_{X(0,1)}\leq 1}} \|Tf\|_{Y(0,1)} = \sup_{\substack{f\geq 0\\ \|f\|_{X(0,1)}\leq 1}} \sup_{\substack{g\geq 0\\ \|g\|_{Y'(0,1)}\leq 1}} \int_0^1 Tf(s)g(s)\,ds$$

$$= \sup_{\substack{g\geq 0\\ \|g\|_{Y'(0,1)}\leq 1}} \sup_{\substack{f\geq 0\\ \|f\|_{X(0,1)}\leq 1}} \int_0^1 f(s)T'g(s)\,ds = \sup_{\substack{g\geq 0\\ \|g\|_{Y'(0,1)}\leq 1}} \left\|T'g\right\|_{X'(0,1)} = \|T'\|. \qquad \square$$

Let $I : [0,1] \to [0,\infty)$ be a measurable function satisfying (5.2). We define the operators H_I and R_I from $\mathcal{M}_+(0,1)$ into $\mathcal{M}_+(0,1)$ by

$$H_I f(t) = \int_t^1 \frac{f(s)}{I(s)}\,ds \quad \text{for } t \in (0,1], \tag{8.2}$$

and

$$R_I f(t) = \frac{1}{I(t)} \int_0^t f(s)\,ds \quad \text{for } t \in (0,1], \tag{8.3}$$

for $f \in \mathcal{M}_+(0,1)$. Moreover, given $j \in \mathbb{N}$, we set

$$H_I^j = \underbrace{H_I \circ H_I \circ \ldots \circ H_I}_{j\text{-times}} \quad \text{and} \quad R_I^j = \underbrace{R_I \circ R_I \circ \ldots \circ R_I}_{j\text{-times}}. \tag{8.4}$$

We also set $H_I^0 = R_I^0 = \mathrm{Id}$.

Remarks 8.2. (i) The operators H_I and R_I are mutually associate. Hence, H_I^j and R_I^j are also mutually associate for $j \in \mathbb{N}$.

(ii) By the Hardy–Littlewood inequality (3.7), we have, for every $f \in \mathcal{M}_+(0,1)$,

$$R_I f(t) \leq R_I f^*(t) \quad \text{for } t \in (0,1].$$

More generally, for every $f \in \mathcal{M}_+(0,1)$ and $j \in \mathbb{N}$, one has that

$$R_I^j f(t) \leq R_I^j f^*(t) \quad \text{for } t \in (0,1]. \tag{8.5}$$

(iii) For every $j \in \mathbb{N}$ and $f \in \mathcal{M}_+(0,1)$, we have that

$$H_I^j f(t) = \frac{1}{(j-1)!} \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds \quad \text{for } t \in (0,1]. \tag{8.6}$$

Equation (8.6) holds for $j=1$ by the very definition of H_I. On the other hand, if (8.6) is assumed to hold for some $j \in \mathbb{N}$, then

$$
\begin{aligned}
H_I^{j+1} f(t) &= \int_t^1 \frac{H_I^j f(s)}{I(s)}\, ds = \frac{1}{(j-1)!} \int_t^1 \frac{1}{I(s)} \int_s^1 \frac{f(r)}{I(r)} \left(\int_s^r \frac{d\tau}{I(\tau)} \right)^{j-1} dr\, ds \\
&= \frac{1}{(j-1)!} \int_t^1 \frac{f(r)}{I(r)} \int_t^r \frac{1}{I(s)} \left(\int_s^r \frac{d\tau}{I(\tau)} \right)^{j-1} ds\, dr \\
&= \frac{1}{j!} \int_t^1 \frac{f(r)}{I(r)} \left(\int_t^r \frac{d\tau}{I(\tau)} \right)^{j} dr \quad \text{for } t \in (0,1].
\end{aligned}
$$

Hence, (8.6) follows by induction. Similarly, for every $j \in \mathbb{N}$ and $f \in \mathcal{M}_+(0,1)$, we also have that

$$
R_I^j f(t) = \frac{1}{(j-1)!} \frac{1}{I(t)} \int_0^t f(s) \left(\int_s^t \frac{dr}{I(r)} \right)^{j-1} ds \quad \text{for } t \in (0,1]. \tag{8.7}
$$

Given any $j \in \mathbb{N}$ and any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, Eq. (8.7) implies that

$$
\|f\|_{X'_{j,I}(0,1)} = (j-1)! \big\| R_I^j f^* \big\|_{X'(0,1)}, \tag{8.8}
$$

for $f \in \mathcal{M}_+(0,1)$, where $\|\cdot\|_{X'_{j,I}(0,1)}$ is the functional introduced in (5.6). We also formally set $\|\cdot\|_{X'_{0,I}} = \|\cdot\|_{X'(0,1)}$.

Proposition 8.3. *Let $I : [0,1] \to [0,\infty)$ be a measurable function satisfying (5.2). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm and let $j \in \mathbb{N}$. Then the functional $\|\cdot\|_{X'_{j,I}(0,1)}$ defined in (8.8) is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{j,I}(0,1)}$ fulfills*

$$
H_I^j : X(0,1) \to X_{j,I}(0,1). \tag{8.9}
$$

Moreover, the space $X_{j,I}(0,1)$ is the optimal target in (8.9) among all rearrangement-invariant spaces.

Proof. We begin by showing that the functional $\|\cdot\|_{X'_{j,I}(0,1)}$ is a rearrangement-invariant function norm. Let $f, g \in \mathcal{M}_+(0,1)$. By (3.6), $\int_0^t (f+g)^*(s)\, ds \le \int_0^t f^*(s)\, ds + \int_0^t g^*(s)\, ds$ for $t \in (0,1)$. Thus, by Hardy's lemma (see Section 3) applied, for each fixed $t \in (0,1)$, with $f_1(s) = (f+g)^*(s)$, $f_2(s) = f^*(s) + g^*(s)$ and $h(s) = \chi_{(0,t)}(s) (\int_s^t \frac{dr}{I(r)})^{j-1}$, we obtain the triangle inequality

$$\|f+g\|_{X'_{j,I}(0,1)} \le \|f\|_{X'_{j,I}(0,1)} + \|g\|_{X'_{j,I}(0,1)}.$$

Other properties in the axiom (P1) of the definition of rearrangement-invariant function norm, as well as the axioms (P2), (P3) and (P6) are obviously satisfied. Next, it follows from (5.2) that there exists a positive constant C such that $\frac{1}{I(t)} \le \frac{C}{t}$ for $t \in (0,1)$. Therefore,

$$\begin{aligned}\|1\|_{X'_{j,I}(0,1)} &= \left\| \frac{1}{I(t)} \int_0^t \left(\int_s^t \frac{dr}{I(r)} \right)^{j-1} ds \right\|_{X'(0,1)} \le C^j \left\| \frac{1}{t} \int_0^t \left(\int_s^t \frac{dr}{r} \right)^{j-1} ds \right\|_{X'(0,1)} \\ &= C^j \left\| \frac{1}{t} \int_0^t \left(\log \frac{t}{s} \right)^{j-1} ds \right\|_{X'(0,1)} = (j-1)! C^j \|1\|_{X'(0,1)},\end{aligned}$$

and (P4) follows. As far as (P5) is concerned, note that

$$\int_0^1 f^*(s)\,ds \le 2 \int_0^{\frac{1}{2}} f^*(s)\,ds$$

for every $f \in \mathcal{M}_+(0,1)$. Thus, by (P5) for the norm $\|\cdot\|_{X'(0,1)}$, there exists a positive constant C such that, if $f \in \mathcal{M}_+(0,1)$, then

$$\begin{aligned}&\left\| \frac{1}{I(t)} \int_0^t f^*(s) \left(\int_s^t \frac{dr}{I(r)} \right)^{j-1} ds \right\|_{X'(0,1)} \\ &\ge C \int_0^1 \frac{1}{I(t)} \int_0^t f^*(s) \left(\int_s^t \frac{dr}{I(r)} \right)^{j-1} ds\,dt \\ &= \frac{C}{j} \int_0^1 f^*(s) \left(\int_s^1 \frac{dr}{I(r)} \right)^j ds \ge \frac{C}{j} \left(\int_{\frac{1}{2}}^1 \frac{dr}{I(r)} \right)^j \int_0^{\frac{1}{2}} f^*(s)\,ds \ge C' \|f\|_{L^1(0,1)}\end{aligned}$$

where $C' = \frac{C}{2j} (\int_{\frac{1}{2}}^1 \frac{dr}{I(r)})^j$. Hence, property (P5) follows.

To prove (8.9), note that, by (8.5) and (8.8), we have

$$\|R_I^j f\|_{X'(0,1)} \le \|R_I^j f^*\|_{X'(0,1)} = \frac{1}{(j-1)!} \|f\|_{X'_{j,I}(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. Hence,

$$R_I^j : X'_{j,I}(0,1) \to X'(0,1).$$

Since R_I^j and H_I^j are mutually associate, Eq. (8.9) follows via Lemma 8.1.

It remains to prove that $X_{j,I}(0,1)$ is optimal in (8.9) among all rearrangement-invariant spaces. To this purpose, assume that $Y(0,1)$ is another rearrangement-invariant space such that $H_I^j : X(0,1) \to Y(0,1)$. Then, by Lemma 8.1 again, $R_I^j : Y'(0,1) \to X'(0,1)$, namely

$$\|R_I^j f\|_{X'(0,1)} \le C\|f\|_{Y'(0,1)}$$

for some positive constant C, and every $f \in \mathcal{M}_+(0,1)$. Thus, in particular, by (8.8),

$$\|f\|_{X'_{j,I}(0,1)} = (j-1)!\|R_I^j f^*\|_{X'(0,1)} \le (j-1)!C\|f^*\|_{Y'(0,1)} = (j-1)!C\|f\|_{Y'(0,1)}$$

for every $f \in \mathcal{M}_+(0,1)$. Hence, $Y'(0,1) \to X'_{j,I}(0,1)$, and, equivalently, $X_{j,I}(0,1) \to Y(0,1)$. This shows that $X_{j,I}(0,1)$ is optimal in (8.9) among all rearrangement-invariant spaces. □

We introduce one more sequence of function norms, based on the iteration of the first-order function norm $\|\cdot\|_{X'_{1,I}(0,1)}$. Let $I : [0,1] \to [0,\infty)$ be a measurable function satisfying (5.2). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $j \in \mathbb{N} \cup \{0\}$. We define $\|\cdot\|_{X_j(0,1)}$ as the rearrangement-invariant function norm whose associate norm $\|\cdot\|_{X'_j(0,1)}$ is given, via iteration, by $\|\cdot\|_{X'_0(0,1)} = \|\cdot\|_{X'(0,1)}$, and, for $j \ge 1$, by

$$\|f\|_{X'_j(0,1)} = \|R_I f^*\|_{X'_{j-1}(0,1)} \tag{8.10}$$

for $f \in \mathcal{M}_+(0,1)$. Note that

$$\|f\|_{X_1(0,1)} = \|f\|_{X_{1,I}(0,1)}. \tag{8.11}$$

Remark 8.4. By Proposition 8.3, applied j times, with $j = 1$, we obtain that, for every $j \in \mathbb{N} \cup \{0\}$, the functional $\|\cdot\|_{X'_j(0,1)}$ is actually a rearrangement-invariant function norm. Moreover, its associate function norm $\|\cdot\|_{X_j(0,1)}$ fulfills

$$H_I : X_j(0,1) \to X_{j+1}(0,1), \tag{8.12}$$

and $\|\cdot\|_{X_{j+1}(0,1)}$ is the optimal target function norm in (8.12) among all rearrangement-invariant function norms. By Lemma 8.1, we also have

$$R_I : X'_{j+1}(0,1) \to X'_j(0,1).$$

Remark 8.5. Note that, by the very definition of $X_j(0,1)$,

$$X_j(0,1) = \underbrace{(\ldots(X_{1,I})_{1,I}\ldots)_{1,I}}_{j\text{-times}}(0,1)$$

for $j \in \mathbb{N}$. In particular,

$$(X_k)_h(0,1) = X_{k+h}(0,1) \tag{8.13}$$

for every $k, h \in \mathbb{N}$.

We now turn our attention to the special situation when I satisfies, in addition, condition (5.12). In this case, most of the results take a simpler form. We start with a result concerned with the equivalence of two couples of functionals under (5.12).

Proposition 8.6. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.12) and let $\|\cdot\|_{X(0,1)}$ be any rearrangement-invariant function norm. Then the following assertions hold.*

(i) *For every $j \in \mathbb{N}$, and $f \in \mathcal{M}_+(0,1)$,*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds \right\|_{X(0,1)} \approx \left\| \int_t^1 f(s) \frac{s^{j-1}}{I(s)^j}\, ds \right\|_{X(0,1)}, \tag{8.14}$$

up to multiplicative constants independent of $\|\cdot\|_{X(0,1)}$ and f.

(ii) *For every $j \in \mathbb{N}$, and $f \in \mathcal{M}_+(0,1)$,*

$$\left\| \frac{1}{I(s)} \int_0^s f(t) \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} dt \right\|_{X(0,1)} \approx \left\| \frac{s^{j-1}}{I(s)^j} \int_0^s f(t)dt \right\|_{X(0,1)},$$

up to multiplicative constants independent of $\|\cdot\|_{X(0,1)}$ and f.

Proof. We first note that, owing to the monotonicity of I, we have, for every $j \in \mathbb{N}$,

$$\left(\frac{s}{I(s)}\right)^{j-1} = \frac{2^{j-1}}{I(s)^{j-1}} \left(\int_{\frac{s}{2}}^s dr \right)^{j-1} \le 2^{j-1} \left(\int_{\frac{s}{2}}^s \frac{dr}{I(r)} \right)^{j-1} \quad \text{for } s \in (0,1).$$

Thus,

$$\begin{aligned}
\int_{2t}^1 \frac{f(s)}{I(s)} \left(\frac{s}{I(s)}\right)^{j-1} ds &\le 2^{j-1} \int_{2t}^1 \frac{f(s)}{I(s)} \left(\int_{\frac{s}{2}}^s \frac{dr}{I(r)} \right)^{j-1} ds \\
&\le 2^{j-1} \int_{2t}^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds
\end{aligned}$$

$$\leq 2^{j-1} \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds \quad \text{for } t \in \left(0, \frac{1}{2}\right].$$

Hence, the right-hand side of (8.14) does not exceed a constant times its left-hand side, owing to the boundedness of the dilation operator in rearrangement-invariant spaces. Note that this inequality holds even without assumption (5.12). On the other hand, (5.12) implies

$$\int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{j-1} ds \leq \int_t^1 \frac{f(s)}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{j-1} ds$$
$$\leq C^{j-1} \int_t^1 \frac{f(s)}{I(s)} \left(\frac{s}{I(s)} \right)^{j-1} ds \quad \text{for } t \in (0,1),$$

hence the converse inequality in (8.14) follows. This proves (i).

The proof of (ii) is similar. □

Given $j \in \mathbb{N}$ and a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, let $\|\cdot\|_{(X^\sharp_{j,I})'(0,1)}$ be the functional defined as in (5.16).

Remark 8.7. It follows from Proposition 8.6 and its proof that for every rearrangement-invariant norm $\|\cdot\|_{X(0,1)}$ and every $j \in \mathbb{N}$, we have

$$\left(X^\sharp_{j,I}\right)'(0,1) \to X'_{j,I}(0,1),$$

and if moreover (5.12) is satisfied, then, in fact,

$$\left(X^\sharp_{j,I}\right)'(0,1) = X'_{j,I}(0,1).$$

This observation has a straightforward consequence.

Proposition 8.8. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.12) and let $\|\cdot\|_{X(0,1)}$ be any rearrangement-invariant function norm. Then*

$$X_{j,I}(0,1) = X^\sharp_{j,I}(0,1),$$

up to equivalent norms.

The following result is a counterpart of Proposition 8.3 under (5.12). It follows from Proposition 8.3, with $j = 1$ and I replaced with the function $(0,1) \ni t \mapsto \frac{I(t)^j}{t^{j-1}}$, which obviously satisfies (5.2).

Proposition 8.9. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.12). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant norm and let $j \in \mathbb{N}$. Then the functional $\|\cdot\|_{(X^\sharp_{j,I})'(0,1)}$ defined as in (5.16) is a rearrangement-invariant function norm. Moreover,*

$$H_I^j : X(0,1) \to X^\sharp_{j,I}(0,1),$$

and $X^\sharp_{j,I}(0,1)$ is optimal in (8.9) among all rearrangement-invariant spaces.

9. Proofs of the main results

Here we are concerned with the proof of the results of Section 5. In what follows, R_I^m denotes the operator defined as in (8.4).

Lemma 9.1. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function fulfilling (5.2), and let $m \in \mathbb{N} \cup \{0\}$. Then, for every $f \in \mathcal{M}_+(0,1)$,*

$$R_I^m f^*(t) \le 2^m R_I^m f^*(s) \quad \text{if } 0 < \frac{t}{2} \le s \le t \le 1. \tag{9.1}$$

Consequently, for every $f \in \mathcal{M}_+(0,1)$,

$$(d-c) R_I^m f^*(d) \le 2^{m+1} \int_c^d R_I^m f^*(s)\, ds \quad \text{if } 0 \le c < d \le 1. \tag{9.2}$$

Proof. We prove inequality (9.1) by induction. Fix any $f \in \mathcal{M}_+(0,1)$. If $m = 0$, then (9.1) is satisfied thanks to the monotonicity of f^*. Next, let $m \ge 1$, and assume that (9.1) is fulfilled with m replaced with $m-1$. If $0 < \frac{t}{2} \le s \le t \le 1$, then

$$\begin{aligned} R_I^m f^*(t) &= \frac{1}{I(t)} \int_0^t R_I^{m-1} f^*(r)\, dr \le \frac{2^{m-1}}{I(s)} \int_0^t R_I^{m-1} f^*\left(\frac{r}{2}\right) dr \\ &= \frac{2^m}{I(s)} \int_0^{\frac{t}{2}} R_I^{m-1} f^*(r)\, dr \\ &\le \frac{2^m}{I(s)} \int_0^s R_I^{m-1} f^*(r)\, dr = 2^m R_I^m f^*(s), \end{aligned}$$

where the first inequality holds according to the induction assumption and to the fact that I is non-decreasing on $[0,1]$. Inequality (9.1) follows.

Now, let $0 \le c < d \le 1$, $m \in \mathbb{N} \cup \{0\}$ and $f \in \mathcal{M}_+(0,1)$. Thanks to (9.1),

$$\int_c^d R_I^m f^*(d)\,ds = 2\int_{\frac{c+d}{2}}^d R_I^m f^*(d)\,ds \le 2^{m+1}\int_{\frac{c+d}{2}}^d R_I^m f^*(s)\,ds \le 2^{m+1}\int_c^d R_I^m f^*(s)\,ds.$$

This proves (9.2). □

Given $m \in \mathbb{N}$ and a non-decreasing function $I : [0,1] \to [0,\infty)$ fulfilling (5.2), we define the operator G_I^m at every $f \in \mathcal{M}_+(0,1)$ by

$$G_I^m f(t) = \sup_{t \le s \le 1} R_I^m f^*(s) \quad \text{for } t \in (0,1). \tag{9.3}$$

When $m = 1$ we simply denote G_I^1 by G_I. Note that, trivially, $R_I^m f^* \le G_I^m f$ for every $f \in \mathcal{M}_+(0,1)$. Moreover, $G_I^m f$ is a non-increasing function, and hence $(R_I^m f^*)^* \le G_I^m f$ as well.

The following lemma tells us that the operator G_I^m does not essentially change if I is replaced with its left-continuous representative.

Lemma 9.2. *Let $m \in \mathbb{N}$, let $I : [0,1] \to [0,\infty)$ be a non-decreasing function fulfilling (5.2), and let $I_0 : [0,1] \to [0,\infty)$ be the left-continuous function which agrees with I a.e. in $[0,1]$. Then, for every $f \in \mathcal{M}_+(0,1)$,*

$$G_I^m f = G_{I_0}^m f$$

up to a countable subset of $(0,1)$.

Proof. Define $M = \{t \in (0,1) : I(t) \ne I_0(t)\}$. The set M is at most countable. We shall prove that, for every $g \in \mathcal{M}_+(0,1)$,

$$\sup_{t \le s \le 1} \frac{1}{I(s)} \int_0^s g(r)\,dr = \sup_{t \le s \le 1} \frac{1}{I_0(s)} \int_0^s g(r)\,dr \quad \text{for } t \in (0,1) \setminus M. \tag{9.4}$$

The conclusion will then follow by applying (9.4) to the function $g = R_{I_0}^{m-1} f^*$, and by the fact that $\frac{1}{I(s)} \int_0^s (R_I^{m-1} f^*)(r)\,dr = \frac{1}{I(s)} \int_0^s (R_{I_0}^{m-1} f^*)(r)\,dr$ for $s \in (0,1]$. Fix $g \in \mathcal{M}_+(0,1)$ and $t \in (0,1)$. Given $s \in (t,1]$, we have that

$$\begin{aligned} \frac{1}{I(s)} \int_0^s g(r)\,dr &\le \left(\lim_{\tau \to s_-} \frac{1}{I(\tau)}\right) \int_0^s g(r)\,dr \\ &= \lim_{\tau \to s_-} \frac{1}{I(\tau)} \int_0^\tau g(r)\,dr \le \sup_{t < \tau \le 1} \frac{1}{I(\tau)} \int_0^\tau g(r)\,dr. \end{aligned}$$

On taking the supremum over all $s \in (t, 1]$, we get that

$$\sup_{t<s\le 1} \frac{1}{I(s)} \int_0^s g(r)\,dr \le \sup_{t<s\le 1} \left(\lim_{\tau\to s_-} \frac{1}{I(\tau)} \right) \int_0^s g(r)\,dr \le \sup_{t<\tau\le 1} \frac{1}{I(\tau)} \int_0^\tau g(r)\,dr.$$

Hence, since $I_0(s) = \lim_{\tau\to s_-} I(\tau)$ for $s \in (0, 1]$,

$$\sup_{t<s\le 1} \frac{1}{I(s)} \int_0^s g(r)\,dr = \sup_{t<s\le 1} \frac{1}{I_0(s)} \int_0^s g(r)\,dr \quad \text{for } t \in (0, 1).$$

This yields (9.4). □

Proposition 9.3. *Let $m \in \mathbb{N}$, let $I : [0, 1] \to [0, \infty)$ be a left-continuous non-decreasing function fulfilling (5.2), and let $f \in \mathcal{M}_+(0, 1)$. Define*

$$E = \big\{t \in (0, 1) : R_I^m f^*(t) < G_I^m f(t)\big\}. \tag{9.5}$$

Then E is an open subset of $(0, 1)$. Hence, there exists an at most countable collection $\{(c_k, d_k)\}_{k\in S}$ of pairwise disjoint open intervals in $(0, 1)$ such that

$$E = \bigcup_{k\in S} (c_k, d_k). \tag{9.6}$$

Moreover,

$$G_I^m f(t) = R_I^m f^*(t) \quad \text{if } t \in (0, 1) \setminus E, \tag{9.7}$$

and

$$G_I^m f(t) = R_I^m f^*(d_k) \quad \text{if } t \in (c_k, d_k) \text{ for some } k \in S. \tag{9.8}$$

Proof. Fix $t \in (0, 1)$. If $G_I^m f(t) = \infty$, then both functions $G_I^m f$ and $R_I^m f^*$ are identically equal to ∞, and hence there is nothing to prove. Assume that $G_I^m f(t) < \infty$. Then we claim that $\sup_{t\le s\le 1} R_I^m f^*(s)$ is attained. This follows from the fact that the function $R_I^m f^*(s)$ is upper-semicontinuous, since $I(s)R_I^m f^*(s)$ is continuous, and $\frac{1}{I(s)}$ is upper-semicontinuous. Notice that this latter property holds since I is left-continuous and non-decreasing, and hence lower-semicontinuous.

Suppose now that $t \in E$. Then, due to the upper-semicontinuity of $R_I^m f^*$, there exists $\delta > 0$ such that

$$R_I^m f^*(r) < G_I^m f(t) \quad \text{if } r \in (t - \delta, t + \delta). \tag{9.9}$$

Let $c \in [t, 1]$ be such that $R_I^m f^*(c) = G_I^m f(t)$. Then, thanks to (9.9), $c \in [t + \delta, 1]$. It easily follows that $G_I^m f(t) = G_I^m f(r)$ for every $r \in (t - \delta, t + \delta)$, a piece of information that, combined with (9.9), yields $r \in E$. This shows that E is an open set. Assertion (9.7) is trivial and (9.8) is an easy consequence of the definition of $G_I^m f$. □

Proposition 9.4. *Let $m \in \mathbb{N}$, let $I : [0, 1] \to [0, \infty)$ be a left-continuous non-decreasing function fulfilling (5.2), and let $f \in \mathcal{M}_+(0, 1)$. Then*

$$G_I^m G_I f \approx G_I^{m+1} f, \tag{9.10}$$

up to multiplicative constants depending on m.

Proof. Fix any $f \in \mathcal{M}_+(0, 1)$. Since $R_I f^* \le G_I f$, for every $m \in \mathbb{N}$

$$\begin{aligned} G_I^{m+1} f(t) &= \sup_{t \le s \le 1} R_I^m R_I f^*(s) \\ &\le \sup_{t \le s \le 1} R_I^m G_I f^*(s) = G_I^m G_I f(t) \quad \text{for } t \in (0, 1). \end{aligned} \tag{9.11}$$

This shows that the right-hand side of (9.10) does not exceed the left-hand side. To show a converse inequality, consider the set E defined as in (9.5), with $m = 1$. By Proposition 9.3, the set E is open. Let $\{(c_k, d_k)\}_{k \in S}$ be open intervals as in (9.6). If $t \in (c_k, d_k)$ for some $k \in S$, then, by (9.8) with $m = 1$,

$$\frac{d_k}{I(d_k)} f^{**}(d_k) = R_I f^*(d_k) = G_I f(t) \ge R_I f^*(t) \ge \frac{t}{I(t)} f^{**}(d_k). \tag{9.12}$$

Observe that $f^{**}(d_k) > 0$. Indeed, if $f^{**}(d_k) = 0$, then $R_I f^*(t) = R_I f^*(d_k) = G_I f(t) = 0$, and hence $t \notin E$, a contradiction. Thus, we obtain from (9.12)

$$\frac{d_k}{I(d_k)} \ge \frac{t}{I(t)} \quad \text{for } t \in (c_k, d_k). \tag{9.13}$$

We shall now prove by induction that, given $m \in \mathbb{N} \cup \{0\}$, there exists a constant $C = C(m)$ such that

$$R_I^m G_I f(t) \le C \Big(R_I^{m+1} f^*(t) + \sum_{k \in S} \chi_{(c_k, d_k)}(t) R_I^{m+1} f^*(d_k) \Big) \quad \text{for } t \in (0, 1). \tag{9.14}$$

Let $m = 0$. Then (9.14) holds with $C = 1$, by (9.7) and (9.8) (with $m = 1$). Next, suppose that (9.14) holds for some $m \in \mathbb{N} \cup \{0\}$. Fix any $t \in (0, 1)$. Then

$$
\begin{aligned}
R_I^{m+1} G_I f(t) &= \frac{1}{I(t)} \int_0^t R_I^m G_I f(r)\,dr \le \frac{C}{I(t)} \int_0^t R_I^{m+1} f^*(r)\,dr \\
&\quad + \frac{C}{I(t)} \sum_{\{\ell\in S: d_\ell \le t\}} \int_{c_\ell}^{d_\ell} R_I^{m+1} f^*(d_\ell)\,dr \\
&\quad + \frac{C}{I(t)} \sum_{k\in S} \chi_{(c_k,d_k)}(t) \int_{c_k}^{t} R_I^{m+1} f^*(d_k)\,dr \\
&\le C R_I^{m+2} f^*(t) + \frac{2^{m+2}C}{I(t)} \sum_{\{\ell\in S: d_\ell \le t\}} \int_{c_\ell}^{d_\ell} R_I^{m+1} f^*(r)\,dr \\
&\quad + C \frac{t}{I(t)} \sum_{k\in S} \chi_{(c_k,d_k)}(t) R_I^{m+1} f^*(d_k) \quad \text{(by (9.2))} \\
&\le C R_I^{m+2} f^*(t) + \frac{2^{m+2}C}{I(t)} \int_0^t R_I^{m+1} f^*(r)\,dr \\
&\quad + C \sum_{k\in S} \chi_{(c_k,d_k)}(t) \frac{d_k}{I(d_k)} R_I^{m+1} f^*(d_k) \quad \text{(by (9.13))} \\
&\le \big(C + 2^{m+2}C\big) R_I^{m+2} f^*(t) \\
&\quad + C 2^{m+2} \sum_{k\in S} \chi_{(c_k,d_k)}(t) \frac{1}{I(d_k)} \int_0^{d_k} R_I^{m+1} f^*(r)\,dr \quad \text{(by (9.2))} \\
&= \big(C + 2^{m+2}C\big) R_I^{m+2} f^*(t) + C 2^{m+2} \sum_{k\in S} \chi_{(c_k,d_k)}(t) R_I^{m+2} f^*(d_k) \\
&\le C' \bigg(R_I^{m+2} f^*(t) + \sum_{k\in S} \chi_{(c_k,d_k)}(t) R_I^{m+2} f^*(d_k) \bigg),
\end{aligned}
$$

where $C' = C + 2^{m+2}C$. This proves (9.14).

Owing to (9.14), for every $m \in \mathbb{N}$ we have that

$$
G_I^m G_I f(t) = \sup_{t\le s\le 1} R_I^m G_I f(s) \le 2C G_I^{m+1} f(t) \quad \text{for } t \in (0,1).
$$

Combining this inequality with (9.11) yields (9.10). $\square$

Theorem 9.5. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.2) and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $m \in \mathbb{N} \cup \{0\}$. Then*

$$\|R_I^{m+1}f^*\|_{X'(0,1)} \approx \|R_I^m\big((R_If^*)^*\big)\|_{X'(0,1)} \approx \|G_I^{m+1}f\|_{X'(0,1)} \approx \|R_I^{m+1}f^*\|_{X'_d(0,1)} \tag{9.15}$$

for every $f \in \mathcal{M}_+(0,1)$, *up to multiplicative constants depending on* m.

Proof. We may assume, without loss of generality, that I is left-continuous. Indeed, equation (9.15) is not affected by a replacement of I with its left-continuous representative, since the latter can differ from I at most on a countable subset of $[0,1]$, and since Lemma 9.2 holds.

Fix any $f \in \mathcal{M}_+(0,1)$, and let $m \geq 1$. By (8.5) and Proposition 9.4, there exists a constant $C = C(m)$ such that

$$R_I^{m+1}f^*(t) \leq R_I^m\big((R_If^*)^*\big)(t) \leq R_I^m(G_If)(t) \leq G_I^mG_If(t) \leq CG_I^{m+1}f(t)$$
$$\text{for } t \in (0,1).$$

Hence,

$$\|R_I^{m+1}f^*\|_{X'(0,1)} \leq \|R_I^m\big((R_If^*)^*\big)\|_{X'(0,1)} \leq C\|G_I^{m+1}f\|_{X'(0,1)}. \tag{9.16}$$

Observe that (9.16) trivially holds also when $m = 0$.

Let E be defined as in (9.5), with m replaced with $m+1$, and let $\{(c_k,d_k)\}_{k\in S}$ be as in (9.6). For every $g \in X(0,1)$, define

$$A(g) = \chi_{(0,1)\setminus E}\,g^* + \sum_{k\in S}\chi_{(c_k,d_k)}\frac{1}{d_k-c_k}\int_{c_k}^{d_k} g^*(t)\,dt. \tag{9.17}$$

Then $A(g)$ is non-increasing on $(0,1)$. Moreover, if $\|g\|_{X(0,1)} \leq 1$, then by [7, Theorem 4.8, Chapter 2],

$$\|A(g)\|_{X(0,1)} \leq \|g^*\|_{X(0,1)} = \|g\|_{X(0,1)} \leq 1. \tag{9.18}$$

Therefore,

$$\begin{aligned}\int_0^1 g^*(t)G_I^{m+1}f(t)\,dt &= \int_{(0,1)\setminus E} g^*(t)R_I^{m+1}f^*(t)\,dt + \sum_{k\in S}\int_{c_k}^{d_k} g^*(t)R_I^{m+1}f^*(d_k)\,dt\\ &= \int_{(0,1)\setminus E} g^*(t)R_I^{m+1}f^*(t)\,dt\\ &\quad + \sum_{k\in S}\frac{1}{d_k-c_k}\left(\int_{c_k}^{d_k} g^*(t)\,dt\right)(d_k-c_k)R_I^{m+1}f^*(d_k)\end{aligned}$$

$$\leq \int\limits_{(0,1)\setminus E} A(g)(t)R_I^{m+1}f^*(t)\,dt$$

$$+ 2^{m+2}\sum_{k\in S}\int\limits_{c_k}^{d_k} A(g)(t)R_I^{m+1}f^*(t)\,dt \quad (\text{by (9.2)})$$

$$\leq 2^{m+2}\int\limits_0^1 A(g)(t)R_I^{m+1}f^*(t)\,dt$$

$$\leq 2^{m+2}\sup_{\|h\|_{X(0,1)}\leq 1}\int\limits_0^1 h^*(t)R_I^{m+1}f^*(t)\,dt \quad (\text{by (9.18)})$$

$$= 2^{m+2}\|R_I^{m+1}f^*\|_{X'_d(0,1)}.$$

On taking the supremum over all g from the unit ball of $X(0,1)$, we get

$$\|G_I^{m+1}f\|_{X'(0,1)} = \|G_I^{m+1}f\|_{X'_d(0,1)} \leq 2^{m+2}\|R_I^{m+1}f^*\|_{X'_d(0,1)}. \tag{9.19}$$

On the other hand, by the very definition of $\|\cdot\|_{X'_d(0,1)}$,

$$\|R_I^{m+1}f^*\|_{X'_d(0,1)} \leq \|R_I^{m+1}f^*\|_{X'(0,1)}. \tag{9.20}$$

Eq. (9.15) follows from (9.16), (9.19) and (9.20). $\square$

Corollary 9.6. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.2), and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Let $m \in \mathbb{N}$. Then*

$$(X_{m,I})_1(0,1) = X_{m+1,I}(0,1) \tag{9.21}$$

(up to equivalent norms).

Proof. By (8.10) and (8.8), if $f \in \mathcal{M}_+(0,1)$, then

$$\|f\|_{((X_{m,I})_1)'(0,1)} = \|R_If^*\|_{X'_{m,I}(0,1)} = (m-1)!\|R_I^m\big((R_If^*)^*\big)\|_{X'(0,1)},$$

and

$$\|f\|_{X'_{m+1,I}(0,1)} = m!\|R_I^{m+1}f^*\|_{X'(0,1)}.$$

Hence, it follows from Theorem 9.5 that

$$\|f\|_{((X_{m,I})_1)'(0,1)} \approx \|f\|_{X'_{m+1,I}(0,1)}.$$

By (3.14), this establishes (9.21). □

Theorem 9.7. *Let $I : [0,1] \to [0,\infty)$ be a non-decreasing function satisfying (5.2) and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then, for every $m \in \mathbb{N}$,*

$$X_{m,I}(0,1) = X_m(0,1). \tag{9.22}$$

Proof. We argue by induction. As noted in (8.11), we have $X_1(0,1) = X_{1,I}(0,1)$. Assume now that (9.22) holds for some $m \in \mathbb{N}$. By (8.13), the induction assumption, and (9.21),

$$X_{m+1}(0,1) = (X_m)_1(0,1) = (X_{m,I})_1(0,1) = X_{m+1,I}(0,1).$$

The conclusion follows. □

One consequence of Theorem 9.5, specifically of the equivalence of the leftmost and the rightmost side of (9.15), is the following feature of inequality (5.3), which was already mentioned in Remark 5.2.

Corollary 9.8. *Assume that (Ω,ν) fulfills (5.1) for some non-decreasing function I satisfying (5.2). Let $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following two assertions are equivalent:*

(i) *There exists a constant C_1 such that inequality (5.3) holds for every nonnegative $f \in X(0,1)$.*

(ii) *There exists a constant C_1' such that inequality (5.3) holds for every nonnegative non-increasing $f \in X(0,1)$.*

Proof. The fact that (i) implies (ii) is trivial. Conversely, assume that (ii) holds. Fix $f \in \mathcal{M}_+(0,1)$. Equation (8.6) with $j = m$ reads

$$\int_t^1 \frac{f(s)}{I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{m-1} ds = (m-1)!\, H_I^m f(t) \quad \text{for } t \in (0,1). \tag{9.23}$$

Now, the function $H_I^m f$ is non-increasing on $(0,1)$. Therefore, it follows from (3.10) and the Hardy–Littlewood inequality (3.7) that

$$\left\|H_I^m f\right\|_{Y(0,1)} = \sup_{\|g\|_{Y'(0,1)} \le 1} \int_0^1 g^*(t) H_I^m f(t)\, dt.$$

Consequently, by Fubini's theorem, we have

$$\big\|H_I^m f\big\|_{Y(0,1)} = \sup_{\|g\|_{Y'(0,1)}\le 1} \int_0^1 f(t) R_I^m g^*(t)\, dt. \tag{9.24}$$

Owing to (9.23) and to the rearrangement-invariance of the norm $\|\cdot\|_{X(0,1)}$, assertion (ii) tells us that

$$C_1' \ge (m-1)! \sup_{\|f\|_{X(0,1)}\le 1} \big\|H_I^m f^*\big\|_{Y(0,1)}.$$

Hence, on applying (9.24) with f replaced with f^*, interchanging the suprema and recalling the definition of the norm $\|\cdot\|_{X_d'(0,1)}$, we get

$$\begin{aligned}
C_1' &\ge (m-1)! \sup_{\|f\|_{X(0,1)}\le 1} \sup_{\|g\|_{Y'(0,1)}\le 1} \int_0^1 f^*(t) R_I^m g^*(t)\, dt \\
&= (m-1)! \sup_{\|g\|_{Y'(0,1)}\le 1} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^1 f^*(t) R_I^m g^*(t)\, dt \\
&= (m-1)! \sup_{\|g\|_{Y'(0,1)}\le 1} \big\|R_I^m g^*\big\|_{X_d'(0,1)}.
\end{aligned}$$

It follows from the equivalence of the first and the last term in (9.15) that there exists a constant C such that

$$\big\|R_I^m g^*\big\|_{X'(0,1)} \le C \big\|R_I^m g^*\big\|_{X_d'(0,1)}.$$

Therefore,

$$C_1' C \ge (m-1)! \sup_{\|g\|_{Y'(0,1)}\le 1} \big\|R_I^m g^*\big\|_{X'(0,1)},$$

namely, by the definition of the norm $\|\cdot\|_{X'(0,1)}$,

$$C_1' C \ge (m-1)! \sup_{\|g\|_{Y'(0,1)}\le 1} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^1 f(t) R_I^m g^*(t)\, dt.$$

Interchanging suprema again and using Fubini's theorem and (9.24) yields

$$\begin{aligned}
C_1' C &\ge (m-1)! \sup_{\|f\|_{X(0,1)}\le 1} \sup_{\|g\|_{Y'(0,1)}\le 1} \int_0^1 g^*(t) H_I^m f(t)\, dt \\
&= (m-1)! \sup_{\|f\|_{X(0,1)}\le 1} \big\|H_I^m f\big\|_{Y(0,1)}.
\end{aligned}$$

Hence, inequality (5.3), or equivalently assertion (i), follows. □

Proof of Theorem 5.1. As observed in Section 5, the case when $m = 1$ is already well-known, and is in fact the point of departure of our approach. We thus focus on the case when $m \geq 2$. On applying Proposition 8.3 with $j = 1$, we get that

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \, ds \right\|_{X_{1,I}(0,1)} \leq \|f\|_{X(0,1)}$$

for every $f \in \mathcal{M}_+(0,1)$. Thus, (5.3) holds with $m = 1$ and $Y(0,1) = X_{1,I}(0,1)$. Hence, by the result for $m = 1$,

$$V^1 X(\Omega,\nu) \to X_1(\Omega,\nu). \tag{9.25}$$

Note that here we have also made use of (8.11). By embedding (9.25) applied to each of the spaces $X_j(\Omega,\nu)$, for $j = 0,\ldots,m-1$, we get

$$V^1 X_j(\Omega,\nu) \to X_{j+1}(\Omega,\nu),$$

whence

$$V^m X(\Omega,\nu) \to V^{m-1} X_1(\Omega,\nu) \to V^{m-2} X_2(\Omega,\nu) \to \ldots \to V^1 X_{m-1}(\Omega,\nu) \to X_m(\Omega,\nu). \tag{9.26}$$

Inequality (5.3) tells us that

$$H_I^m : X(0,1) \to Y(0,1). \tag{9.27}$$

The optimality of the space $X_{m,I}(0,1)$ as a target in (9.27), proved in Proposition 8.3, entails that

$$X_{m,I}(0,1) \to Y(0,1). \tag{9.28}$$

A combination of (9.26), (9.22) and (9.28) yields

$$V^m X(\Omega,\nu) \to X_m(\Omega,\nu) = X_{m,I}(\Omega,\nu) \to Y(\Omega,\nu), \tag{9.29}$$

and (5.4) follows.

Finally, (5.5) is equivalent to (5.4) by Proposition 4.4. Note that assumption (4.9) of that Proposition is satisfied, owing to (5.2). □

Proof of Theorem 5.4. Embedding (5.7) is a straightforward consequence of (9.29). In turn, Proposition 4.4 yields the Poincaré inequality (5.8).

Assume now that the validity of (5.4) implies (5.3). Let $\|\cdot\|_{Y(0,1)}$ be any rearrangement-invariant function norm such that (5.4) holds. Then, by our assumption, inequality (5.3) holds as well, namely

$$H_I^m : X(0,1) \to Y(0,1). \tag{9.30}$$

Since, by Proposition 8.3, $X_{m,I}(0,1)$ is the optimal rearrangement-invariant target space in (9.30), we necessarily have

$$X_{m,I}(0,1) \to Y(0,1).$$

This implies the optimality of the norm $\|\cdot\|_{X_{m,I}(0,1)}$ in (5.7). □

Proof of Corollary 5.5. Observe that

$$\sup_{\substack{f\geq 0\\ \|f\|_{X(0,1)}\leq 1}} \left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{L^\infty(0,1)} = \sup_{\substack{f\geq 0\\ \|f\|_{X(0,1)}\leq 1}} \int_0^1 \frac{f(s)}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{m-1} ds$$
$$= \left\| \frac{1}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{m-1} \right\|_{X'(0,1)}.$$

Hence, (5.9) is equivalent to (5.3) with $Y(0,1) = L^\infty(0,1)$. The assertion thus follows from Theorem 5.1. □

Proof of Theorem 5.7. By Theorem 9.7 and (8.13),

$$(X_{k,I})_{h,I}(0,1) = (X_k)_h(0,1) = X_{k+h}(0,1) = X_{k+h,I}(0,1),$$

and the claim follows. □

Corollaries 5.8, 5.9 and 5.10 follow from Theorems 5.1, 5.4 and 5.7, respectively (via Propositions 8.6–8.9).

Proof of Proposition 4.5. Owing to (4.23), Eq. (4.25) will follow if we show that

$$V^m X(\Omega,\nu) \to W^m X(\Omega,\nu). \tag{9.31}$$

The isoperimetric function $I_{\Omega,\nu}$ is non-decreasing on $[0,\frac{1}{3}]$ by definition. Let us define the function I by

$$I(s) = \begin{cases} I_{\Omega,\nu}(s) & \text{if } s \in [0,\frac{1}{3}], \\ I_{\Omega,\nu}(\frac{1}{3}) & \text{if } s \in [\frac{1}{3},1]. \end{cases} \tag{9.32}$$

Then I is non-decreasing on $[0,1]$. Moreover, by (4.24), it satisfies (5.2). Let H_I be the operator defined as in (8.2), with I given by (9.32). Then,

$$\|H_I f\|_{L^1(0,1)} \leq \int_0^1 f(t)\frac{t}{I(t)}\,dt \leq C\|f\|_{L^1(0,1)},$$

and

$$\|H_I f\|_{L^\infty(0,1)} \leq \int_0^1 \frac{f(t)}{I(t)}\,dt \leq \int_0^1 \frac{ds}{I(s)}\|f\|_{L^\infty(0,1)} \leq C\|f\|_{L^\infty(0,1)},$$

for every $f \in \mathcal{M}_+(0,1)$. Thus, H_I is well defined and bounded both on $L^1(0,1)$ and on $L^\infty(0,1)$. Owing to an interpolation theorem of Calderón [7, Chapter 3, Theorem 2.12], the operator H_I is bounded on every rearrangement-invariant space $X(0,1)$. Hence, from Theorem 5.1 applied with $Y(0,1) = X(0,1)$ and $m = 1$, we obtain that

$$V^1X(\Omega,\nu) \to X(\Omega,\nu). \tag{9.33}$$

Iterating (9.33) tells us that there exists a constant C such that

$$\|\nabla^h u\|_{X(\Omega,\nu)} \leq C\left(\sum_{k=h}^{m-1}\|\nabla^k u\|_{L^1(\Omega,\nu)} + \|\nabla^m u\|_{X(\Omega,\nu)}\right) \tag{9.34}$$

for every $h = 0,\dots,m-1$, and $u \in V^mX(\Omega,\nu)$. Embedding (9.31) is a consequence of (9.34). □

10. Proofs of the Euclidean Sobolev embeddings

In what follows, we shall make use of the fact that the function $I(t) = t^\alpha$ satisfies (5.12) if $\alpha \in (0,1)$.

Proof of Theorem 6.1. If the one-dimensional inequality (6.1) holds, then the Sobolev embedding (6.2) and the Poincaré inequality (6.3) hold as well, owing to (2.4) and to Corollary 5.8. This shows that (i) implies (ii) and (iii). The equivalence of (ii) and (iii) is a consequence of Proposition 4.4.

It thus only remains to prove that (ii) implies (i). Assume that the Sobolev embedding (6.2) holds. If $m \geq n$, then there is nothing to prove, since (6.1) holds for every rearrangement-invariant spaces $X(0,1)$ and $Y(0,1)$. Indeed,

$$\left\|\int_t^1 f(s)s^{-1+\frac{m}{n}}\,ds\right\|_{L^\infty(0,1)} = \int_0^1 f(s)s^{-1+\frac{m}{n}}\,ds \leq \|f\|_{L^1(0,1)}$$

for every nonnegative $f \in L^1(0,1)$, and hence (6.1) follows from (3.16). In the case when $m \leq n-1$, the validity of (6.1) was proved in [53, Theorem A]. Note that the proof is given in [53] for Lipschitz domains, and with the space $W^m X(\Omega)$ in the place of $V^m X(\Omega)$. However, by Proposition 4.5, $W^m X(\Omega) = V^m X(\Omega)$ if Ω is a John domain, since (4.24) is fulfilled for any such domain. Moreover, the Lipschitz property of the domain is immaterial, since the proof does not involve any property of the boundary and hence applies, in fact, to any open set Ω. $\square$

Proof of Theorem 6.2. By Theorem 6.1, every John domain has the property that (5.14) implies (5.13). Consequently, the conclusion follows from Corollary 5.9. $\square$

Proof of Theorem 6.3. The assertion is a consequence of Corollary 5.10. $\square$

The following result provides us with model Euclidean domains of revolution in $\mathbb{R}^n$ in the class $\mathcal{J}_\alpha$. It is an easy consequence of a special case of [70, Section 5.3.3]. In the statement, ω_{n-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

Proposition 10.1. (i) *Given $\alpha \in [\frac{1}{n'}, 1)$, define $\eta_\alpha : [0, \frac{1}{1-\alpha}] \to [0, \infty)$ as*

$$\eta_\alpha(r) = \omega_{n-1}^{-\frac{1}{n-1}} \big(1 - (1-\alpha)r\big)^{\frac{\alpha}{(1-\alpha)(n-1)}} \quad \text{for } r \in \left[0, \frac{1}{1-\alpha}\right].$$

Let Ω be the Euclidean domain in $\mathbb{R}^n$ given by

$$\Omega = \left\{ \big(x', x_n\big) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, 0 < x_n < \frac{1}{1-\alpha}, \big|x'\big| < \eta_\alpha(x_n) \right\}.$$

Then $|\Omega| = 1$, and

$$I_\Omega(s) \approx s^\alpha \quad \text{for } s \in \left[0, \frac{1}{2}\right]. \tag{10.1}$$

(ii) *Define $\eta_1 : [0, \infty) \to [0, \infty)$ as*

$$\eta_1(r) = \omega_{n-1}^{-\frac{1}{n-1}} e^{-\frac{r}{n-1}} \quad \text{for } r \geq 0.$$

Let Ω be the Euclidean domain in $\mathbb{R}^n$ given by

$$\Omega = \big\{ \big(x', x_n\big) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0, \big|x'\big| < \eta_1(x_n) \big\}.$$

Then $|\Omega| = 1$, and

$$I_\Omega(s) \approx s \quad \text{for } s \in \left[0, \frac{1}{2}\right]. \tag{10.2}$$

Proof of Theorem 6.4. The Sobolev embedding (6.10) and the Poincaré inequality (6.11) are equivalent, owing to Theorem 4.4. If $\alpha \in [\frac{1}{n'}, 1)$, then inequality (6.8) implies (6.10) and (6.11), via Corollary 5.8, whereas if $\alpha = 1$, then inequality (6.9) implies (6.10) and (6.11) via Theorem 5.1.

It thus remains to exhibit a domain $\Omega \in \mathcal{J}_\alpha$ such that the Sobolev embedding (6.10) implies either (6.8), or (6.9), according to whether $\alpha \in [\frac{1}{n'}, 1)$ or $\alpha = 1$.
If $\alpha \in [\frac{1}{n'}, 1)$, let Ω be the set given by Proposition 10.1, Part (i), whereas, if $\alpha = 1$, let Ω be the set given by Proposition 10.1, Part (ii). By either (10.1) or (10.2), one has that $\Omega \in \mathcal{J}_\alpha$. Consequently, embedding (6.10) entails that there exists a constant C such that

$$\|u\|_{Y(\Omega)} \leq C\left(\big\|\nabla^m u\big\|_{X(\Omega)} + \sum_{k=0}^{m-1} \big\|\nabla^k u\big\|_{L^1(\Omega)}\right) \tag{10.3}$$

for every $u \in V^m X(\Omega)$. Let us fix any nonnegative function $f \in X(0,1)$, and define $u : \Omega \to [0, \infty)$ as

$$u(x) = \int_{M_\alpha(x_n)}^{1} \frac{1}{r_1^\alpha} \int_{r_1}^{1} \frac{1}{r_2^\alpha} \cdots \int_{r_{m-1}}^{1} \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_1 \quad \text{for } x \in \Omega,$$

where M_α is given by

$$M_\alpha(r) = \begin{cases} (1-(1-\alpha)r)^{\frac{1}{1-\alpha}} & \text{for } r \in [0, \frac{1}{1-\alpha}], \text{ if } \alpha \in [\frac{1}{n'}, 1), \\ e^{-r} & \text{for } r \in [0, \infty), \text{ if } \alpha = 1. \end{cases}$$

The function u is m-times weakly differentiable in Ω, and, since $-M_\alpha' = (M_\alpha)^\alpha$,

$$\big|\nabla^k u(x)\big| = \frac{\partial^k u}{\partial x_n^k}(x) = \int_{M_\alpha(x_n)}^{1} \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^{1} \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^{1} \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_{k+1}$$
$$\text{for a.e. } x \in \Omega,$$

for $k = 1, \ldots, m-1$, and

$$\big|\nabla^m u(x)\big| = \frac{\partial^m u}{\partial x_n^m}(x) = f\big(M_\alpha(x_n)\big) \quad \text{for a.e. } x \in \Omega.$$

Moreover, on setting $L_\alpha = \frac{1}{1-\alpha}$ if $\alpha \in [\frac{1}{n'}, 1)$, and $L_\alpha = \infty$ if $\alpha = 1$, we have that

$$
\begin{aligned}
\left|\left\{\left(x', x_n\right) \in \Omega : x_n > t\right\}\right| &= \omega_{n-1} \int_t^{L_\alpha} \eta_\alpha(r)^{n-1}\, dr = \int_t^{L_\alpha} M_\alpha(r)^\alpha\, dr \\
&= \int_t^{L_\alpha} -M'_\alpha(r)\, dr = M_\alpha(t) \quad \text{for } t \in (0, L_\alpha).
\end{aligned}
$$

Thus,

$$
u^*(s) = \int_s^1 \frac{1}{r_1^\alpha} \int_{r_1}^1 \frac{1}{r_2^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_1 \quad \text{for } s \in (0,1), \tag{10.4}
$$

$$
\left|\nabla^k u\right|^*(s) = \int_s^1 \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^1 \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_{k+1} \quad \text{for } s \in (0,1), \tag{10.5}
$$

for $1 \le k \le m-1$, and

$$
\left|\nabla^m u\right|^*(s) = f^*(s) \quad \text{for } s \in (0,1). \tag{10.6}
$$

Eq. (10.6) ensures that $u \in V^m X(\Omega)$. On the other hand, by (10.3) and (10.4)–(10.6),

$$
\begin{aligned}
&\left\| \int_s^1 \frac{1}{r_1^\alpha} \int_{r_1}^1 \frac{1}{r_2^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1}\, \ldots dr_1 \right\|_{Y(0,1)} \\
&\quad \le C\|f\|_{X(0,1)} + C \sum_{k=0}^{m-1} \int_0^1 \int_s^1 \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^1 \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1}\, \ldots dr_{k+1}\, ds.
\end{aligned} \tag{10.7}
$$

Subsequent applications of Fubini's theorem tell us that

$$
\begin{aligned}
&\int_s^1 \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^1 \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1}\, \ldots dr_{k+1} \\
&\quad = \frac{1}{(m-k-1)!} \int_s^1 \frac{1}{r^\alpha} \left(\int_s^r \frac{dt}{t^\alpha} \right)^{m-k-1} f(r) dr \quad \text{for } s \in (0,1).
\end{aligned} \tag{10.8}
$$

By (10.8), (3.16) and (8.9) applied with $I(t) = t^\alpha$, one has that

$$
\begin{aligned}
&\int_0^1 \int_s^1 \frac{1}{r_{k+1}^\alpha} \int_{r_{k+1}}^1 \frac{1}{r_{k+2}^\alpha} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{r_m^\alpha} dr_m\, dr_{m-1} \ldots dr_{k+1}\, ds \\
&= \frac{1}{(m-k-1)!} \int_0^1 \int_s^1 \frac{1}{r^\alpha} \left(\int_s^r \frac{dt}{t^\alpha} \right)^{m-k-1} f(r) dr\, ds = \left\| H_I^{m-k} f \right\|_{L^1(0,1)} \\
&\leq C \left\| H_I^{m-k} f \right\|_{(L^1)_{m-k,I}(0,1)} \leq C' \|f\|_{L^1(0,1)} \leq C'' \|f\|_{X(0,1)} \qquad (10.9)
\end{aligned}
$$

for $k = 0, \ldots, m-1$, for some constants C, C' and C''.

When $\alpha = 1$, inequality (6.9) follows from (10.7), (10.8) and (10.9). When $\alpha \in [\frac{1}{n'}, 1)$, inequality (6.8) follows from (10.7), (10.8) and (10.9), via Proposition 8.6, Part (i). $\square$

Theorem 10.2. *Let $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in (3.21) is satisfied. Let $I : [0,1] \to [0, \infty)$ be a non-decreasing function such that $\frac{t}{I(t)}$ is non-decreasing. Then*

$$
\left\| R_I f^* \right\|_{(L^{p,q;\alpha})'(0,1)} \approx \left\| t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} R_I f^*(t) \right\|_{L^{q'}(0,1)}
$$

for every $f \in \mathcal{M}_+(0,1)$, up to multiplicative constants depending on p, q, α.

Proof. Fix $f \in \mathcal{M}_+(0,1)$. By Theorem 9.5, applied with $m = 0$ and $X(0,1) = L^{p,q;\alpha}(0,1)$, and Hölder's inequality, there exists a universal constant C such that

$$
\begin{aligned}
&\left\| R_I f^*(t) \right\|_{(L^{p,q;\alpha})'(0,1)} \leq C \left\| R_I f^*(t) \right\|_{(L^{p,q;\alpha})'_d(0,1)} \\
&= C \sup_{\|g\|_{L^{p,q;\alpha}(0,1)} \leq 1} \int_0^1 g^*(t) R_I f^*(t)\, dt \\
&= C \sup_{\|g\|_{L^{p,q;\alpha}(0,1)} \leq 1} \int_0^1 g^*(t) t^{\frac{1}{p} - \frac{1}{q}} \left(\log \frac{2}{t} \right)^{\alpha} t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} R_I f^*(t)\, dt \\
&\leq C \left\| t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} R_I f^*(t) \right\|_{L^{q'}(0,1)}.
\end{aligned}
$$

In order to prove the reverse inequality, assume first that either $1 < p < \infty$ or $p = q = 1$ and $\alpha \geq 0$ or $p = q = \infty$ and $\alpha \leq 0$. By Theorem 9.5 (with $m = 0$) and (3.23),

$$
\begin{aligned}
&\left\| t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} R_I f^*(t) \right\|_{L^{q'}(0,1)} \\
&\leq \left\| t^{\frac{1}{p'} - \frac{1}{q'}} \left(\log \frac{2}{t} \right)^{-\alpha} \sup_{t \leq s \leq 1} R_I f^*(t) \right\|_{L^{q'}(0,1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \|G_I f\|_{L^{p',q';-\alpha}(0,1)} \approx \|G_I f\|_{(L^{p,q;\alpha})'(0,1)} \\
&\approx \left\|R_I f^*\right\|_{(L^{p,q;\alpha})'(0,1)},
\end{aligned}
$$

where the last but one equivalence holds up to multiplicative constants depending on p, q, α.

It remains to consider the case when $p = \infty$, $q \in [1, \infty)$ and $\alpha + \frac{1}{q} < 0$. We have that

$$
\begin{aligned}
&\left\|R_I f^*(t)\right\|_{(L^{p,q;\alpha})'(0,1)} \approx \left\|R_I f^*(t)\right\|_{L^{(1,q';-\alpha-1)}(0,1)} \\
&= \left\| t^{1-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \frac{1}{t} \int_0^t \left(R_I f^*\right)^*(s)\, ds \right\|_{L^{q'}(0,1)} \\
&\geq \left\| t^{1-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \frac{1}{t} \int_0^t R_I f^*(s)\, ds \right\|_{L^{q'}(0,1)} \\
&= \left\| t^{-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \int_0^t f^*(r) \int_r^t \frac{ds}{I(s)}\, dr \right\|_{L^{q'}(0,1)} \\
&\geq \left\| t^{-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \int_0^{t^2} f^*(r) \int_r^t \frac{ds}{I(s)}\, dr \right\|_{L^{q'}(0,1)} \\
&\geq \left\| t^{-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} \int_0^{t^2} f^*(r)\, dr \int_{t^2}^t \frac{s}{I(s)} \frac{ds}{s} \right\|_{L^{q'}(0,1)} \\
&\geq \left\| t^{2-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha-1} f^{**}(t^2) \frac{t^2}{I(t^2)} \left(\log \frac{1}{t}\right) \right\|_{L^{q'}(0,1)} \\
&\geq \frac{1}{2} \left\| \chi_{(0,\frac{1}{2})}(t) t^{2-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha} \frac{t^2}{I(t^2)} f^{**}(t^2) \right\|_{L^{q'}(0,1)} \\
&\geq C \left\| t^{1-\frac{1}{q'}} \left(\log \frac{2}{t}\right)^{-\alpha} R_I f^*(t) \right\|_{L^{q'}(0,1)},
\end{aligned}
$$

for some constant $C = C(\alpha, q)$. The proof is complete. $\square$

Proof of Theorem 6.8. By Corollary 5.9,

$$
\|f\|_{((L^p)_{m,\alpha})'(0,1)} = \left\|s^{m(1-\alpha)} f^{**}(s)\right\|_{L^{p'}(0,1)}
$$

for $f \in \mathcal{M}_+(0,1)$. If $m(1-\alpha) < 1$ and $p < \frac{1}{m(1-\alpha)}$ and r is given by $\frac{1}{r} = \frac{1}{p} - m(1-\alpha)$ (note that $1 < r < \infty$), then this equality of norms, (3.24) and (3.23) yield

$$\big(L^p\big)_{m,\alpha}(0,1) = \big(L^{(r',p')}\big)'(0,1) = \big(L^{r',p'}\big)'(0,1) = L^{r,p}(0,1).$$

Since $p < r$, we have $L^{r,p}(0,1) \to L^r(0,1)$, and the claim follows. The optimality of $L^r(0,1)$ is a consequence of the fact that $L^q(0,1) \subsetneq L^{r,p}(0,1)$ if $q > r$. If $m(1-\alpha) < 1$ and $p = \frac{1}{m(1-\alpha)}$, then $L^{r'}(0,1) \to L^{(1,p)}(0,1)$ for every $r \in [1,\infty)$, and hence

$$\big(L^p\big)_{m,\alpha}(0,1) = \big(L^{(1,p')}\big)'(0,1) \to L^r(0,1).$$

Finally, if either $m(1-\alpha) \geq 1$, or $m(1-\alpha) < 1$ and $p > \frac{1}{m(1-\alpha)}$, then (5.9) is satisfied. The conclusion thus follows from Corollary 5.5. □

Proof of Theorem 6.9. First, assume that either $m(1-\alpha) \geq 1$, or $m(1-\alpha) < 1$, $p = \frac{1}{m(1-\alpha)}$ and $q = 1$, or $m(1-\alpha) < 1$ and $p > \frac{1}{m(1-\alpha)}$. In each of these cases, condition (5.9) is satisfied with $I(t) = t^\alpha$ and $X(0,1) = L^{p,q}(0,1)$. Hence, by Corollary 5.5, $V^m L^{p,q}(\Omega) \to L^\infty(\Omega)$.

Next, assume that $m(1-\alpha) < 1$, and either $1 \leq p < \frac{1}{m(1-\alpha)}$, or $p = \frac{1}{m(1-\alpha)}$ and $q > 1$. Set $J(t) = t^{-m(1-\alpha)+1}$ for $t \in [0,1]$. Then J is a non-decreasing function such that $\frac{t}{J(t)}$ is non-decreasing on $(0,1)$. Given $f \in \mathcal{M}_+(0,1)$, by Theorem 10.2 (with $\alpha = 0$),

$$\begin{aligned}
\|f\|_{(L^{p,q}_{m,\alpha})'(0,1)} &= \Bigg\| s^{-1+m(1-\alpha)} \int_0^s f^*(r)\,dr \Bigg\|_{(L^{p,q})'(0,1)} = \big\|R_J f^*\big\|_{(L^{p,q})'(0,1)} \\
&\approx \big\|t^{\frac{1}{p'}-\frac{1}{q'}} R_J f^*(t)\big\|_{L^{q'}(0,1)} = \big\|t^{\frac{1}{p'}-\frac{1}{q'}+m(1-\alpha)} f^{**}(t)\big\|_{L^{q'}(0,1)} \\
&= \|f\|_{L^{(r',q')}(0,1)},
\end{aligned}$$

where the equivalence holds up to constants depending on p and q, and r' satisfies $\frac{1}{r'} = \frac{1}{p'} + m(1-\alpha)$. Owing to (3.24), (3.23) and (3.18), $L^{(r',q')}(0,1) = (L^{\frac{p}{1-mp(1-\alpha)},q})'(0,1)$ if $m(1-\alpha) < 1$ and $1 \leq p < \frac{1}{m(1-\alpha)}$, and $L^{(r',q')}(0,1) = (L^{\infty,q;-1})'(0,1)$ if $m(1-\alpha) < 1$, $p = \frac{1}{m(1-\alpha)}$ and $q > 1$. The conclusion follows. □

Proof of Theorem 6.10, sketched. Since (5.1) holds with $I(t) = t$, in the case when $1 \leq p < \infty$ the assertion follows from an analogous argument as in the proof of Theorem 7.12 applied with $\beta = 1$ (see Section 11 below for the proof of Theorem 7.12). If $p = \infty$, Theorem 7.12 has to be combined with an appropriate embedding between Lebesgue spaces. □

Proof of Theorem 6.11, sketched. Since (5.1) holds with $I(t) = t$, the conclusion is a consequence of an analogous argument as in the proof of Theorem 7.12 applied with $\beta = 1$. □

11. Proofs of the Sobolev embeddings in product probability spaces

This final section is devoted to the proof of the results of Section 7.

Lemma 11.1. *Let Φ be as in (7.1). Then:*

(i) *The function L_Φ defined by (7.9) is non-decreasing on $[0,1]$;*

(ii) *The inequality*

$$s\Phi'\left(\Phi^{-1}\left(\log\left(\frac{1}{s}\right)\right)\right) \le L_\Phi(s) \le 2s\Phi'\left(\Phi^{-1}\left(\log\left(\frac{1}{s}\right)\right)\right) \tag{11.1}$$

holds for every $s \in (0, \frac{1}{2}]$;

(iii) *The inequality*

$$\frac{\Phi^{-1}(s)}{2s} \le \frac{1}{\Phi'(\Phi^{-1}(s))} \le \frac{\Phi^{-1}(s) - \Phi^{-1}(t)}{s-t} \le \frac{\Phi^{-1}(s)}{s} \tag{11.2}$$

holds whenever $0 \le t < s < \infty$.

Proof. (i) The convexity of Φ and the concavity of $\sqrt{\Phi}$ imply that

$$0 \le \Phi''(t) \le \frac{\Phi'(t)^2}{2\Phi(t)} \quad \text{for } t > 0.$$

Therefore,

$$L'_\Phi(s) = \Phi'\left(\Phi^{-1}\left(\log\frac{2}{s}\right)\right) - \frac{\Phi''(\Phi^{-1}(\log\frac{2}{s}))}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))} \ge \frac{\Phi''(\Phi^{-1}(\log\frac{2}{s}))}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))}\left(2\log\frac{2}{s} - 1\right) > 0$$

for $s \in (0,1)$.

Hence, (i) follows.

(ii) The first inequality in (11.1) trivially holds, since both Φ' and Φ^{-1} are non-decreasing functions. The second inequality follows from (i) and from the fact that

$$2s\Phi'\left(\Phi^{-1}\left(\log\left(\frac{1}{s}\right)\right)\right) = L_\Phi(2s) \quad \text{for } s \in \left(0, \frac{1}{2}\right].$$

(iii) Let $0 \le r_1 < r_2 < \infty$. Owing to the convexity of Φ and to the fact that $\Phi(0) = 0$, we obtain that

$$\frac{\Phi(r_2)}{r_2} \le \frac{\Phi(r_2) - \Phi(r_1)}{r_2 - r_1} \le \Phi'(r_2). \tag{11.3}$$

Furthermore, by the concavity of $\sqrt{\Phi}$ and the fact that $\sqrt{\Phi(0)} = 0$,

$$(\sqrt{\Phi})'(r_2) = \frac{\Phi'(r_2)}{2\sqrt{\Phi(r_2)}} \leq \frac{\sqrt{\Phi(r_2)}}{r_2},$$

and, therefore,

$$\Phi'(r_2) \leq \frac{2\Phi(r_2)}{r_2}. \tag{11.4}$$

Let $0 \leq t < s < \infty$. If we set $r_1 = \Phi^{-1}(t)$, $r_2 = \Phi^{-1}(s)$, then $0 \leq r_1 < r_2 < \infty$. Hence, inequalities (11.3) and (11.4) yield

$$\frac{s}{\Phi^{-1}(s)} \leq \frac{s-t}{\Phi^{-1}(s) - \Phi^{-1}(t)} \leq \Phi'\big(\Phi^{-1}(s)\big) \leq \frac{2s}{\Phi^{-1}(s)}.$$

Assertion (11.2) follows. $\square$

Let $m \in \mathbb{N}$. We define the operator P_Φ^m from $\mathcal{M}_+(0,1)$ into $\mathcal{M}_+(0,1)$ by

$$P_\Phi^m f(t) = \left(\frac{\Phi^{-1}(\log\frac{2}{t})}{\log\frac{2}{t}}\right)^m \int_t^1 \frac{f(s)}{s}\left(\log\frac{s}{t}\right)^{m-1} ds \quad \text{for } t \in (0,1), \tag{11.5}$$

and for $f \in \mathcal{M}_+(0,1)$. Moreover, let $H_{L_\Phi}^m$ be the operator defined as in (8.4) (see also (8.6)), with $I = L_\Phi$, namely

$$H_{L_\Phi}^m f(t) = \frac{1}{(m-1)!}\int_t^1 \frac{f(s)}{L_\Phi(s)}\left(\int_t^s \frac{dr}{L_\Phi(r)}\right)^{m-1} ds \quad \text{for } t \in (0,1), \tag{11.6}$$

and for $f \in \mathcal{M}_+(0,1)$. Observe that, by the change of variables $\tau \mapsto \Phi^{-1}(\log\frac{2}{t})$, we have

$$\begin{aligned}\frac{1}{L_\Phi(r)}\left(\int_s^r \frac{dt}{L_\Phi(t)}\right)^{m-1} &= \frac{1}{r\Phi'(\Phi^{-1}(\log\frac{2}{r}))}\left(\int_s^r \frac{dt}{t\Phi'(\Phi^{-1}(\log\frac{2}{t}))}\right)^{m-1} \\ &= \frac{(\Phi^{-1}(\log\frac{2}{s}) - \Phi^{-1}(\log\frac{2}{r}))^{m-1}}{r\Phi'(\Phi^{-1}(\log\frac{2}{r}))} \quad \text{for } 0 < s \leq r < 1.\end{aligned} \tag{11.7}$$

In particular, this yields

$$H_{L_\Phi}^m f(t) = \frac{1}{(m-1)!}\int_t^1 \frac{f(s)}{s\Phi'(\Phi^{-1}(\log\frac{2}{s}))}\left(\Phi^{-1}\left(\log\frac{2}{t}\right) - \Phi^{-1}\left(\log\frac{2}{s}\right)\right)^{m-1} ds \quad \text{for } t \in (0,1), \tag{11.8}$$

and $f \in \mathcal{M}_+(0,1)$.

A connection between the operators P^m_Φ and $H^m_{L_\Phi}$ is described in the following proposition.

Proposition 11.2. *Suppose that Φ is as in (7.1), $m \in \mathbb{N}$ and $f \in \mathcal{M}_+(0,1)$. Then*

$$\frac{1}{2^m(m-1)!} P^m_\Phi f(t) \le H^m_{L_\Phi} f(t) \quad \text{for } t \in (0,1). \tag{11.9}$$

Moreover, if f is non-increasing on $(0,1)$, then

$$H^m_{L_\Phi} f(t) \le \frac{1}{(m-1)!} P^m_\Phi f(t) \quad \text{for } t \in (0,1). \tag{11.10}$$

Proof. Let $f \in \mathcal{M}_+(0,1)$. Since the function $s \mapsto \frac{1}{\Phi'(\Phi^{-1}(\log \frac{2}{s}))}$ is non-decreasing on $(0,1)$, from the first inequality in (11.2) we obtain that

$$\begin{aligned}
H^m_{L_\Phi} f(t) &= \frac{1}{(m-1)!} \int_t^1 \frac{f(s)}{s\Phi'(\Phi^{-1}(\log \frac{2}{s}))} \left(\int_t^s \frac{dr}{r\Phi'(\Phi^{-1}(\log \frac{2}{r}))} \right)^{m-1} ds \\
&\ge \frac{1}{(m-1)!} \frac{1}{(\Phi'(\Phi^{-1}(\log \frac{2}{t})))^m} \int_t^1 \frac{f(s)}{s} \left(\int_t^s \frac{dr}{r} \right)^{m-1} ds \\
&= \frac{1}{(m-1)!} \frac{1}{(\Phi'(\Phi^{-1}(\log \frac{2}{t})))^m} \int_t^1 \frac{f(s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \\
&\ge \frac{1}{(m-1)!} \frac{1}{2^m} \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \int_t^1 \frac{f(s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \\
&= \frac{1}{(m-1)!} \frac{1}{2^m} P^m_\Phi f(t) \quad \text{for } t \in (0,1).
\end{aligned}$$

Now, assume that f is non-increasing on $(0,1)$. In the special case when f is a characteristic function of an open interval, namely, $f = \chi_{(0,b)}$ for some $b \in (0,1]$, Eq. (11.8) tells us that

$$H^m_{L_\Phi}(\chi_{(0,b)})(t) = \frac{1}{m!} \chi_{(0,b)}(t) \left(\Phi^{-1}\left(\log \frac{2}{t}\right) - \Phi^{-1}\left(\log \frac{2}{b}\right) \right)^m$$

and

$$P^m_\Phi(\chi_{(0,b)})(t) = \chi_{(0,b)}(t) \frac{1}{m} \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \left(\log \frac{2}{t} - \log \frac{2}{b} \right)^m$$

for $t \in (0,1)$. By the last inequality in (11.2),

$$\left(\frac{\Phi^{-1}(\log\frac{2}{t})-\Phi^{-1}(\log\frac{2}{b})}{\log\frac{2}{t}-\log\frac{2}{b}}\right)^m \le \left(\frac{\Phi^{-1}(\log\frac{2}{t})}{\log\frac{2}{t}}\right)^m \quad \text{for } t\in(0,b).$$

Hence,

$$H^m_{L_\Phi}(\chi_{(0,b)}) \le \frac{1}{(m-1)!}P^m_\Phi(\chi_{(0,b)}). \tag{11.11}$$

Assume next that f is a nonnegative non-increasing simple function on $(0,1)$. Then there exist $k\in\mathbb{N}$, nonnegative numbers $a_1,a_2,\dots,a_k\in\mathbb{R}$ and $0<b_1<b_2<\ldots<b_k\le 1$ such that $f=\sum_{i=1}^k a_i\chi_{(0,b_i)}$ a.e. on $(0,1)$. Hence, owing to (11.11),

$$\begin{aligned} H^m_{L_\Phi}f(t) &= \sum_{i=1}^k a_i H^m_{L_\Phi}(\chi_{(0,b_i)})(t) \le \frac{1}{(m-1)!}\sum_{i=1}^k a_i P^m_\Phi(\chi_{(0,b_i)})(t) \\ &= \frac{1}{(m-1)!}P^m_\Phi f(t) \quad \text{for } t\in(0,1). \end{aligned}$$

Finally, if $f\in\mathcal{M}_+(0,1)$ is non-increasing on $(0,1)$, then there exists a sequence f_k of nonnegative non-increasing simple functions on $(0,1)$ such that $f_n\uparrow f$. Clearly,

$$H^m_{L_\Phi}f(t)=\lim_{n\to\infty}H^m_{L_\Phi}f_n(t)\le\frac{1}{(m-1)!}\lim_{n\to\infty}P^m_\Phi f_n(t)=\frac{1}{(m-1)!}P^m_\Phi f(t) \quad \text{for } t\in(0,1),$$

whence (11.10) follows. □

Proposition 11.2 has an important consequence.

Proposition 11.3. *Let Φ be as in (7.1), let $m\in\mathbb{N}$ and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then*

$$P^m_\Phi : X(0,1)\to Y(0,1) \quad \textit{if and only if} \quad H^m_{L_\Phi} : X(0,1)\to Y(0,1).$$

Proof. By (11.9), the boundedness of the operator $H^m_{L_\Phi}$ implies the boundedness of P^m_Φ. Conversely, if P^m_Φ is bounded from $X(0,1)$ into $Y(0,1)$ then, in particular, there exists a constant C such that

$$\left\|P^m_\Phi f\right\|_{Y(0,1)} \le C\|f\|_{X(0,1)}$$

for every nonnegative non-increasing function $f\in X(0,1)$. Combining this inequality with (11.10), we obtain that

$$\left\|H^m_{L_\Phi} f\right\|_{Y(0,1)} \le C\|f\|_{X(0,1)}$$

for every nonnegative non-increasing $f \in X(0,1)$. In view of Corollary 9.8, this is equivalent to the boundedness of $H^m_{L_\Phi}$ from $X(0,1)$ into $Y(0,1)$. $\square$

Proof of Theorem 7.1. Properties (ii) and (iii) are equivalent, by Proposition 4.4. Let us show that (i) and (ii) are equivalent as well. First, assume that (i) is satisfied. Owing to Proposition 11.3, there exists a constant C such that

$$\left\| \int_t^1 \frac{f(s)}{L_\Phi(s)} \left(\int_t^s \frac{dr}{L_\Phi(r)} \right)^{m-1} ds \right\|_{Y(0,1)} \le C \|f\|_{X(0,1)}$$

for every nonnegative $f \in X(0,1)$. By Lemma 11.1(i), the function L_Φ is non-decreasing on $[0,1]$. Furthermore, condition (5.2) is clearly satisfied with $I = L_\Phi$. Thanks to these facts and to (7.10), the assumptions of Theorem 5.1 are fulfilled with $(\Omega, \nu) = (\mathbb{R}^n, \mu_{\Phi,n})$ and $I = L_\Phi$. Hence, (ii) follows.

It only remains to prove that (ii) implies (i). Assume that (ii) holds, namely, there exists a constant C, such that

$$\|u\|_{Y(\mathbb{R}^n,\mu_{\Phi,n})} \le C\left(\left\|\nabla^m u\right\|_{X(\mathbb{R}^n,\mu_{\Phi,n})} + \sum_{k=0}^{m-1} \left\|\nabla^k u\right\|_{L^1(\mathbb{R}^n,\mu_{\Phi,n})} \right) \tag{11.12}$$

for every $u \in V^m X(\mathbb{R}^n, \mu_{\Phi,n})$.

Given any nonnegative function $f \in X(0,1)$ such that $f(s) = 0$ if $s \in (\frac{1}{2}, 1)$, consider the function $u : \mathbb{R}^n \to \mathbb{R}$ defined as

$$u(x) = \int_{H(x_1)}^1 \frac{1}{F_\Phi(r_1)} \int_{r_1}^1 \frac{1}{F_\Phi(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \dots dr_1 \qquad \text{for } x \in \mathbb{R}^n,$$

where H is given by (7.6). Note that, since $H'(t) = -F_\Phi(H(t))$, then

$$\begin{aligned}\left|\nabla^k u(x)\right| &= \frac{\partial^k u}{\partial x_1^k}(x)\\ &= \int_{H(x_1)}^1 \frac{1}{F_\Phi(r_{k+1})} \int_{r_{k+1}}^1 \frac{1}{F_\Phi(r_{k+2})} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \dots dr_{k+1}\end{aligned}$$

for a.e. $x \in \mathbb{R}^n$, for $k = 1, \dots, m-1$, and

$$\left|\nabla^m u(x)\right| = \frac{\partial^m u}{\partial x_1^m}(x) = f\big(H(x_1)\big) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Thus, by (7.8),

$$
\left|\nabla^k u\right|^*(s) = \int_s^1 \frac{1}{F_\Phi(r_{k+1})} \int_{r_{k+1}}^1 \frac{1}{F_\Phi(r_{k+2})} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \dots dr_{k+1} \quad \text{for } s \in (0,1), \tag{11.13}
$$

for $k = 0, \dots, m-1$, and

$$
\left|\nabla^m u\right|^*(s) = f^*(s) \quad \text{for } s \in (0,1). \tag{11.14}
$$

By (11.14), $u \in V^m X(\mathbb{R}^n, \mu_{\Phi,n})$. From (11.12), (11.13) and (11.14) we thus deduce that

$$
\begin{aligned}
&\left\| \int_s^1 \frac{1}{F_\Phi(r_1)} \int_{r_1}^1 \frac{1}{F_\Phi(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \dots dr_1 \right\|_{Y(0,1)} \\
&\quad \le C \|f\|_{X(0,1)} \\
&\qquad + C \sum_{k=0}^{m-1} \int_0^1 \int_s^1 \frac{1}{F_\Phi(r_{k+1})} \int_{r_{k+1}}^1 \frac{1}{F_\Phi(r_{k+2})} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \dots dr_{k+1}\, ds.
\end{aligned} \tag{11.15}
$$

Owing to Fubini's Theorem, (7.10) and (11.7),

$$
\begin{aligned}
&\int_s^1 \frac{1}{F_\Phi(r_1)} \int_{r_1}^1 \frac{1}{F_\Phi(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \dots dr_1 \\
&\quad \approx \int_s^1 \frac{f(r)}{F_\Phi(r)} \left(\int_s^r \frac{dt}{F_\Phi(t)} \right)^{m-1} dr \\
&\quad \approx \int_s^1 \frac{f(r)}{L_\Phi(r)} \left(\int_s^r \frac{dt}{L_\Phi(t)} \right)^{m-1} dr \\
&\quad = \int_s^1 f(r) \frac{(\Phi^{-1}(\log \frac{2}{s}) - \Phi^{-1}(\log \frac{2}{r}))^{m-1}}{r\Phi'(\Phi^{-1}(\log \frac{2}{r}))}\, dr \quad \text{for } s \in (0,1).
\end{aligned} \tag{11.16}
$$

Note that the second equivalence makes use of the fact that f vanishes in $(\frac{1}{2}, 1)$. On the other hand, by (11.16) (with m replaced with $m-k$), (3.16), and (8.9) (with I replaced with L_Φ),

$$
\int_0^1 \int_s^1 \frac{1}{F_\Phi(r_{k+1})} \int_{r_{k+1}}^1 \frac{1}{F_\Phi(r_{k+2})} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{F_\Phi(r_m)}\, dr_m\, dr_{m-1} \dots dr_{k+1}\, ds
$$

$$\approx \int_0^1 \int_s^1 \frac{f(r)}{L_\Phi(r)} \left(\int_s^r \frac{dt}{L_\Phi(t)} \right)^{m-k-1} dr\, ds \approx \|H_{L_\Phi}^{m-k} f\|_{L^1(0,1)}$$
$$\le C \big\| H_{L_\Phi}^{m-k} f \big\|_{(L^1)_{m-k,L_\Phi}(0,1)} \le C' \|f\|_{L^1(0,1)} \le C'' \|f\|_{X(0,1)} \tag{11.17}$$

for some constants C, C' and C''. From inequalities (11.15)–(11.17), we deduce that there exists a constant C such that

$$\left\| \int_s^1 f(r) \frac{(\Phi^{-1}(\log \frac{2}{s}) - \Phi^{-1}(\log \frac{2}{r}))^{m-1}}{r\Phi'(\Phi^{-1}(\log \frac{2}{r}))} dr \right\|_{Y(0,1)} \le C\|f\|_{X(0,1)}$$

for every nonnegative function $f \in X(0,1)$ such that $f(s) = 0$ if $s \in (\frac{1}{2}, 1)$. By Proposition 11.2, for each such function f we also have

$$\left\| \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \int_t^1 \frac{f(s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \right\|_{Y(0,1)} \le 2^m C \|f\|_{X(0,1)}. \tag{11.18}$$

Finally, assume that f is any nonnegative function from $X(0,1)$ (which need not vanish in $(\frac{1}{2}, 1)$). Then, by the boundedness of the dilation operator on $Y(0,1)$, there exists a constant C such that

$$\left\| \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \int_t^1 \frac{f(s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \right\|_{Y(0,1)}$$
$$\le C \left\| \chi_{(0,\frac{1}{2})}(t) \left(\frac{\Phi^{-1}(\log \frac{1}{t})}{\log \frac{1}{t}} \right)^m \int_{2t}^1 \frac{f(s)}{s} \left(\log \frac{s}{2t} \right)^{m-1} ds \right\|_{Y(0,1)}. \tag{11.19}$$

Furthermore, since

$$\frac{\Phi^{-1}(\log \frac{1}{t})}{\log \frac{1}{t}} \le \frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{1}{t}} \le \frac{2\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \quad \text{for } t \in \left(0, \frac{1}{2}\right),$$

from inequality (11.18) with f replaced with $\chi_{(0,\frac{1}{2})}(t) f(2t)$, and the boundedness of the dilation operator, we obtain

$$\left\| \chi_{(0,\frac{1}{2})}(t) \left(\frac{\Phi^{-1}(\log \frac{1}{t})}{\log \frac{1}{t}} \right)^m \int_{2t}^1 \frac{f(s)}{s} \left(\log \frac{s}{2t} \right)^{m-1} ds \right\|_{Y(0,1)}$$
$$\le 2^m \left\| \chi_{(0,\frac{1}{2})}(t) \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \int_t^{\frac{1}{2}} \frac{f(2s)}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \right\|_{Y(0,1)}$$
$$\le C' \big\| \chi_{(0,\frac{1}{2})}(t) f(2t) \big\|_{X(0,1)} \le C'' \|f\|_{X(0,1)} \tag{11.20}$$

for some constants C' and C'' independent of f. Coupling (11.19) with (11.20) yields (7.11). $\square$

Proof of Theorem 7.3. Set $J(s) = s$ for $s \in [0,1]$. Then condition (5.2) is obviously fulfilled with $I = J$. The norm $\|\cdot\|_{\widetilde{X}_{m,J}(0,1)}$ is thus well defined and, moreover, $\|\cdot\|_{\widetilde{X}_m(0,1)} = \|\cdot\|_{X_{m,J}(0,1)}$. Therefore, Proposition 8.3 tells us that $\|\cdot\|_{\widetilde{X}_m(0,1)}$ is a rearrangement-invariant function norm. We shall now verify that $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is a rearrangement-invariant function norm as well. The first two properties in (P1) and properties (P2) and (P3) are straightforward consequences of the corresponding properties for $\|\cdot\|_{\widetilde{X}_m(0,1)}$. To prove the triangle inequality, fix $f, g \in \mathcal{M}_+(0,1)$. By (3.6), $\int_0^s (f+g)^*(r)\,dr \leq \int_0^s (f^*(r)+g^*(r))\,dr$ for $s \in (0,1)$. We observe that for each $t \in (0,1)$, the function $s \mapsto \chi_{(0,t)}(s)(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})})^m$ is nonnegative and non-increasing on $(0,1)$. Hardy's lemma therefore yields that

$$\int_0^t \left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m (f+g)^*(s)\,ds$$
$$\leq \int_0^t \left(\left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m f^*(s) + \left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m g^*(s)\right)ds$$

for $t \in (0,1)$. The triangle inequality now follows using the Hardy–Littlewood–Pólya principle and the triangle inequality for $\|\cdot\|_{\widetilde{X}_m(0,1)}$.

One has that

$$\exp L^{\frac{1}{m}}(0,1) = \left(L^\infty\right)_m(0,1) \to \widetilde{X}_m(0,1),$$

where the equality is a consequence of Theorem 6.11. Thus, there exists a constant C such that

$$\|1\|_{X_{m,\Phi}(0,1)} = \left\|\left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m\right\|_{\widetilde{X}_m(0,1)} \leq C\left\|\left(\frac{\log\frac{2}{s}}{\Phi^{-1}(\log\frac{2}{s})}\right)^m\right\|_{\exp L^{\frac{1}{m}}(0,1)}$$
$$\approx \left\|\frac{1}{(\Phi^{-1}(\log\frac{2}{s}))^m}\right\|_{L^\infty(0,1)} = \frac{1}{(\Phi^{-1}(\log 2))^m} < \infty.$$

This proves (P4).

Finally, by property (P5) for $\|\cdot\|_{\widetilde{X}_m(0,1)}$, there exists a positive constant C such that for all $f \in \mathcal{M}_+(0,1)$,

$$\|f\|_{X_{m,\Phi}(0,1)} \geq \left(\frac{\log 2}{\Phi^{-1}(\log 2)}\right)^m \|f^*\|_{\widetilde{X}_m(0,1)} \geq \left(\frac{C\log 2}{\Phi^{-1}(\log 2)}\right)^m \int_0^1 f^*(s)\,ds.$$

Therefore, $\|\cdot\|_{X_{m,\Phi}(0,1)}$ satisfies (P5). Since the property (P6) holds trivially, $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is actually a rearrangement-invariant norm.

It follows from the proof of Theorem 7.1 that the assumptions of Theorem 5.4 are fulfilled with $(\Omega, \nu) = (\mathbb{R}^n, \mu_{\Phi,n})$ and $I = L_\Phi$. Therefore, $\|\cdot\|_{X_{m,L_\Phi}(0,1)}$ is the optimal rearrangement-invariant target function norm for $\|\cdot\|_{X(0,1)}$ in the Sobolev embedding (7.12). Thus, the proof will be complete if we show that $X_{m,\Phi}(0,1) = X_{m,L_\Phi}(0,1)$. We have that

$$\begin{aligned}
\|f\|_{X'_{m,L_\Phi}(0,1)} &= (m-1)!\big\|R^m_{L_\Phi} f^*\big\|_{X'(0,1)} \approx \big\|R^m_{L_\Phi} f^*\big\|_{X'_d(0,1)} \quad \text{(by Theorem 9.5)} \\
&= \sup_{\|g\|_{X(0,1)}\le 1} \int_0^1 g^*(t) R^m_{L_\Phi} f^*(t)\,dt \\
&= \sup_{\|g\|_{X(0,1)}\le 1} \int_0^1 f^*(t) H^m_{L_\Phi} g^*(t)\,dt \\
&\approx \sup_{\|g\|_{X(0,1)}\le 1} \int_0^1 f^*(t) P^m_\Phi g^*(t)\,dt \quad \text{(by Proposition 11.2)} \\
&\approx \sup_{\|g\|_{X(0,1)}\le 1} \int_0^1 f^*(t) \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}}\right)^m H^m_J g^*(t)\,dt \\
&= \sup_{\|g\|_{X(0,1)}\le 1} \int_0^1 g^*(t) R^m_J \left(f^*(s)\left(\frac{\Phi^{-1}(\log \frac{2}{s})}{\log \frac{2}{s}}\right)^m\right)(t)\,dt \\
&= \left\| R^m_J \left(f^*(s)\left(\frac{\Phi^{-1}(\log \frac{2}{s})}{\log \frac{2}{s}}\right)^m\right)\right\|_{X'_d(0,1)} \quad \text{for } f \in L^1(0,1),
\end{aligned} \tag{11.21}$$

up to multiplicative constants depending on m.

We now claim that, given $f \in L^1(0,1)$, there exists a non-decreasing function I on $[0,1]$ fulfilling (5.2) and a function $h \in \mathcal{M}_+(0,1)$ such that

$$f^*(s)\left(\frac{\Phi^{-1}(\log \frac{2}{s})}{\log \frac{2}{s}}\right)^m \approx R_I h^*(s) \quad \text{for } s \in (0,1), \tag{11.22}$$

up to multiplicative constants depending on m. Indeed, let $s_0 \in (0,1)$ be chosen in such a way that the function $s \mapsto s(\log \frac{2}{s})^{m+1}$ is non-decreasing on $(0, s_0)$. Then we set

$$I(s) = \frac{1}{f^*(s)} \quad \text{for } s \in (0,1] \quad \text{and} \quad I(0) = 0,$$

and

$$h(s)=\begin{cases}\frac{(\Phi^{-1}(\log\frac{2}{s}))^{m-1}(\Phi^{-1}(\log\frac{2}{s})-\frac{\log\frac{2}{s}}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))})}{s(\log\frac{2}{s})^{m+1}}, & s\in(0,s_0]\\ \frac{(\Phi^{-1}(\log\frac{2}{s}))^{m-1}(\Phi^{-1}(\log\frac{2}{s})-\frac{\log\frac{2}{s}}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))})}{s_0(\log\frac{2}{s_0})^{m+1}}, & s\in(s_0,1).\end{cases}$$

It follows from (11.2) that the function h is non-negative on $(0,1)$. To verify (11.22) we first show that h is non-increasing on $(0,1)$. The function Φ^{-1} is clearly non-decreasing on $(0,\infty)$. Furthermore, we deduce from the convexity of Φ that the function $s\mapsto\Phi^{-1}(s)-\frac{s}{\Phi'(\Phi^{-1}(s))}$ is non-decreasing on $(0,\infty)$. Altogether, this ensures that the function

$$s\mapsto\left(\Phi^{-1}\left(\log\frac{2}{s}\right)\right)^{m-1}\left(\Phi^{-1}\left(\log\frac{2}{s}\right)-\frac{\log\frac{2}{s}}{\Phi'(\Phi^{-1}(\log\frac{2}{s}))}\right)$$

is non-increasing on $(0,1)$. By the definition of s_0, the function

$$s\mapsto\begin{cases}s(\log\frac{2}{s})^{m+1}, & s\in(0,s_0]\\ s_0(\log\frac{2}{s_0})^{m+1}, & s\in(s_0,1)\end{cases}$$

is non-decreasing (and continuous) on $(0,1)$, and therefore, in particular, $h=h^*$.

Consequently, we have

$$\begin{aligned}f^*(s)\left(\frac{\Phi^{-1}(\log\frac{2}{s})}{\log\frac{2}{s}}\right)^m&=\frac{m}{I(s)}\int_0^s\frac{(\Phi^{-1}(\log\frac{2}{r}))^{m-1}(\Phi^{-1}(\log\frac{2}{r})-\frac{\log\frac{2}{r}}{\Phi'(\Phi^{-1}(\log\frac{2}{r}))})}{r(\log\frac{2}{r})^{m+1}}\,dr\\&\approx\frac{1}{I(s)}\int_0^s h^*(r)\,dr=R_Ih^*(s)\quad\text{for } s\in(0,1),\end{aligned}$$

up to multiplicative constants depending on m. This proves (11.22). Furthermore, it can be easily verified that the function I fulfills also the remaining required properties.

Coupling (11.21) with (11.22) entails that

$$\|f\|_{X'_{m,L_\Phi}(0,1)}\approx\left\|R_J^mR_Ih^*\right\|_{X'_d(0,1)},\tag{11.23}$$

up to multiplicative constants depending on m.

Now, the same proof as that of Theorem 9.5 yields that

$$\left\|R_J^mR_Ih^*\right\|_{X'_d(0,1)}\approx\left\|R_J^m\left(R_Ih^*\right)^*\right\|_{X'(0,1)},\tag{11.24}$$

up to multiplicative constants still depending only on m.

On combining (11.23), (11.24) and (11.22), we obtain that for every $f\in L^1(0,1)$,

$$\|f\|_{X'_{m,L_\Phi}(0,1)}\approx\left\|R_J^m\left(f^*(\cdot)\left(\frac{\Phi^{-1}(\log\frac{2}{(\cdot)})}{\log\frac{2}{(\cdot)}}\right)^m\right)^*(t)\right\|_{X'(0,1)}$$

$$\approx \left\| f^*(t) \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \right\|_{\widetilde{X}'_m(0,1)} \tag{11.25}$$

up to multiplicative constants depending on m. Consequently, by (11.25), we have that, for every $g \in \mathcal{M}_+(0,1)$,

$$\begin{aligned}
\|g\|_{X_{m,L_\Phi}(0,1)} &= \sup \left\{ \int_0^1 f^*(s) g^*(s)\, ds : \|f\|_{X'_{m,L_\Phi}(0,1)} \le 1 \right\} \\
&\approx \sup \left\{ \int_0^1 f^*(s) g^*(s)\, ds : \left\| f^*(t) \left(\frac{\Phi^{-1}(\log \frac{2}{t})}{\log \frac{2}{t}} \right)^m \right\|_{\widetilde{X}'_m(0,1)} \le 1 \right\} \\
&\le \left\| g^*(t) \left(\frac{\log \frac{2}{t}}{\Phi^{-1}(\log \frac{2}{t})} \right)^m \right\|_{\widetilde{X}_m(0,1)},
\end{aligned}$$

up to multiplicative constants depending on m.

Conversely,

$$\begin{aligned}
&\left\| g^*(t) \left(\frac{\log \frac{2}{t}}{\Phi^{-1}(\log \frac{2}{t})} \right)^m \right\|_{\widetilde{X}_m(0,1)} \\
&= \sup \left\{ \int_0^1 g^*(t) \left(\frac{\log \frac{2}{t}}{\Phi^{-1}(\log \frac{2}{t})} \right)^m f^*(t)\, dt : \|f\|_{\widetilde{X}'_m(0,1)} \le 1 \right\} \\
&\approx \sup \left\{ \int_0^1 g^*(t) \left(\frac{\log \frac{2}{t}}{\Phi^{-1}(\log \frac{2}{t})} \right)^m f^*(t)\, dt : \left\| f^*(t) \left(\frac{\log \frac{2}{t}}{\Phi^{-1}(\log \frac{2}{t})} \right)^m \right\|_{X'_{m,L_\Phi}(0,1)} \le 1 \right\} \\
&\le \|g\|_{X_{m,L_\Phi}(0,1)},
\end{aligned} \tag{11.26}$$

up to multiplicative constants depending on m. Note that the equivalence in (11.26) holds by (11.25) and the fact that the function $t \mapsto (\frac{\log \frac{2}{t}}{\Phi^{-1}(\log \frac{2}{t})})^m$ is non-increasing. Hence, $X_{m,\Phi}(0,1) = X_{m,L_\Phi}(0,1)$. The proof is complete. □

Proof of Proposition 7.5. Since the m-th iteration of the double-star operator $g \mapsto g^{**}$ associates a function g with $\frac{1}{s} \int_0^s (\log \frac{s}{r})^{m-1} g^*(r)\, dr$ for $s \in (0,1)$, we obtain from the boundedness of the double-star operator on $X'(0,1)$ that

$$\|g\|_{\widetilde{X}'_m(0,1)} \approx \|g\|_{X'(0,1)}.$$

Thus, $\widetilde{X}_m(0,1) = X(0,1)$. Consequently, the assertion follows from (7.15). □

Proof of Theorem 7.6. This is a consequence of Theorem 5.7 and of the fact that $X_{m,\Phi}(0,1) = X_{m,L_\Phi}(0,1)$. □

Proof of Theorem 7.12. Set $X(0,1)=L^{p,q;\alpha}(0,1)$. We claim that

$$\widetilde{X}_m(0,1)=\begin{cases} L^{p,q;\alpha}(0,1) & \text{if } p<\infty, \\ L^{\infty,q;\alpha-m}(0,1) & \text{if } p=\infty. \end{cases}$$

Indeed, let $p<\infty$ and set $\Phi(t)=t$ for $t\in[0,\infty)$. Then, by Remark 7.2,

$$\widetilde{X}_m(0,1)=X_{m,\Phi}(0,1).$$

By (3.23) and (3.24), the operator $f\mapsto f^{**}$ is bounded on $X'(0,1)$. Therefore, by Proposition 7.5,

$$X_{m,\Phi}(0,1)=X(0,1)=L^{p,q;\alpha}(0,1).$$

Now, let $p=\infty$, and set $I(s)=s$ for $s\in[0,1]$. Then $R_I f^*=f^{**}$, whence, by Theorem 10.2,

$$\|f\|_{(\widetilde{X}_1)'(0,1)}=\|f^{**}\|_{X'(0,1)}\approx\left\|t^{1-\frac{1}{q'}}\left(\log\frac{2}{t}\right)^{-\alpha}f^{**}(t)\right\|_{L^{q'}(0,1)}=\|f\|_{L^{(1,q';-\alpha)}(0,1)}.$$

Owing to (3.23) and (3.24),

$$\left(L^{(1,q';-\alpha)}\right)'(0,1)=L^{\infty,q;\alpha-1}(0,1).$$

Thus,

$$\widetilde{X}_1(0,1)=L^{\infty,q;\alpha-1}(0,1).$$

By making use of Theorem 7.6 combined with Remark 7.2, we obtain that

$$\widetilde{X}_m(0,1)=L^{\infty,q;\alpha-m}(0,1).$$

The conclusion is now a consequence of Theorem 7.11. $\square$

12. List of symbols

Function spaces

$\mathcal{M}(\Omega,\nu)$ (3.1)
$\mathcal{M}_+(\Omega,\nu)$ (3.2)
$\mathcal{M}_0(\Omega,\nu)$ (3.3)
$X(\Omega,\nu)$ (3.12)
$X_{\mathrm{loc}}(\Omega,\nu)$ (3.13)
$L^p\log^\alpha L(\Omega,\nu)$ below (3.27)

$\exp L^\beta(\Omega,\nu)$ below (3.27)
$\exp\exp L^\beta(\Omega,\nu)$ below (3.27)
$V_\perp^m X(\Omega,\nu)$ (4.15)

Function norms

$\|\cdot\|_{X(0,1)}$ (3.8)
$\|\cdot\|_{X'(0,1)}$ (3.9)
$\|\cdot\|_{X'_d(0,1)}$ (3.11)
$\|\cdot\|_{L^{p,q}(0,1)}$ (3.17)
$\|\cdot\|_{L^{(p,q)}(0,1)}$ (3.17)
$\|\cdot\|_{L^{p,q;\alpha}(0,1)}$ (3.20)
$\|\cdot\|_{L^{(p,q;\alpha)}(0,1)}$ (3.20)
$\|\cdot\|_{L^{p,q;\alpha,\beta}(0,1)}$ (3.22)
$\|\cdot\|_{L^A(0,1)}$ (3.26)
$\|\cdot\|_{L(p,q,D)(0,1)}$ (3.28)
$\|\cdot\|_{V^m X(\Omega,\nu)}$ (4.14)
$\|\cdot\|_{W^m X(\Omega,\nu)}$ (4.22)
$\|\cdot\|_{X_{m,I}(0,1)}$ (5.6)
$\|\cdot\|_{X^\sharp_{m,I}(0,1)}$ (5.16)
$\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ (6.4)
$\|\cdot\|_{X_{m,\alpha}(0,1)}$ (6.12)
$\|\cdot\|_{\widetilde{X}_m(0,1)}$ (7.14)
$\|\cdot\|_{X_{m,\Phi}(0,1)}$ (7.15)
$\|\cdot\|_{X_{m,G}(0,1)}$ (7.18)
$\|\cdot\|_{X_{m,B,\beta}(0,1)}$ (7.21)
$\|\cdot\|_{X_j(0,1)}$ (8.10)

Operators

E_λ (3.15)
H_I (8.2)
H_I^j (8.4)
R_I (8.3)
R_I^j (8.4)
G_I^m (9.3)
P_Φ^m (11.5)
$H_{L_\Phi}^m$ (11.6)

Miscellaneous

ν (2.1)
ω (2.1)
u^* (3.4)

u^{**} (3.5)
$P_\nu(E,\Omega)$ (4.1)
$I_{\Omega,\nu}$ (4.2)
$\mathrm{med}(u)$ (4.11)
$\mathcal{J}_\alpha$ (6.7)
$A_{m,\alpha}$ (6.21)
Φ above (7.1)
μ_Φ (7.1)
c_Φ (7.1)
$\mu_{\Phi,n}$ (7.2)
$d\gamma_n$ (7.4)
$d\gamma_{n,\beta}$ (7.5)
F_Φ (7.7)
L_Φ (7.9)
ω_{n-1} above Proposition 10.1

Acknowledgments

This research was partly supported by the research project of MIUR (Italian Ministry of Education, University and Research) Prin 2008 2008ZPC95C "Geometric aspects of partial differential equations and related topics", by GNAMPA of the Italian INdAM (National Institute of High Mathematics), by the grants 201/08/0383 and P201/13/14743S of the Grant Agency of the Czech Republic, and by the grant SVV-2013-267316.

The authors wish to thank the referee for his valuable comments and suggestions.

References

[1] A. Alvino, V. Ferone, G. Trombetti, Moser-type inequalities in Lorentz spaces, Potential Anal. 5 (1996) 273–299.
[2] T. Aubin, Problèmes isopérimetriques et espaces de Sobolev, J. Differential Geom. 11 (1976) 573–598.
[3] F. Barthe, P. Cattiaux, C. Roberto, Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry, Rev. Mat. Iberoam. 22 (2006) 993–1067.
[4] F. Barthe, P. Cattiaux, C. Roberto, Isoperimetry between exponential and Gaussian, Electron. J. Probab. 12 (2007) 1212–1237.
[5] T. Bartsch, T. Weth, M. Willem, A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator, Calc. Var. Partial Differential Equations 18 (2003) 253–268.
[6] C. Bennett, K. Rudnick, On Lorentz–Zygmund spaces, Dissertationes Math. 175 (1980) 1–72.
[7] C. Bennett, R. Sharpley, Interpolation of Operators, Pure and Applied Mathematics, vol. 129, Academic Press, Boston, 1988.
[8] S.G. Bobkov, C. Houdré, Isoperimetric constants for product probability measures, Ann. Probab. 25 (1997) 184–205.
[9] S.G. Bobkov, C. Houdré, Some Connections Between Isoperimetric and Sobolev-Type Inequalities, Mem. Amer. Math. Soc., vol. 129, 1997.
[10] S.G. Bobkov, M. Ledoux, From Brunn–Minkowski to sharp Sobolev inequalities, Ann. Mat. Pura Appl. 187 (2008) 384–389.

[11] B. Bojarski, Remarks on Sobolev imbedding inequalities, in: Proc. Conference on Complex Analysis, Joensuu 1987, in: Lecture Notes in Math., vol. 1351, Springer, 1988.
[12] C. Borell, The Brunn–Minkowski inequality in Gauss space, Invent. Math. 30 (1975) 207–216.
[13] H. Brézis, E. Lieb, Sobolev inequalities with remainder terms, J. Funct. Anal. 62 (1985) 73–86.
[14] S. Buckley, P. Koskela, Sobolev–Poincaré implies John, Math. Res. Lett. 2 (1995) 577–593.
[15] S. Buckley, P. Koskela, Criteria for embeddings of Sobolev–Poincaré type, Int. Math. Res. Not. 18 (1996) 881–902.
[16] Yu.D. Burago, V.A. Zalgaller, Geometric Inequalities, Springer, Berlin, 1988.
[17] A. Cañete, M. Miranda Jr., D. Vittone, Some isoperimetric problems in planes with density, J. Geom. Anal. 20 (2010) 243–290.
[18] L. Capogna, D. Danielli, S.D. Pauls, J.T. Tyson, An Introduction to the Heisenberg Group and the sub-Riemannian Isoperimetric Problem, Birkhäuser, Basel, 2007.
[19] E.A. Carlen, C. Kerce, On the cases of equality in Bobkov's inequality and Gaussian rearrangement, Calc. Var. Partial Differential Equations 13 (2001) 1–18.
[20] I. Chavel, Isoperimetric Inequalities: Differential Geometric Aspects and Analytic Perspectives, Cambridge University Press, Cambridge, 2001.
[21] J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian, in: Problems in Analysis, Princeton Univ. Press, Princeton, 1970, pp. 195–199.
[22] A. Cianchi, On relative isoperimetric inequalities in the plane, Boll. Un. Mat. Ital. 3-B (1989) 289–326.
[23] A. Cianchi, A sharp embedding theorem for Orlicz–Sobolev spaces, Indiana Univ. Math. J. 45 (1996) 39–65.
[24] A. Cianchi, Second-order derivatives and rearrangements, Duke Math. J. 105 (2000) 355–385.
[25] A. Cianchi, Optimal Orlicz–Sobolev embeddings, Rev. Mat. Iberoam. 20 (2004) 427–474.
[26] A. Cianchi, Symmetrization and second-order Sobolev inequalities, Ann. Mat. Pura Appl. 183 (2004) 45–77.
[27] A. Cianchi, Higher-order Sobolev and Poincaré inequalities in Orlicz spaces, Forum Math. 18 (2006) 745–767.
[28] A. Cianchi, Orlicz–Sobolev boundary trace embeddings, Math. Z. 266 (2010) 431–449.
[29] A. Cianchi, D.E. Edmunds, P. Gurka, On weighted Poincaré inequalities, Math. Nachr. 180 (1996) 15–41.
[30] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, The sharp Sobolev inequality in quantitative form, J. Eur. Math. Soc. 11 (2009) 1105–1139.
[31] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, On the isoperimetric deficit in the Gauss space, Amer. J. Math. 133 (2011) 131–186.
[32] A. Cianchi, L. Pick, Optimal Gaussian Sobolev embeddings, J. Funct. Anal. 256 (2009) 3588–3642.
[33] D. Cordero-Erausquin, B. Nazaret, C. Villani, A mass-transportation approach to sharp Sobolev and Gagliardo–Nirenberg inequalities, Adv. Math. 182 (2004) 307–332.
[34] E. De Giorgi, Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita, Atti. Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. 5 (1958) 33–44.
[35] A. Diaz, N. Harman, S. Howe, D. Thompson, Isoperimetric problems in sectors with density, Adv. Geom. 12 (2012) 589–619.
[36] D.E. Edmunds, W.D. Evans, Spectral Theory and Differential Operators, Clarendon Press, Oxford, 1987.
[37] D.E. Edmunds, R. Kerman, L. Pick, Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms, J. Funct. Anal. 170 (2000) 307–355.
[38] D.E. Edmunds, V. Kokilashvili, A. Meskhi, Bounded and Compact Integral Operators, Kluwer Academic Publishers, Dordrecht, 2002.
[39] L. Esposito, V. Ferone, B. Kawohl, C. Nitsch, C. Trombetti, The longest shortest fence and sharp Poincaré–Sobolev inequalities, Arch. Ration. Mech. Anal. 206 (2012) 821–851.
[40] W.D. Evans, B. Opic, L. Pick, Interpolation of integral operators on scales of generalized Lorentz–Zygmund spaces, Math. Nachr. 182 (1996) 127–181.
[41] H. Federer, W. Fleming, Normal and integral currents, Ann. of Math. 72 (1960) 458–520.
[42] A. Figalli, F. Maggi, On the isoperimetric problem for radial log-convex densities, Calc. Var. Partial Differential Equations 48 (2013) 447–489.
[43] E. Gagliardo, Proprietà di alcune classi di funzioni di più variabili, Ric. Mat. 7 (1958) 102–137.
[44] F. Gazzola, H.-C. Grunau, Critical dimensions and higher order Sobolev inequalities with remainder terms, NoDEA Nonlinear Differential Equations Appl. 8 (2001) 35–44.

[45] F. Gazzola, H.-C. Grunau, G. Sweers, Polyharmonic Boundary Value Problems. Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains, Lecture Notes in Math., vol. 1991, Springer-Verlag, Berlin, 2010.
[46] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975) 1061–1083.
[47] A. Grygor'yan, Isoperimetric inequalities and capacities on Riemannian manifolds, in: The Maz'ya Anniversary Collection, vol. 1, Operator Theory, in: Advances and Applications, vol. 110, Birkhäuser, Basel, 1999, pp. 139–153.
[48] P. Haiłasz, P. Koskela, Isoperimetric inequalities and imbedding theorems in irregular domains, J. Lond. Math. Soc. 58 (1998) 425–450.
[49] E. Hebey, Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lecture Notes in Mathematics, vol. 5, AMS, Providence, 1999.
[50] J.A. Hempel, G.R. Morris, N.S. Trudinger, On the sharpness of a limiting case of the Sobolev imbedding theorem, Bull. Aust. Math. Soc. 3 (1970) 369–373.
[51] D. Hoffman, J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds, Comm. Pure Appl. Math. 27 (1974) 715–727.
[52] D. Hoffmann, J. Spruck, A correction to: "Sobolev and isoperimetric inequalities for Riemannian submanifolds", Comm. Pure Appl. Math. 27 (1974) 715–725, Comm. Pure Appl. Math. 28 (1975) 765–766.
[53] R. Kerman, L. Pick, Optimal Sobolev imbeddings, Forum Math. 18 (4) (2006) 535–570.
[54] T. Kilpeläinen, J. Malý, Sobolev inequalities on sets with irregular boundaries, Z. Anal. Anwend. 19 (2000) 369–380.
[55] V.S. Klimov, Imbedding theorems and geometric inequalities, Izv. Akad. Nauk SSSR 40 (1976) 645–671 (in Russian); English translation: Math. USSR Izv. 10 (1976) 615–638.
[56] V.I. Kolyada, Estimates on rearrangements and embedding theorems, Mat. Sb. 136 (1988) 3–23 (in Russian); English translation: Math. USSR Sb. 64 (1989) 1–21.
[57] M. Krbec, B. Opic, L. Pick, J. Rákosník, Some recent results on Hardy type operators in weighted function spaces and related topics, in: Function Spaces, Differential Operators and Nonlinear Analysis, Friedrichroda, 1992, in: Teubner-Texte Math., vol. 133, Teubner, Stuttgart, 1993, pp. 158–184.
[58] D.A. Labutin, Embedding of Sobolev spaces on Hölder domain, Proc. Steklov Inst. Math. 227 (1999) 163–172.
[59] M. Ledoux, The Concentration of Measure Phenomenon, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001.
[60] M. Ledoux, Spectral gap, logarithmic Sobolev constant and geometric bounds, in: Surveys in Differential Geometry, vol. IX, Somerville, MA, 2004, pp. 219–240.
[61] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. II, Rev. Mat. Iberoam. 1 (1985) 45–121.
[62] P.-L. Lions, F. Pacella, M. Tricarico, Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions, Indiana Univ. Math. J. 37 (1988) 301–324.
[63] E. Lutwak, D. Yang, G. Zhang, Sharp affine L_p Sobolev inequalities, J. Differential Geom. 62 (2002) 17–38.
[64] J. Martín, M. Milman, Higher-order symmetrization inequalities and applications, J. Math. Anal. Appl. 330 (2007) 91–113.
[65] J. Martín, M. Milman, Isoperimetry and symmetrization for logarithmic Sobolev inequalities, J. Funct. Anal. 256 (2009) 149–178.
[66] J. Martín, M. Milman, Pointwise symmetrization inequalities for Sobolev functions and applications, Adv. Math. 225 (2010) 121–199.
[67] F. Martín-Reyes, Weights, one-sided operators, singular integrals and ergodic theorems, in: Nonlinear Analysis, Function Spaces and Applications, vol. 5, Prague, 1994, Prometheus, Prague, 1994, pp. 103–137.
[68] V.G. Maz'ya, Classes of regions and imbedding theorems for function spaces, Dokl. Akad. Nauk SSSR 133 (1960) 527–530 (in Russian); English translation: Soviet Math. Dokl. 1 (1960) 882–885.
[69] V.G. Maz'ya, On p-conductivity and theorems on embedding certain functional spaces into a C-space, Dokl. Akad. Nauk SSSR 140 (1961) 299–302 (in Russian); English translation: Soviet Math. Dokl. 3 (1962).
[70] V.G. Maz'ya, Sobolev Spaces with Applications to Elliptic Partial Differential Equations, Springer, Berlin, 2011.
[71] E. Milman, On the role of convexity in functional and isoperimetric inequalities, Proc. Lond. Math. Soc. 99 (2009) 32–66.
[72] J. Moser, A sharp form of an inequality by Trudinger, Indiana Univ. Math. J. 20 (1971) 1077–1092.

[73] L. Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Super. Pisa 13 (1959) 115–162.
[74] R. O'Neil, Convolution operators in $L(p,q)$ spaces, Duke Math. J. 30 (1963) 129–142.
[75] B. Opic, L. Pick, On generalized Lorentz–Zygmund spaces, Math. Inequal. Appl. 2 (1999) 391–467.
[76] J. Peetre, Espaces d' interpolation et théorème de Soboleff, Ann. Inst. Fourier 16 (1966) 279–317.
[77] E. Pelliccia, G. Talenti, A proof of a logarithmic Sobolev inequality, Calc. Var. Partial Differential Equations 1 (1993) 237–242.
[78] S.I. Pohozaev, On the imbedding Sobolev theorem for $pl = n$, Dokl. Conf., Sect. Math. Moscow Power Inst. (1965) 158–170 (in Russian).
[79] C. Rosales, A. Cañete, V. Bayle, F. Morgan, On the isoperimetric problem in Euclidean space with density, Calc. Var. Partial Differential Equations 31 (2008) 27–46.
[80] L. Saloff-Coste, Aspects of Sobolev-Type Inequalities, Cambridge University Press, Cambridge, 2002.
[81] S.L. Sobolev, On some estimates relating to families of functions having derivatives that are square integrable, Dokl. Akad. Nauk SSSR 1 (1936) 267–270 (in Russian).
[82] S.L. Sobolev, On a theorem in functional analysis, Sb. Math. 4 (1938) 471–497 (in Russian); English translation: Amer. Math. Soc. Trans. 34 (1963) 39–68.
[83] V.D. Stepanov, Weighted norm inequalities for integral operators and related topics, in: Nonlinear Analysis, Function Spaces and Applications, vol. 5, Prague, 1994, Prometheus, Prague, 1994, pp. 139–175.
[84] R.S. Strichartz, A note on Trudinger's extension of Sobolev's inequality, Indiana Univ. Math. J. 21 (1972) 841–842.
[85] V.N. Sudakov, B.S. Tsirel'son, Extremal properties of half-spaces for spherically invariant measures, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 41 (1974) 14–24 (in Russian); English translation: J. Soviet. Math. 9 (1978) 9–18.
[86] C.A. Swanson, The best Sobolev constant, Appl. Anal. 47 (1992) 227–239.
[87] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 110 (1976) 353–372.
[88] G. Talenti, Some inequalities of Sobolev type on two-dimensional spheres, in: General Inequalities 5, in: Internat. Schriftenreihe Numer. Math., vol. 80, Oberwolfach, 1986, Birkhäuser, Basel, 1987, pp. 401–408.
[89] N.S. Trudinger, On imbeddings into Orlicz spaces and some applications, J. Mech. Anal. 17 (1967) 473–483.
[90] V.I. Yudovich, Some estimates connected with integral operators and with solutions of elliptic equations, Soviet Math. Dokl. 2 (1961) 746–749 (in Russian).
[91] G. Zhang, The affine Sobolev inequality, J. Differential Geom. 53 (1999) 183–202.
[92] W.P. Ziemer, Weakly Differentiable Functions, Graduate Texts in Math., vol. 120, Springer, Berlin, 1989.

Paper II

A. Cianchi, L. Pick, and L. Slavíková. Banach algebras of weakly differentiable functions. Preprint, 2015.

BANACH ALGEBRAS OF WEAKLY DIFFERENTIABLE FUNCTIONS

ANDREA CIANCHI, LUBOŠ PICK AND LENKA SLAVÍKOVÁ

Abstract. The question is addressed of when a Sobolev type space, built upon a general rearrangement-invariant norm, on an n-dimensional domain, is a Banach algebra under pointwise multiplication of functions. A sharp balance condition among the order of the Sobolev space, the strength of the norm, and the (ir)regularity of the domain is provided for the relevant Sobolev space to be a Banach algebra. The regularity of the domain is described in terms of its isoperimetric function. Related results on the boundedness of the multiplication operator into lower-order Sobolev type spaces are also established. The special cases of Orlicz-Sobolev and Lorentz-Sobolev spaces are discussed in detail. New results for classical Sobolev spaces on possibly irregular domains follow as well.

1. Introduction and main results

The Sobolev space $W^{m,p}(\Omega)$ of those functions in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, whose weak derivatives up to the order m belong to $L^p(\Omega)$, is classically well known to be a Banach space for every $m \in \mathbb{N}$ and $p \in [1, \infty]$. In particular, the sum of any two functions from $W^{m,p}(\Omega)$ always still belongs to $W^{m,p}(\Omega)$. The situation is quite different if the operation of sum is replaced by product. In fact, membership of functions to a Sobolev space need not be preserved under multiplication. Hence, $W^{m,p}(\Omega)$ is not a Banach algebra in general. A standard result in the theory of Sobolev spaces tells us that if Ω is regular, say a bounded domain with the cone property, then $W^{m,p}(\Omega)$ is indeed a Banach algebra if and only if either $p > 1$ and $pm > n$, or $p = 1$ and $m \geq n$. Recall that this amounts to the existence of a constant C such that

$$\|uv\|_{W^mX(\Omega)} \leq C\|u\|_{W^mX(\Omega)}\|v\|_{W^mX(\Omega)} \tag{1.1}$$

for every $u, v \in W^mX(\Omega)$. We refer to Section 6.1 of the monograph [48] for this result, where a comprehensive updated treatment of properties of Sobolev functions under product can be found. See also [1, Theorem 5.23] for a proof of the sufficiency part of the result.

In the present paper abandon this classical setting, and address the question of the validity of an inequality of the form (1.1) in a much more general framework. Assume that Ω is just a domain in $\mathbb{R}^n$, namely an open connected set, with finite Lebesgue measure $|\Omega|$, which, without loss of generality, will be assumed to be equal to 1. Moreover, suppose that $L^p(\Omega)$ is replaced with an arbitrary rearrangement-invariant space $X(\Omega)$, loosely speaking, a Banach space of measurable functions endowed with a norm depending only on the measure of level sets of functions. We refer to the next section for precise definitions concerning function spaces. Let us just recall here that, besides Lebesgue spaces, Lorentz and Orlicz spaces are classical instances of rearrangement-invariant spaces.

2000 *Mathematics Subject Classification.* 46E35, 46E30.

Key words and phrases. Sobolev spaces, Banach algebras, rearrangement-invariant spaces, isoperimetric inequalities, Orlicz spaces, Lorentz spaces.

This research was partly supported by the Research Project of Italian Ministry of University and Research (MIUR) "Elliptic and parabolic partial differential equations: geometric aspects, related inequalities, and applications" 2012, by GNAMPA of Italian INdAM (National Institute of High Mathematics), and by the grant P201-13-14743S of the Grant Agency of the Czech Republic.

Given any $m \in \mathbb{N}$ and any rearrangement-invariant space $X(\Omega)$, consider the m-th order Sobolev type space $\mathcal{V}^m X(\Omega)$ built upon $X(\Omega)$, and defined as the collection of all m times weakly differentiable functions $u : \Omega \to \mathbb{R}$ such that $|\nabla^m u| \in X(\Omega)$. Here, $\nabla^m u$ denotes the vector of all m-th order weak derivatives of u, and $|\nabla^m u|$ stands for its length. For notational convenience, we also set $\nabla^0 u = u$ and $\mathcal{V}^0 X(\Omega) = X(\Omega)$. Given any fixed ball $B \subset \Omega$, we define the functional $\|\cdot\|_{\mathcal{V}^m X(\Omega)}$ by

$$\|u\|_{\mathcal{V}^m X(\Omega)} = \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(B)} + \|\nabla^m u\|_{X(\Omega)} \tag{1.2}$$

for $u \in \mathcal{V}^m X(\Omega)$. Observe that in the definition of $\mathcal{V}^m X(\Omega)$ it is only required that the derivatives of the highest order m of u belong to $X(\Omega)$. This assumption does not ensure, for an arbitrary domain Ω, that also u and its derivatives up to the order $m-1$ belong to $X(\Omega)$, or even to $L^1(\Omega)$. However, owing to a standard Poincaré inequality, if $u \in \mathcal{V}^m X(\Omega)$, then $|\nabla^k u| \in L^1(B)$ for $k = 0, \dots, m-1$, for every ball $B \subset \Omega$. It follows that the functional $\|\cdot\|_{\mathcal{V}^m X(\Omega)}$ is a norm on $\mathcal{V}^m X(\Omega)$. Furthermore, a standard argument shows that $\mathcal{V}^m X(\Omega)$ is a Banach space equipped with this norm, which results in equivalent norms under replacements of B with other balls.

We shall exhibit minimal conditions on m, Ω and $\|\cdot\|_{X(\Omega)}$ for $\mathcal{V}^m X(\Omega)$ to be a Banach algebra under pointwise multiplication of functions, namely for an inequality of the form

$$\|uv\|_{\mathcal{V}^m X(\Omega)} \leq C \|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)}$$

to hold for some constant C and every $u, v \in \mathcal{V}^m X(\Omega)$. Variants of this inequality, where $\mathcal{V}^m X(\Omega)$ is replaced by a lower-order Sobolev space on the left-hand side, are also dealt with.

In our discussion, we neither a priori assume any regularity on Ω, nor we assume that $X(\Omega)$ is a Lebesgue space (or any other specific space). We shall exhibit a balance condition between the degree of regularity of Ω, the order of differentiation m, and the strength of the norm in $X(\Omega)$ ensuring that $\mathcal{V}^m X(\Omega)$ be a Banach algebra. The dependence on $X(\Omega)$ is only through the representation norm $\|\cdot\|_{X(0,1)}$ of $\|\cdot\|_{X(\Omega)}$. In particular, the associate norm $\|\cdot\|_{X'(0,1)}$ of $\|\cdot\|_{X(0,1)}$, a kind of measure theoretic dual norm of $\|\cdot\|_{X(0,1)}$, will be relevant.
As for our assumptions on the domain Ω, a key role in their formulation will be played by the relative isoperimetric inequality. Let us recall that the discovery of the link between isoperimetric inequalities and Sobolev type inequalities can be traced back to the work of Maz'ya on one hand ([45, 46]), who proved the equivalence of general Sobolev inequalities to either isoperimetric or isocapacitary inequalities, and that of Federer and Fleming on the other hand ([29]) who used the standard isoperimetric inequality by De Giorgi ([26]) to exhibit the best constant in the Sobolev inequality for $W^{1,1}(\mathbb{R}^n)$. The detection of optimal constants in classical Sobolev inequalities continued in the contributions [50], [55], [4], where crucial use of De Giorgi's isoperimetric inequality was again made. An extensive research followed, along diverse directions, on the interplay between isoperimetric and Sobolev inequalities. We just mention the papers [2, 5, 6, 9, 10, 11, 12, 15, 17, 18, 19, 22, 24, 27, 28, 30, 32, 33, 35, 36, 37, 38, 40, 41, 42, 43, 49, 54, 56] and the monographs [13, 14, 16, 31, 34, 47, 53].

Before stating our most general result, let us focus on the situation when m and $X(\Omega)$ are arbitrary, but Ω is still, in a sense, a best possible domain. This is the case when Ω is a John domain. Recall that a bounded open set Ω in $\mathbb{R}^n$ is called a *John domain* if there exist a constant $c \in (0,1)$, an $l \in (0,\infty)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parameterized by arclength, such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \geq cr \qquad \text{for } r \in [0,l].$$

Lipschitz domains, and domains with the cone property are customary instances of John domains.

When Ω is any John domain, a necessary and sufficient condition for $\mathcal{V}^m X(\Omega)$ to be a Banach algebra is provided by the following result.

Theorem 1.1. *Let $m, n \in \mathbb{N}$, $n \geq 2$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then $\mathcal{V}^m X(\Omega)$ is a Banach algebra if and only if*

$$\|r^{-1+\frac{m}{n}}\|_{X'(0,1)} < \infty. \tag{1.3}$$

As a consequence of Theorem 1.1, and of the characterization of Sobolev embeddings into $L^\infty(\Omega)$, we have the following corollary.

Corollary 1.2. *Let $m, n \in \mathbb{N}$, $n \geq 2$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the Sobolev space $\mathcal{V}^m X(\Omega)$ is a Banach algebra if and only if $\mathcal{V}^m X(\Omega) \to L^\infty(\Omega)$.*

Let us now turn to the general case. Regularity on Ω will be imposed in terms of its isoperimetric function $I_\Omega : [0,1] \to [0,\infty]$, introduced in [45], and given by

$$I_\Omega(s) = \inf\left\{P(E,\Omega) : E \subset \Omega, s \leq |E| \leq \tfrac{1}{2}\right\} \quad \text{if } s \in [0, \tfrac{1}{2}], \tag{1.4}$$

and $I_\Omega(s) = I_\Omega(1-s)$ if $s \in (\frac{1}{2}, 1]$. Here, $P(E,\Omega)$ denotes the perimeter of a measurable set E relative to Ω, which agrees with $\mathcal{H}^{n-1}(\Omega \cap \partial^M E)$, where $\partial^M E$ denotes the essential boundary of E, in the sense of geometric measure theory, and $\mathcal{H}^{n-1}$ stands for $(n-1)$-dimensional Hausdorff measure. The very definition of I_Ω implies the relative isoperimetric inequality in Ω, which tells us that

$$P(E,\Omega) \geq I_\Omega(|E|), \tag{1.5}$$

for every measurable set $E \subset \Omega$. In other words, I_Ω is the largest non-decreasing function in $[0, \frac{1}{2}]$, symmetric about $\frac{1}{2}$, which renders (1.5) true.

The degree of regularity of Ω can be described in terms of the rate of decay of $I_\Omega(s)$ to 0 as $s \to 0$. Heuristically speaking, the faster I_Ω decays to 0, the less regular Ω is. For instance, the isoperimetric function I_Ω of any John domain $\Omega \subset \mathbb{R}^n$ is known to satisfy

$$I_\Omega(s) \approx s^{\frac{1}{n'}} \tag{1.6}$$

near 0, where $n' = \frac{n}{n-1}$. Here, and in what follows, the notation $f \approx g$ mans that the real-valued functions f and g are equivalent, in the sense that there exist positive constants c, C such that $cf(c\cdot) \leq g(\cdot) \leq Cf(C\cdot)$. Notice that (1.6) is the best (i.e. slowest) possible decay of I_Ω, since, if Ω is any domain, then

$$\frac{I_\Omega(s)}{s^{\frac{1}{n'}}} \leq C \quad \text{for } s \in (0,1], \tag{1.7}$$

for some constant C [25, Proposition 4.1].

What enters in our characterization of Sobolev algebras is, in fact, just a lower bound for I_Ω. We shall thus work with classes of domains whose isoperimetric function admits a lower bound in terms of some non-decreasing function $I : (0,1) \to (0,\infty)$. The function I will be continued by continuity at 0 when needed. Given any such function I, we denote by $\mathcal{J}_I$ the collection of all domains $\Omega \subset \mathbb{R}^n$ such that

$$I_\Omega(s) \geq cI(cs) \quad \text{for} \quad s \in (0, \tfrac{1}{2}], \tag{1.8}$$

for some constant $c > 0$. The assumption that $I(t) > 0$ for $t \in (0,1)$ is consistent with the fact that $I_\Omega(t) > 0$ for $t \in (0,1)$, owing to the connectedness of Ω [47, Lemma 5.2.4].

In particular, if $I(s) = s^\alpha$ for $s \in (0,1)$, for some $\alpha \in [\frac{1}{n'}, \infty)$, we denote $\mathcal{J}_I$ simply by $\mathcal{J}_\alpha$, and call it a *Maz'ya class.* Thus, a domain $\Omega \in \mathcal{J}_\alpha$ if there exists a positive constant C such that

$$I_\Omega(s) \geq C s^\alpha \quad \text{for every } s \in (0, \tfrac{1}{2}].$$

Observe that, thanks to (1.6), any John domain belongs to the class $\mathcal{J}_{\frac{1}{n'}}$.

Our most general result about Banach algebras of Sobolev spaces is stated in the next theorem. Let us emphasize that, if Ω is not a regular domain, it brings new information even in the standard case when $X(\Omega) = L^p(\Omega)$.

Theorem 1.3. *Assume that $m, n \in \mathbb{N}$, $n \geq 2$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$. If $\Omega \in \mathcal{J}_I$ and*

$$\left\| \frac{1}{I(t)} \left(\int_0^t \frac{ds}{I(s)} \right)^{m-1} \right\|_{X'(0,1)} < \infty, \tag{1.9}$$

then

$$\mathcal{V}^m X(\Omega) \quad \text{is a Banach algebra,}$$

or, equivalently, there exists a constant C such that

$$\|uv\|_{\mathcal{V}^m X(\Omega)} \leq C \|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)} \tag{1.10}$$

for every $u, v \in \mathcal{V}^m X(\Omega)$.
Conversely, if, in addition,

$$\frac{I(t)}{t^{\frac{1}{n'}}} \quad \text{is equivalent to a non-decreasing function on } (0,1), \tag{1.11}$$

then (1.9) is sharp, in the sense that if $\mathcal{V}^m X(\Omega)$ is a Banach algebra for every $\Omega \in \mathcal{J}_I$, then (1.9) holds.

Remark 1.4. Assumption (1.11) is not restrictive in view of (1.7), and can just be regarded as a qualification of the latter.

Remark 1.5. If condition (1.9) holds for some $X(0,1)$ and m, then necessarily

$$\int_0 \frac{ds}{I(s)} < \infty.$$

This is obvious for $m \geq 2$, whereas it follows from (2.5) below for $m = 1$.

An analogue of Corollary 1.2 is provided by the following statement.

Corollary 1.6. *Let $m, n \in \mathbb{N}$, $n \geq 2$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$ satisfying (1.11). Then the Sobolev space $\mathcal{V}^m X(\Omega)$ is a Banach algebra for all domains Ω satisfying (1.8) if and only if $\mathcal{V}^m X(\Omega) \to L^\infty(\Omega)$ for every $\Omega \in \mathcal{J}_I$.*

The next corollary of Theorem 1.3 tells us that $\mathcal{V}^m X(\Omega)$ is always a Banach algebra, whatever $X(\Omega)$ is, provided that I_Ω is sufficiently well behaved near 0, depending on m.

Corollary 1.7. *Let $m, n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_I$ for some positive non-decreasing function I on $(0,1)$. Suppose that*

$$\limsup_{t \to 0_+} \frac{1}{I(t)} \left(\int_0^t \frac{dr}{I(r)} \right)^{m-1} < \infty. \tag{1.12}$$

Then the Sobolev space $\mathcal{V}^m X(\Omega)$ is a Banach algebra for every rearrangement-invariant space $X(\Omega)$.

Remark 1.8. Besides $\mathcal{V}^m X(\Omega)$, one can consider the m-th order Sobolev type space $V^m X(\Omega)$ of those functions u such that the norm

$$\|u\|_{V^m X(\Omega)} = \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(\Omega)} + \|\nabla^m u\|_{X(\Omega)} \tag{1.13}$$

is finite, and the space $W^m X(\Omega)$ of those functions whose norm

$$\|u\|_{W^m X(\Omega)} = \sum_{k=0}^{m} \|\nabla^k u\|_{X(\Omega)}$$

is finite. Both $V^m X(\Omega)$ and $W^m X(\Omega)$ are Banach spaces. Since any rearrangement-invariant space is embedded into $L^1(\Omega)$, one has that

$$W^m X(\Omega) \to V^m X(\Omega) \to \mathcal{V}^m X(\Omega), \tag{1.14}$$

and the inclusions are strict, as noticed above, unless Ω satisfies some additional regularity assumption. In particular, if

$$\inf_{t\in(0,1)} \frac{I_\Omega(t)}{t} > 0, \tag{1.15}$$

then

$$V^m X(\Omega) = \mathcal{V}^m X(\Omega), \tag{1.16}$$

with equivalent norms, as a consequence of [47, Theorem 5.2.3] and of the closed graph theorem. If assumption (1.15) is strengthened to

$$\int_0 \frac{ds}{I_\Omega(s)} < \infty, \tag{1.17}$$

then, in fact,

$$W^m X(\Omega) = V^m X(\Omega) = \mathcal{V}^m X(\Omega), \tag{1.18}$$

up to equivalent norms [25, Proposition 4.5].
If $\Omega \in \mathcal{J}_I$ for some I fulfilling (1.9), then, by Remark 1.5, condition (1.17) is certainly satisfied. Thus, by the first part of Theorem 1.3, assumption (1.9) is sufficient also for $V^m X(\Omega)$ and $W^m X(\Omega)$ to be Banach algebras. Under (1.11), such assumption is also necessary, provided that the function I is a priori known to fulfil either

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0, \tag{1.19}$$

or

$$\int_0 \frac{ds}{I(s)} < \infty, \tag{1.20}$$

according to whether $V^m X(\Omega)$ or $W^m X(\Omega)$ is in question. Let us emphasize that these additional assumptions cannot be dispensed with in general. For instance, it is easily seen that $W^m L^\infty(\Omega)$ is always a Banach algebra, whatever m and Ω are.

So far we have analyzed the question of whether $\mathcal{V}^m X(\Omega)$ is a Banach algebra, namely of the validity of inequality (1.10). We now focus on inequalities in the spirit of (1.10), where the space $\mathcal{V}^m X(\Omega)$ is replaced with a lower-order Sobolev space $\mathcal{V}^{m-k} X(\Omega)$ on the left-hand side.

The statement of our result in this connection requires the notion of the fundamental function $\varphi_X : [0,1] \to [0,\infty)$ of a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$. Recall that

$$\varphi_X(t) = \|\chi_{(0,t)}\|_{X(0,1)} \quad \text{for } t \in (0,1]. \tag{1.21}$$

Theorem 1.9. *Let m, n, $k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq m$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$. If $\Omega \in \mathcal{J}_I$ and*

$$\sup_{t\in(0,1)} \frac{1}{\varphi_X(t)} \left(\int_0^t \frac{ds}{I(s)}\right)^{m+k} < \infty, \tag{1.22}$$

then

$$\|uv\|_{\mathcal{V}^{m-k}X(\Omega)} \leq C\|u\|_{\mathcal{V}^m X(\Omega)}\|v\|_{\mathcal{V}^m X(\Omega)} \tag{1.23}$$

for every $u, v \in \mathcal{V}^m X(\Omega)$.

Conversely, if, in addition, (1.11) *is fulfilled, then* (1.22) *is sharp, in the sense that if* (1.23) *is satisfied for all domains $\Omega \in \mathcal{J}_I$, then* (1.22) *holds.*

Remark 1.10. Considerations on the replacement of the space $\mathcal{V}^m X(\Omega)$ with either the space $V^m X(\Omega)$, or $W^m X(\Omega)$ in Theorem 1.9 can be made, which are analogous to those of Remark 1.8 about Theorem 1.3.

In the borderline case when $k = 0$, condition (1.22) is (essentially) weaker than (1.9) - see Proposition 3.3 and Remark 3.4, Section 3. The next result asserts that it "almost" implies inequality (1.10), in that it yields an inequality of that form, with the borderline terms $u\nabla^m v$ and $v\nabla^m u$ missing in the Leibniz formula for the m-th order derivative of the product uv.

Theorem 1.11. *Let $m, n \in \mathbb{N}$, $m, n \geq 2$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$. If $\Omega \in \mathcal{J}_I$ and*

$$\sup_{t\in(0,1)} \frac{1}{\varphi_X(t)} \left(\int_0^t \frac{ds}{I(s)}\right)^{m} < \infty\,, \tag{1.24}$$

then there exists a positive constant C such that

$$\sum_{k=1}^{m-1} \||\nabla^k u||\nabla^{m-k} v|\|_{X(\Omega)} \leq C\|u\|_{\mathcal{V}^m X(\Omega)}\|v\|_{\mathcal{V}^m X(\Omega)} \tag{1.25}$$

for every $u, v \in \mathcal{V}^m X(\Omega)$.
Conversely, if, in addition, (1.11) *is fulfilled, then* (1.24) *is sharp, in the sense that if* (1.25) *is satisfied for all domains $\Omega \in \mathcal{J}_I$, then* (1.24) *holds.*

Theorem 1.3 enables us, for instance, to characterize the Lorentz–Zygmund-Sobolev spaces and Orlicz-Sobolev spaces which are Banach algebras for all domains $\Omega \in \mathcal{J}_\alpha$.
Let us first focus on the Lorentz–Zygmund-Sobolev spaces $\mathcal{V}^m L^{p,q;\beta}(\Omega)$. Recall (see e.g. [52, Theorem 9.10.4] or [51, Theorem 7.4]) that a necessary and sufficient condition for $L^{p,q;\beta}(\Omega)$ to be a rearrangement-invariant space is that the parameters p, q, β satisfy either of the following conditions:

$$\begin{cases} 1 < p < \infty, \ 1 \leq q \leq \infty, \ \beta \in \mathbb{R}; \\ p = 1, \ q = 1, \ \beta \geq 0; \\ p = \infty, \ q = \infty, \ \beta \leq 0; \\ p = \infty, \ 1 \leq q < \infty, \ \beta + \frac{1}{q} < 0. \end{cases} \tag{1.26}$$

Proposition 1.12. *Let $m, n \in \mathbb{N}$, $n \geq 2$, and let $\alpha \in [\frac{1}{n'}, \infty)$. Assume that $1 \leq p, q \leq \infty$, $\beta \in \mathbb{R}$ and one of the conditions in* (1.26) *is in force. Then $\mathcal{V}^m L^{p,q;\beta}(\Omega)$ is a Banach algebra for every domain $\Omega \in \mathcal{J}_\alpha$ if and only if $\alpha < 1$ and either of the following conditions is satisfied:*

$$\begin{cases} m(1-\alpha) > \frac{1}{p}, \\ m(1-\alpha) = \frac{1}{p}, \ q = 1, \ \beta \geq 0, \\ m(1-\alpha) = \frac{1}{p}, \ q > 1, \ \beta > \frac{1}{q'}. \end{cases} \tag{1.27}$$

The Orlicz-Sobolev spaces $\mathcal{V}^m L^A(\Omega)$ are the object of the next result. In particular, it recovers a result from [21], dealing with the case of regular domains.

Proposition 1.13. *Let $m \in \mathbb{N}$, $n \geq 2$, and $\alpha \in [\frac{1}{n'}, \infty)$. Let A be a Young function. Then $\mathcal{V}^m L^A(\Omega)$ is a Banach algebra for every domain $\Omega \in \mathcal{J}_\alpha$ if and only if $\alpha < 1$ and either of the following conditions is satisfied:*

$$\begin{cases} m \geq \frac{1}{1-\alpha}, \\ m < \frac{1}{1-\alpha} \quad \text{and} \quad \int^\infty \left(\frac{t}{A(t)}\right)^{\frac{(1-\alpha)m}{1-(1-\alpha)m}} dt < \infty. \end{cases} \tag{1.28}$$

The Lorentz–Zygmund-Sobolev and Orlicz-Sobolev spaces for which the product operator is bounded into a lower-order space for every $\Omega \in \mathcal{J}_\alpha$ can be characterized via Theorem 1.9.

Proposition 1.14. *Let $n, m, k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq m$, and let $\alpha \in [\frac{1}{n'}, \infty)$. Assume that $1 \leq p, q \leq \infty$, $\beta \in \mathbb{R}$ and one of the conditions in* (1.26) *is in force. Then, for every domain $\Omega \in \mathcal{J}_\alpha$ there exists a constant C such that*

$$\|uv\|_{\mathcal{V}^{m-k}L^{p,q;\beta}(\Omega)} \leq C\|u\|_{\mathcal{V}^m L^{p,q;\beta}(\Omega)}\|v\|_{\mathcal{V}^m L^{p,q;\beta}(\Omega)}$$

for every $u, v \in \mathcal{V}^m L^{p,q;\beta}(\Omega)$ if and only if $\alpha < 1$, and either of the following conditions is satisfied:

$$\begin{cases} (m+k)(1-\alpha) > \frac{1}{p}, \\ (m+k)(1-\alpha) = \frac{1}{p}, \ \beta \geq 0. \end{cases} \tag{1.29}$$

Proposition 1.15. *Let $n, m, k \in \mathbb{N}$, $n \geq 2$, $1 \leq k \leq m$, and let $\alpha \in [\frac{1}{n'}, \infty)$. Let A be a Young function. Then, for every domain $\Omega \in \mathcal{J}_\alpha$ there exists a constant C such that*

$$\|uv\|_{\mathcal{V}^{m-k}L^A(\Omega)} \leq C\|u\|_{\mathcal{V}^m L^A(\Omega)}\|v\|_{\mathcal{V}^m L^A(\Omega)}$$

for every $u, v \in \mathcal{V}^m L^{p,q;\beta}(\Omega)$ if and only if $\alpha < 1$ and either of the following conditions is satisfied:

$$\begin{cases} (m+k) \geq \frac{1}{1-\alpha}, \\ (m+k) < \frac{1}{1-\alpha} \quad \text{and} \quad A(t) \geq Ct^{\frac{1}{(1-\alpha)(m+k)}} \quad \text{for large } t, \end{cases} \tag{1.30}$$

for some positive constant C.

2. Background

We denote by $\mathcal{M}(\Omega)$ the set of all Lebesgue measurable functions from Ω into $[-\infty, \infty]$. We also define $\mathcal{M}_+(\Omega) = \{u \in \mathcal{M}(\Omega) : u \geq 0\}$, and $\mathcal{M}_0(\Omega) = \{u \in \mathcal{M}(\Omega) : u \text{ is finite a.e. in } \Omega\}$.

The *decreasing rearrangement* $u^* : (0,1) \to [0, \infty]$ of a function $u \in \mathcal{M}(\Omega)$ is defined as

$$u^*(s) = \sup\{t \in \mathbb{R} : |\{x \in \Omega : |u(x)| > t\}| > s\} \quad \text{for } s \in (0,1).$$

We also define $u^{**}:(0,1)\to[0,\infty]$ as

$$u^{**}(s)=\frac{1}{s}\int_0^s u^*(r)\,dr.$$

We say that a functional $\|\cdot\|_{X(0,1)}:\mathcal{M}_+(0,1)\to[0,\infty]$ is a *function norm*, if, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}_+(0,1)$, and every $\lambda\geq 0$, the following properties hold:

(P1) $\|f\|_{X(0,1)}=0$ if and only if $f=0$ a.e.; $\|\lambda f\|_{X(0,1)}=\lambda\|f\|_{X(0,1)}$; $\|f+g\|_{X(0,1)}\leq\|f\|_{X(0,1)}+\|g\|_{X(0,1)}$;

(P2) $f\leq g$ a.e. implies $\|f\|_{X(0,1)}\leq\|g\|_{X(0,1)}$;

(P3) $f_j\nearrow f$ a.e. implies $\|f_j\|_{X(0,1)}\nearrow\|f\|_{X(0,1)}$;

(P4) $\|1\|_{X(0,1)}<\infty$;

(P5) $\int_0^1 f(x)\,dx\leq C\|f\|_{X(0,1)}$ for some constant C independent of f.

If, in addition,

(P6) $\|f\|_{X(0,1)}=\|g\|_{X(0,1)}$ whenever $f^*=g^*$,

we say that $\|\cdot\|_{X(0,1)}$ is a *rearrangement-invariant function norm.*

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, the space $X(\Omega)$ is defined as the collection of all functions $u\in\mathcal{M}(\Omega)$ such that the expression

$$\|u\|_{X(\Omega)}=\|u^*\|_{X(0,1)}$$

is finite. Such expression defines a norm on $X(\Omega)$, and the latter is a Banach space endowed with this norm, called a *rearrangement-invariant space*. Moreover, $X(\Omega)\subset\mathcal{M}_0(\Omega)$ for any rearrangement-invariant space $X(\Omega)$.

With any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, it is associated another functional on $\mathcal{M}_+(0,1)$, denoted by $\|\cdot\|_{X'(0,1)}$, and defined, for $g\in\mathcal{M}_+(0,1)$, by

$$\|g\|_{X'(0,1)}=\sup_{\substack{f\in\mathcal{M}_+(0,1)\\ \|f\|_{X(0,1)}\leq 1}}\int_0^1 f(s)g(s)\,ds. \tag{2.1}$$

It turns out that $\|\cdot\|_{X'(0,1)}$ is also an rearrangement invariant function norm, which is called the *associate function norm* of $\|\cdot\|_{X(0,1)}$. The rearrangement invariant space $X'(\Omega)$ built upon the function norm $\|\cdot\|_{X'(0,1)}$ is called the *associate space* of $X(\Omega)$. Given an rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, the *Hölder inequality*

$$\int_\Omega |u(x)v(x)|\,dx\leq\|u\|_{X(\Omega)}\|v\|_{X'(\Omega)} \tag{2.2}$$

holds for every $u\in X(\Omega)$ and $v\in X'(\Omega)$. For every rearrangement-invariant space $X(\Omega)$ the identity $X''(\Omega)=X(\Omega)$ holds and, moreover, for every $f\in\mathcal{M}(\Omega)$, we have that

$$\|u\|_{X(\Omega)}=\sup_{v\in\mathcal{M}(\Omega);\ \|v\|_{X'(\Omega)}\leq 1}\int_0^1|u(x)v(x)|\,dx. \tag{2.3}$$

The fundamental functions of a rearrangement-invariant space $X(\Omega)$ and its associate space $X'(\Omega)$ satisfy

$$\varphi_X(t)\varphi_{X'}(t)=t\quad\text{for every } t\in(0,1). \tag{2.4}$$

Since we are assuming that Ω has finite measure,

$$L^\infty(\Omega)\to X(\Omega)\to L^1(\Omega) \tag{2.5}$$

for every rearrangement-invariant space $X(\Omega)$.
A basic property of rearrangements is the *Hardy-Littlewood inequality* which tells us that, if $u, v \in \mathcal{M}(\Omega)$, then

$$\int_\Omega |u(x)v(x)|dx \leq \int_0^1 u^*(t)v^*(t)dt. \tag{2.6}$$

A key fact concerning rearrangement-invariant function norms is the *Hardy–Littlewood–Pólya principle* which states that if, for some $u, v \in \mathcal{M}(\Omega)$,

$$\int_0^t u^*(s)\,ds \leq \int_0^t v^*(s)\,ds \text{ for } t \in (0,1), \tag{2.7}$$

then

$$\|u\|_{X(\Omega)} \leq \|v\|_{X(\Omega)}$$

for every rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$. Moreover,

$$\|uv\|_{X(\Omega)} \leq \|u^*v^*\|_{X(0,1)} \tag{2.8}$$

for every rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, and all functions $u, v \in \mathcal{M}(\Omega)$. Inequality (2.8) follows from the inequality

$$\int_0^t (uv)^*(s)\,ds \leq \int_0^t u^*(s)v^*(s)\,ds,$$

and the Hardy–Littlewood–Pólya principle.

We refer the reader to [7] for proofs of the results recalled above, and for a comprehensive treatment of rearrangement-invariant spaces.

Let $1 \leq p, q \leq \infty$ and $\beta \in \mathbb{R}$. We define the functional $\|\cdot\|_{L^{p,q;\beta}(0,1)}$ as

$$\|f\|_{L^{p,q;\beta}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^\beta\left(\tfrac{2}{s}\right) f^*(s) \right\|_{L^q(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. If one of the conditions in (1.26) is satisfied, then $\|\cdot\|_{L^{p,q;\beta}(0,1)}$ is equivalent to a rearrangement-invariant function norm, called a *Lorentz–Zygmund norm* (for details see e.g [8], [51] or [52]). The corresponding rearrangement-invariant space $L^{p,q;\beta}(\Omega)$ is called the *Lorentz–Zygmund space*. When $\beta = 0$, the space $L^{p,q;0}(\Omega)$ is denoted by $L^{p,q}(\Omega)$ and called *Lorentz space*. It is known (e.g. [51, Theorem 6.11] or [52, Theorem 9.6.13]) that

$$(L^{p,q;\beta})'(\Omega) = \begin{cases} L^{p',q';-\beta}(\Omega) & \text{if } p < \infty; \\ L^{(1,q';-\beta-1)}(\Omega) & \text{if } p = \infty,\ 1 \leq q < \infty,\ \beta + \frac{1}{q} < 0; \\ L^1(\Omega) & \text{if } p = \infty,\ q = \infty,\ \beta = 0, \end{cases} \tag{2.9}$$

where $L^{(p,q;\beta)}(\Omega)$ denotes the function space defined analogously to $L^{p,q;\beta}(\Omega)$ but with the functional $\|\cdot\|_{L^{(p,q;\beta)}(0,1)}$ given by

$$\|f\|_{L^{(p,q;\beta)}(0,1)} = \left\| s^{\frac{1}{p}-\frac{1}{q}} \log^\beta\left(\tfrac{2}{s}\right) f^{**}(s) \right\|_{L^q(0,1)}$$

for $f \in \mathcal{M}_+(0,1)$. Note that, if $\beta = 0$, then

$$L^{p,q;0}(\Omega) = L^{p,q}(\Omega),$$

a standard Lorentz space. In particular, if $p = q$, then

$$L^{p,p}(\Omega) = L^p(\Omega),$$

a Lebesgue space.

A generalization of the Lebesgue spaces in a different direction is provided by the Orlicz spaces. Let $A : [0,\infty) \to [0,\infty]$ be a Young function, namely a convex (non-trivial), left-continuous function vanishing at 0. The Orlicz space $L^A(\Omega)$ is the rearrangement-invariant space associated with the *Luxemburg function norm* defined as

$$\|f\|_{L^A(0,1)} = \inf\left\{\lambda > 0 : \int_0^1 A\left(\frac{f(s)}{\lambda}\right) ds \le 1\right\}$$

for $f \in \mathcal{M}_+(0,1)$. In particular, $L^A(\Omega) = L^p(\Omega)$ if $A(t) = t^p$ for some $p \in [1,\infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = \infty\chi_{(1,\infty)}(t)$.
The associate function norm of $\|\cdot\|_{L^A(0,1)}$ is equivalent to the function norm $\|\cdot\|_{L^{\tilde{A}}(0,1)}$, where $\widetilde{A}$ is the Young conjugate of A defined as

$$\widetilde{A}(t) = \sup\{ts - A(s) : s \ge 0\} \qquad \text{for } t \ge 0.$$

3. Key one-dimensional inequalities

In this section we shall state and prove two key assertions concerning one-dimensional inequalities involving non-increasing functions and rearrangement-invariant spaces defined on an interval. Both these results are of independent interest and they constitute a new approach to inequalities involving products of functions.

Assume that I is a positive non-decreasing function on $(0,1)$. We denote by H_I the operator defined at every nonnegative measurable function g on $(0,1)$ by

$$H_I g(t) = \int_t^1 \frac{g(r)}{I(r)}\,dr \quad \text{for } t \in (0,1).$$

Moreover, given $m \in \mathbb{N}$, we set

$$H_I^m = \underbrace{H_I \circ H_I \circ \cdots \circ H_I}_{m-\text{times}}.$$

We also denote by H_I^0 the identity operator. It is easily verified that

$$H_I^m g(t) = \frac{1}{(m-1)!}\int_t^1 \frac{g(s)}{I(s)}\left(\int_t^s \frac{dr}{I(r)}\right)^{m-1} ds \quad \text{for } m \in \mathbb{N} \text{ and } t \in (0,1),$$

see [25, Remark 8.2 (iii)].

Let $X(0,1)$ be a rearrangement-invariant space and let $m \in \mathbb{N}$. If the function I satisfies (1.19), then the optimal (smallest) rearrangement-invariant space $X_m(0,1)$ such that

$$H_I^m : X(0,1) \to X_m(0,1) \tag{3.1}$$

is endowed with the function norm $\|\cdot\|_{X_m(0,1)}$, whose associate function norm is given by

$$\|g\|_{X_m'(0,1)} = \left\| \frac{1}{I(t)}\int_0^t g^*(s)\left(\int_s^t \frac{dr}{I(r)}\right)^{m-1} ds \right\|_{X'(0,1)} \tag{3.2}$$

for $g \in \mathcal{M}(0,1)$ [25, Proposition 8.3].

The following lemma provides us with a pointwise inequality involving the operator H_I^k for $k = 1,\dots,m-1$ for I satisfying (1.20). For every such I we denote by ψ_I the function defined by

$$\psi_I(t) = \left(\int_0^t \frac{dr}{I(r)}\right)^m \quad \text{for } t \in (0,1). \tag{3.3}$$

Lemma 3.1. *Assume that $m, k \in \mathbb{N}$, $m \geq 2$, $1 \leq k \leq m-1$. Let I be positive a non-decreasing function on $(0,1)$ satisfying (1.20) and let ψ_I be the function defined by (3.3). There exists a constant $C = C(m)$ such that, if $g \in \mathcal{M}_+(0,1)$ and*

$$g^*(t) \leq \frac{1}{\psi_I(t)} \quad \text{for } t \in (0,1), \tag{3.4}$$

then

$$H_I^k g^*(t) \leq C g^*(t)^{1-\frac{k}{m}} \quad \text{for } t \in (0,1) \text{ and } k \in \{1, \dots, m\}. \tag{3.5}$$

Proof. Fix $g \in \mathcal{M}_+(0,1)$ such that (3.4) holds, and any $t \in (0,1)$. Define $a \in (0,1]$ by the identity

$$g^*(t) = \frac{1}{\psi_I(a)} \quad \text{if } g^*(t) > \frac{1}{\psi_I(1)}$$

and $a = 1$ otherwise. Note that the definition is correct since ψ_I is continuous and strictly increasing on $(0,1)$, and $\lim_{t \to 0_+} \psi_I(t) = 0$.
Assume first that $g^*(t) > \frac{1}{\psi_I(1)}$. Then (3.4) and the monotonicity of ψ_I imply that $t \leq a \leq 1$. We thus get

$$g^*(s) \leq \begin{cases} g^*(t) & \text{if } s \in [t,a), \\ \frac{1}{\psi_I(s)} & \text{if } s \in [a,1). \end{cases}$$

Fix $k \in \{1, \dots, m-1\}$. Then, consequently,

$$\begin{aligned} (k-1)! H_I^k g^*(t) &= \int_t^a \frac{g^*(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds + \int_a^1 \frac{g^*(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds \\ &\leq g^*(t) \int_t^a \frac{1}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds + \int_a^1 \frac{1}{\psi_I(s) I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds. \end{aligned}$$

By the definition of a,

$$\begin{aligned} g^*(t) \int_t^a \frac{1}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds &= \frac{g^*(t)}{k} \left(\int_t^a \frac{dr}{I(r)} \right)^k \leq \frac{g^*(t)}{k} \left(\int_0^a \frac{dr}{I(r)} \right)^k \\ &= \frac{g^*(t)}{k} \psi_I(a)^{\frac{k}{m}} = \frac{1}{k} g^*(t)^{1-\frac{k}{m}} \end{aligned}$$

and

$$\begin{aligned} \int_a^1 \frac{1}{\psi_I(s) I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{k-1} ds &= \int_a^1 \frac{1}{I(s)} \frac{\left(\int_t^s \frac{dr}{I(r)} \right)^{k-1}}{\left(\int_0^s \frac{dr}{I(r)} \right)^m} ds \leq \int_a^1 \frac{1}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{k-m-1} ds \\ &= \tfrac{1}{m-k} \left[\left(\int_0^a \frac{dr}{I(r)} \right)^{k-m} - \left(\int_0^1 \frac{dr}{I(r)} \right)^{k-m} \right] \\ &\leq \tfrac{1}{m-k} \left(\int_0^a \frac{dr}{I(r)} \right)^{k-m} = \tfrac{1}{m-k} \psi_I(a)^{\frac{k}{m}-1} = \tfrac{1}{m-k} g^*(t)^{1-\frac{k}{m}}. \end{aligned}$$

Assume next that $g^*(t) \le \frac{1}{\psi_I(1)}$. Then $a = 1$. Similarly as above, we have that

$$\begin{aligned}(k-1)! H_I^k g^*(t) &= \int_t^1 \frac{g^*(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)}\right)^{k-1} ds \le g^*(t) \int_t^1 \frac{1}{I(s)} \left(\int_t^s \frac{dr}{I(r)}\right)^{k-1} ds \\ &= \frac{g^*(t)}{k} \left(\int_t^1 \frac{dr}{I(r)}\right)^k \le \frac{g^*(t)}{k} \psi_I(1)^{\frac{k}{m}} \le \frac{1}{k} g^*(t)^{1-\frac{k}{m}}.\end{aligned}$$

Altogether, inequality (3.5) follows. □

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$ and $p \in (1,\infty)$, we define the functional $\|\cdot\|_{X^p(0,1)}$ by

$$\|g\|_{X^p(0,1)} = \|g^p\|_{X(0,1)}^{\frac{1}{p}} \quad \text{for } g \in \mathcal{M}_+(0,1).$$

The functional $\|\cdot\|_{X^p(0,1)}$ is also an rearrangement invariant function norm. Moreover, the inequality

$$\|fg\|_{X(0,1)} \le \|f\|_{X^p(0,1)} \|g\|_{X^{p'}(0,1)} \tag{3.6}$$

holds for every $f, g \in \mathcal{M}(0,1)$ (see e.g. [44, Lemma 1]).

The following lemma, of possible independent interest, is a major tool in the proofs of our main results.

Lemma 3.2. *Assume that $m \in \mathbb{N}$, $m \ge 2$. Let I be a positive non-decreasing function on $(0,1)$, and let ψ_I be the function defined by (3.3). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the following statements are equivalent:*

(i) *Condition (1.20) holds, and there exists a positive constant C such that*

$$\sup_{t\in(0,1)} g^{**}(t)\psi_I(t) \le C\|g\|_{X(0,1)} \tag{3.7}$$

for every $g \in \mathcal{M}_+(0,1)$.

(ii) *Condition (1.20) holds, and there exists a positive constant C such that*

$$\sup_{t\in(0,1)} g^{*}(t)\psi_I(t) \le C\|g\|_{X(0,1)} \tag{3.8}$$

for every $g \in \mathcal{M}_+(0,1)$.

(iii) *For every $k \in \mathbb{N}$, $1 \le k \le m-1$, there is a positive constant C such that*

$$\|g\|_{X^{\frac{m}{m-k}}(0,1)} \le C\|g\|_{X_k(0,1)} \tag{3.9}$$

for every $g \in \mathcal{M}_+(0,1)$.

(iv) *For every $k \in \mathbb{N}$, $1 \le k \le m-1$, there exists a positive constant C such that*

$$\|fg\|_{X(0,1)} \le C\|f\|_{X_k(0,1)}\|g\|_{X_{m-k}(0,1)} \tag{3.10}$$

for every $f, g \in \mathcal{M}_+(0,1)$.

(v) *There exists $k \in \mathbb{N}$, $1 \le k \le m-1$, and a positive constant C such that (3.10) holds.*

(vi) *There exists $k \in \mathbb{N}$, $1 \le k \le m-1$, and a positive constant C such that*

$$\|H_I^k f\, H_I^{m-k} g\|_{X(0,1)} \le C\|f\|_{X(0,1)}\|g\|_{X(0,1)} \tag{3.11}$$

for every $f, g \in \mathcal{M}_+(0,1)$.

(vii) *There exists a positive constant C such that*

$$\left(\int_0^t \frac{ds}{I(s)}\right)^m \le C\varphi_X(t) \quad \textit{for } t \in (0,1). \tag{3.12}$$

Proof. (i)⇒(ii) This implication is trivial thanks to the universal pointwise estimate $g^*(t) \leq g^{**}(t)$ which holds for every $g \in \mathcal{M}(0,1)$ and every $t \in (0,1)$.

(ii)⇒(iii) Fix $k \in \mathbb{N}$, $1 \leq k \leq m-1$. In view of the optimality of the space $X_k(0,1)$ in $H_I^k : X(0,1) \to X_k(0,1)$, mentioned above Lemma 3.1, the assertion will follow once we show that

$$H_I^k : X(0,1) \to X^{\frac{m}{m-k}}(0,1). \tag{3.13}$$

Let $g \in X(0,1)$, $g \not\equiv 0$, and let C be the constant from (3.8). Define $h = \frac{g}{C\|g\|_{X(0,1)}}$. Inequality (3.8) implies that $h^*(t) \leq \frac{1}{\psi_I(t)}$ for $t \in (0,1)$, whence, by Lemma 3.1,

$$H_I^k h^*(t) \leq C' h^*(t)^{1-\frac{k}{m}} \quad \text{for } t \in (0,1),$$

for some constant $C' = C'(m,k)$. Thus,

$$\begin{aligned}\|H_I^k g^*\|_{X^{\frac{m}{m-k}}(0,1)} &= C\|g\|_{X(0,1)}\|H_I^k h^*\|_{X^{\frac{m}{m-k}}(0,1)} \leq C'C\|g\|_{X(0,1)}\|{h^*}^{1-\frac{k}{m}}\|_{X^{\frac{m}{m-k}}(0,1)} \\ &= C'C\|g\|_{X(0,1)}\|h^*\|_{X(0,1)}^{1-\frac{k}{m}} = C'C^{\frac{k}{m}}\|g\|_{X(0,1)}.\end{aligned}$$

By [25, Corollary 9.8], this is equivalent to the existence of a positive constant $C(m,k,X)$ such that

$$\|H_I^k g\|_{X^{\frac{m}{m-k}}(0,1)} \leq C(m,k,X)\|g\|_{X(0,1)}$$

for every $g \in \mathcal{M}_+(0,1)$. Hence, (3.13) follows.

(iii)⇒(iv) Fix $k \in \{1,\dots,m-1\}$ and let $f,g \in \mathcal{M}_+(0,1)$. On applying first (3.9) to f in place of g, and then (3.9) again, this time with k replaced by $m-k$, we obtain

$$\|f\|_{X^{\frac{m}{m-k}}(0,1)} \leq C\|f\|_{X_k(0,1)} \quad \text{and} \quad \|g\|_{X^{\frac{m}{k}}(0,1)} \leq C\|g\|_{X_{m-k}(0,1)}.$$

Combining these estimates with (3.6), with $p = \frac{m}{m-k}$, yields

$$\|fg\|_{X(0,1)} \leq C^2\|f\|_{X_k(0,1)}\|g\|_{X_{m-k}(0,1)},$$

and (iv) follows.

(iv)⇒(v) This implication is trivial.

(v)⇒(vi) Let $k \in \mathbb{N}$, $1 \leq k \leq m-1$, be such that (3.10) holds, and let $f,g \in \mathcal{M}_+(0,1)$. On making use of (3.10) with $H_I^k f$ and $H_I^{m-k} g$ in the place of f and g, respectively, we obtain

$$\|H_I^k f\, H_I^{m-k} g\|_{X(0,1)} \leq C\|H_I^k f\|_{X_k(0,1)}\|H_I^{m-k} g\|_{X_{m-k}(0,1)}.$$

It follows from (3.1) that

$$H_I^k : X(0,1) \to X_k(0,1) \quad \text{and} \quad H_I^{m-k} : X(0,1) \to X_{m-k}(0,1).$$

Coupling these facts with the preceding inequality implies that

$$\|H_I^k f\, H_I^{m-k} g\|_{X(0,1)} \leq C'\|f\|_{X(0,1)}\|g\|_{X(0,1)}$$

for some positive constant $C' = C'(m,k,I,X)$ but independent of $f,g \in \mathcal{M}_+(0,1)$. Thus, the property (vi) follows.

(vi)⇒(vii) Let $k \in \mathbb{N}$, $1 \leq k \leq m-1$, be such that (3.11) holds. Assume, for the time being, that $m < 2k$. On replacing, if necessary, $\|\cdot\|_{X(0,1)}$ with the equivalent norm $C\|\cdot\|_{X(0,1)}$, we may suppose, without loss of generality, that $C = 1$ in (3.11). Thus,

$$\|(H_I^k f)(H_I^{m-k} g)\|_{X(0,1)} \leq \|f\|_{X(0,1)}\|g\|_{X(0,1)} \tag{3.14}$$

for every $f,g \in \mathcal{M}_+(0,1)$.

Let $b > -1$. Fix $a \in (0,1]$ and $\varepsilon \in (0,a)$. Set

$$f(t) = \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)}\right)^b, \quad g(t) = \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{k}{m-k}+k-m} \tag{3.15}$$

for $t \in (0,1)$. One can verify that

$$H_I^k f(t) = \frac{1}{(b+1)\dots(b+k)} \left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)} \quad \text{for } t \in (\varepsilon, a). \tag{3.16}$$

Since we are assuming that $m < 2k$ and $b > -1$,

$$(b+k)\tfrac{k}{m-k} + k - m > (-1+k)\tfrac{k}{2k-k} + k - 2k = -1. \tag{3.17}$$

Hence, analogously to (3.16),

$$H_I^{m-k} g(t) = \frac{1}{[(b+k)\frac{k}{m-k}+k-m+1]\dots[(b+k)\frac{k}{m-k}]} \left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{k}{m-k}} \quad \text{for } t \in (\varepsilon, a). \tag{3.18}$$

Note that

$$(b+k)\tfrac{k}{m-k} + k - m = b\tfrac{m-k}{k} + (b+k)\tfrac{m}{m-k}\tfrac{2k-m}{k}. \tag{3.19}$$

Set $p = \frac{k}{m-k}$. The assumption $m < 2k$ guarantees that $p > 1$. Moreover, $p' = \frac{k}{2k-m}$. Thus, inequality (3.6), applied to this choice of p, and equation (3.19) tell us that

$$\begin{aligned} \|g\|_{X(0,1)} &= \left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{k}{m-k}+k-m} \right\|_{X(0,1)} \\ &\le \left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)}\right)^{b} \right\|_{X(0,1)}^{\frac{m-k}{k}} \left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}} \right\|_{X(0,1)}^{\frac{2k-m}{k}} \\ &= \|f\|_{X(0,1)}^{\frac{m}{k}-1} \left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}} \right\|_{X(0,1)}^{\frac{2k-m}{k}}. \end{aligned} \tag{3.20}$$

Coupling (3.14) with (3.20) yields

$$\|H_I^k f \, H_I^{m-k} g\|_{X(0,1)} \le \|f\|_{X(0,1)}^{\frac{m}{k}} \left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}} \right\|_{X(0,1)}^{\frac{2k-m}{k}}. \tag{3.21}$$

By (3.16) and (3.18),

$$\begin{aligned} &H_I^k f(t)\, H_I^{m-k} g(t) \\ &= \frac{1}{(b+1)\dots(b+k)[(b+k)\frac{k}{m-k}+k-m+1]\dots[(b+k)\frac{k}{m-k}]} \left(\int_t^a \frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}} \end{aligned}$$

for $t \in (\varepsilon, a)$. On setting

$$B(b) = (b+1)\dots(b+k)\big[(b+k)\tfrac{k}{m-k}+k-m+1\big]\dots\big[(b+k)\tfrac{k}{m-k}\big] \tag{3.22}$$

and making use of (3.21), we obtain

(3.23)

$$\begin{aligned}\frac{1}{B(b)}\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)} &= \left\|\chi_{(\varepsilon,a)}\,H_I^kf\,H_I^{m-k}g\right\|_{X(0,1)}\\ &\le \left\|H_I^kf\,H_I^{m-k}g\right\|_{X(0,1)}\\ &\le \|f\|_{X(0,1)}^{\frac{m}{k}}\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{\frac{2k-m}{k}}.\end{aligned}$$

The function

$$(0,1)\ni t\mapsto \chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}$$

is bounded and hence, by (2.5), belongs to $X(0,1)$. Thus,

$$\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}<\infty.$$

Moreover, $1-\frac{2k-m}{k}=\frac{m}{k}-1$. Therefore, (3.23) yields

$$\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{\frac{m}{k}-1}\le B(b)\,\|f\|_{X(0,1)}^{\frac{m}{k}}.$$

On raising this inequality to the power $\frac{k}{m}$, and recalling the definition of f, we get

$$\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{(b+k)\frac{m}{m-k}}\right\|_{X(0,1)}^{1-\frac{k}{m}}\le B(b)^{\frac{k}{m}}\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{b}\right\|_{X(0,1)}. \tag{3.24}$$

Next, assume that $m=2k$. Note that (3.17) holds also in this case. Let f and g be defined by (3.15) again. Then

$$(b+k)\tfrac{k}{m-k}+k-m=b,$$

whence $f=g$. Moreover, since $k=m-k$, we also have that $H_I^kf=H_I^{m-k}g$. Therefore, (3.14), (3.16) and (3.18) imply that

$$\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{2(b+k)}\right\|_{X(0,1)}^{\frac{1}{2}}\le (b+1)\dots(b+k)\left\|\chi_{(\varepsilon,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^{b}\right\|_{X(0,1)}.$$

The assumption $m=2k$ entails that $B(b)=[(b+1)\dots(b+k)]^2$, and $\frac{k}{m}=\frac{1}{2}$. Hence, (3.24) follows. We have thus established (3.24) whenever $m\le 2k$. From now on, we keep this assumption in force.

Define $b_0=0$ and, for $j\in\mathbb{N}$,

$$b_j=(b_{j-1}+k)\tfrac{m}{m-k},$$

namely

$$b_j=m\big[(\tfrac{m}{m-k})^j-1\big]. \tag{3.25}$$

We next set, for $j \in \mathbb{N}$,

$$K_j = \prod_{i=0}^{j-1} B(b_i)^{\frac{k}{m}(\frac{m-k}{m})^i}.$$

Let us note that the assumption $2k \geq m$ implies that $B(b_j) \geq 1$ for $j \in \mathbb{N} \cup \{0\}$, and hence $K_j \geq 1$ as well.

We claim that, for every $j \in \mathbb{N}$,

$$\left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_j} \right\|_{X(0,1)}^{(\frac{m-k}{m})^j} \leq K_j \|\chi_{(\varepsilon,a)}\|_{X(0,1)}. \tag{3.26}$$

Indeed, choosing $b = 0$ in (3.24), yields (3.26) for $j = 1$. Assume now that (3.26) holds for some fixed $j \in \mathbb{N}$. Then, by (3.24) with $b = b_j$,

$$\left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_{j+1}} \right\|_{X(0,1)}^{(\frac{m-k}{m})^{j+1}} \leq B(b_j)^{\frac{k}{m}(\frac{m-k}{m})^j} \left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_j} \right\|_{X(0,1)}^{(\frac{m-k}{m})^j}.$$

Thus, by the induction assumption,

$$\left\| \chi_{(\varepsilon,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_{j+1}} \right\|_{X(0,1)}^{(\frac{m-k}{m})^{j+1}} \leq B(b_j)^{\frac{k}{m}(\frac{m-k}{m})^j} K_j \|\chi_{(\varepsilon,a)}\|_{X(0,1)} = K_{j+1} \|\chi_{(\varepsilon,a)}\|_{X(0,1)},$$

and (3.26) follows.

Letting $\varepsilon \to 0^+$ in (3.26) and making use of property (P3) of rearrangement-invariant function norms yields

$$\left\| \chi_{(0,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_j} \right\|_{X(0,1)}^{(\frac{m-k}{m})^j} \leq K_j \|\chi_{(0,a)}\|_{X(0,1)}. \tag{3.27}$$

Fix $j \in \mathbb{N}$, and set

$$g_j(t) = \chi_{(0,a)}(t) \left(\int_t^a \frac{dr}{I(r)} \right)^{b_j} \quad \text{for } t \in (0,1).$$

Then, owing to (3.27)

$$\|g_j\|_{X(0,1)}^{(\frac{m-k}{m})^j} \leq K_j \|\chi_{(0,a)}\|_{X(0,1)}. \tag{3.28}$$

By the Hölder inequality and (2.4),

$$\int_0^a g_j(t)\, dt \leq \|g_j\|_{X(0,1)} \|\chi_{(0,a)}\|_{X'(0,1)} = \frac{a\|g_j\|_{X(0,1)}}{\|\chi_{(0,a)}\|_{X(0,1)}}.$$

By (3.28),

$$\frac{1}{a} \int_0^a g_j(t)\, dt \leq K_j^{\frac{1}{(\frac{m-k}{m})^j}} \|\chi_{(0,a)}\|_{X(0,1)}^{\frac{1-(\frac{m-k}{m})^j}{(\frac{m-k}{m})^j}}.$$

We have thus shown that

$$\left(\frac{1}{a} \int_0^a g_j(t)\, dt \right)^{\frac{(\frac{m-k}{m})^j}{1-(\frac{m-k}{m})^j}} \leq K_j^{\frac{1}{1-(\frac{m-k}{m})^j}} \|\chi_{(0,a)}\|_{X(0,1)} \tag{3.29}$$

for $j \in \mathbb{N}$.

Observe that

$$\lim_{j\to\infty}\left(\frac{1}{a}\int_0^a g_j(t)\,dt\right)^{\frac{(\frac{m-k}{m})^j}{1-(\frac{m-k}{m})^j}} = \lim_{j\to\infty}\left(\frac{1}{a}\int_0^a\left(\int_t^a\frac{dr}{I(r)}\right)^{b_j}dt\right)^{\frac{1}{(\frac{m}{m-k})^j-1}} \tag{3.30}$$

$$= \lim_{j\to\infty}\left(\frac{1}{a}\int_0^a\left(\int_t^a\frac{dr}{I(r)}\right)^{m((\frac{m}{m-k})^j-1)}dt\right)^{\frac{1}{(\frac{m}{m-k})^j-1}}$$

$$= \left\|\chi_{(0,a)}(t)\left(\int_t^a\frac{dr}{I(r)}\right)^m\right\|_{L^\infty(0,1)} = \left(\int_0^a\frac{dr}{I(r)}\right)^m.$$

On the other hand, since $\frac{m-k}{m}<1$,

$$1-\left(\tfrac{m-k}{m}\right)^j \geq 1-\tfrac{m-k}{m} = \tfrac{k}{m}$$

for $j\in\mathbb{N}$. As observed above, $K_j\geq 1$ for every $j\in\mathbb{N}$. By (3.22), if $b>-1$, then

$$B(b)\leq (b+k)^k((b+k)\tfrac{k}{m-k})^{m-k}\leq (b+k)^m(\tfrac{m}{m-k})^m.$$

With the choice $b=b_j$, the last chain and (3.25) yield

$$B(b_j)\leq (m((\tfrac{m}{m-k})^j-1)+k)^m(\tfrac{m}{m-k})^m\leq (m(\tfrac{m}{m-k})^{j+1})^m$$

for $j\in\mathbb{N}\cup\{0\}$. Altogether, we deduce that

$$K_j^{\frac{1}{1-(\frac{m-k}{m})^j}}\leq K_j^{\frac{m}{k}} = \prod_{i=0}^{j-1}B(b_i)^{(\frac{m-k}{m})^i}\leq \prod_{i=0}^{\infty}\left(\left(m\left(\tfrac{m}{m-k}\right)^{i+1}\right)^m\right)^{(\frac{m-k}{m})^i}<\infty.$$

On setting

$$K=\prod_{i=0}^{\infty}\left(\left(m\left(\tfrac{m}{m-k}\right)^{i+1}\right)^m\right)^{(\frac{m-k}{m})^i},$$

we have that

$$\limsup_{j\to\infty}K_j^{\frac{1}{1-(\frac{m-k}{m})^j}}\leq K. \tag{3.31}$$

Therefore, combining (3.29), (3.30) and (3.31) tells us that

$$\left(\int_0^a\frac{dr}{I(r)}\right)^m\leq K\|\chi_{(0,a)}\|_{X(0,1)}.$$

Hence, inequality (3.12) follows. Thus, property (vii) is proved when $m\leq 2k$. However, if this is not the case, then $m\leq 2(m-k)$. The same argument as above, applied with $m-k$ in the place of k, leads to the conclusion.

(vii)⇒(i) Let $g\in\mathcal{M}(0,1)$. Then, by (3.12),

$$\sup_{t\in(0,1)}g^{**}(t)\psi_I(t)\leq C\sup_{t\in(0,1)}g^{**}(t)\varphi_X(t).$$

One has that

$$\sup_{t\in(0,1)}g^{**}(t)\varphi_X(t)\leq\|g\|_{X(0,1)}$$

for every $g\in\mathcal{M}(0,1)$ (see e.g. It is a classical fact (see e.g. [7, Chapter 2, Proposition 5.9]). Combining the last two estimates yields

$$\sup_{t\in(0,1)}g^{**}(t)\psi_I(t)\leq C\|g\|_{X(0,1)}$$

for $g \in \mathcal{M}(0,1)$, namely (3.7). □

We conclude this section by showing that assumption (1.9) is actually essentially stronger than (1.24).

Proposition 3.3. *Let $m, n \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that I is a positive non-decreasing function on $(0,1)$. If (1.9) holds, then (1.24) holds as well.*

Proof. By the Hölder inequality in rearrangement-invariant spaces,

$$\left(\int_0^t \frac{ds}{I(s)}\right)^m = m\int_0^t \frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1} ds \leq m\left\|\frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1}\right\|_{X'(0,1)} \|\chi_{(0,t)}\|_{X(0,1)}$$
$$= m\left\|\frac{1}{I(s)}\left(\int_0^s \frac{dr}{I(r)}\right)^{m-1}\right\|_{X'(0,1)} \varphi_X(t) \qquad \text{for } t \in (0,1).$$

Hence, the assertion follows. □

Remark 3.4. It is easily seen that (1.9) is in fact essentially stronger than (1.24), in general. Indeed, let $I(t) = t^\alpha$ for some $\alpha \in \mathbb{R}$ such that $\alpha \geq \frac{1}{n'}$ and $\alpha > \frac{1}{m'}$, and let $\|\cdot\|_{X(0,1)} = \|\cdot\|_{L^q(0,1)}$ with $q = \frac{1}{m(1-\alpha)}$. Then (1.24) holds but (1.9) does not. In other words, by Theorems 1.3 and 1.9, $\mathcal{V}^m L^{\frac{1}{m(1-\alpha)}}(\Omega)$ is not a Banach algebra for every $\Omega \in \mathcal{J}_\alpha$, yet it satisfies (1.25) for every $\Omega \in \mathcal{J}_\alpha$.

4. Proofs of the main results

Here, we accomplish the proofs of the results stated in Section 1.
A result to be exploited in our proofs is an embedding theorem for the space $\mathcal{V}^m X(\Omega)$, which tells us that, under assumption (1.19),

$$V^m X(\Omega) \to X_m(\Omega) \tag{4.1}$$

where $X_m(\Omega)$ is the rearrangement-invariant space built upon the function norm $\|\cdot\|_{X_m(0,1)}$ given by (3.1), and that $X_m(\Omega)$ is the optimal (smallest) such rearrangement-invariant space [25, Theorem 5.4] (see also [27], [19] and [39] for earlier proofs in special cases).

The next three lemmas are devoted to certain "worst possible" domains whose isoperimetric function has a prescribed decay. Such domains will be of use in the proof of the necessity of our conditions in the main results.

Lemma 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and let I be a positive , non-decreasing function satisfying (1.11). Then there exists a positive non-decreasing function $\widehat{I}$ in $(0,1)$ such that $\widehat{I} \in C^1(0,1)$, $\widehat{I}^{n'}$ is convex on $(0,1)$, and*

$$\widehat{I}(s) \approx I(s) \quad \textit{for } s \in (0,1). \tag{4.2}$$

Proof. By (1.11), there exists a non-decreasing function $\varsigma : (0,1) \to (0,\infty)$ such that

$$\frac{I(s)^{n'}}{s} \approx \varsigma(s) \qquad \text{for } s \in (0,1). \tag{4.3}$$

Set

$$I_1(s) = \left(\int_0^s \varsigma(r)\, dr\right)^{1/n'} \qquad \text{for } s \in (0,1).$$

Then $I_1 \in C^0(0,1)$, and $I_1^{n'}$ is convex in $(0,1)$. Moreover, we claim that

$$I_1(s) \approx I(s) \qquad \text{for } s \in (0,1). \tag{4.4}$$

Indeed, by the monotonicity of ς,

$$\tfrac{1}{2}\varsigma(s/2) \le \frac{1}{s}\int_{s/2}^{s} \varsigma(r)dr \le \frac{I_1(s)^{n'}}{s} = \frac{1}{s}\int_0^s \varsigma(r)dr \le \varsigma(s) \qquad \text{for } s \in (0,1). \tag{4.5}$$

Equation (4.4) follows from (4.3) and (4.5). Similarly, on setting

$$\widehat{I}(s) = \left(\int_0^s \frac{I_1(r)^{n'}}{r} dr\right)^{1/n'} \qquad \text{for } s \in (0,1),$$

one has that $\widehat{I} \in C^1(0,1)$, $\widehat{I}^{n'}$ is convex in $(0,1)$, and

$$\widehat{I}(s) \approx I_1(s) \qquad \text{for } s \in (0,1). \tag{4.6}$$

Coupling (4.4) with (4.6) yields (4.2). Thus, the function $\widehat{I}$ has the required properties. $\square$

Lemma 4.2. *Let $n \in \mathbb{N}$, $n \ge 2$, and let I be a positive, non-decreasing function on $(0,1)$ satisfying* (1.11)*. Then there exist $L \in (0,\infty]$ and a convex function $\eta : (0,L) \to (0,\infty)$ such that the domain $\Omega_I \subset \mathbb{R}^n$, defined by*

$$\Omega_I = \{x \in \mathbb{R}^n \colon x = (x', x_n),\ x_n \in (0,L),\ x' \in \mathbb{R}^{n-1},\ |x'| < \eta(x_n)\}, \tag{4.7}$$

satisfies $|\Omega_I| = 1$ and

$$I_{\Omega_I}(s) \approx I(s) \quad \textit{for } s \in (0, \tfrac{1}{2}]. \tag{4.8}$$

Proof. By Lemma 4.1, we can assume with no loss of generality that $I \in C^1(0,1)$ and $I^{n'}$ is convex in $(0,1)$. Let $L \in (0,\infty]$ be defined by

$$L = \int_0^1 \frac{dr}{I(r)}, \tag{4.9}$$

and let $M : [0,L) \to (0,1]$ be the function implicitly defined as

$$\int_{M(t)}^1 \frac{dr}{I(r)} = t \qquad \text{for } t \in [0,L). \tag{4.10}$$

The function M strictly decreases from 1 to 0. In particular, M is continuously differentiable in $(0,L)$, and

$$I(M(r)) = -M'(r) \qquad \text{for } r \in (0,L)\,. \tag{4.11}$$

On defining $\eta : (0,L) \to (0,\infty)$ as

$$\eta(r) = \left(\frac{I(M(r))}{\omega_{n-1}}\right)^{\frac{1}{n-1}} \qquad \text{for } r \in (0,L), \tag{4.12}$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional unit ball, we have that $\eta(r) > 0$ for $r \in (0,L)$, and, by (4.11),

$$\begin{aligned} |\{x \in \Omega_I : t < x_n < L\}| &= \omega_{n-1}\int_t^L \eta(r)^{n-1}\,dr = \int_t^L I(M(r))\,dr \\ &= \int_t^L -M'(r)\,dr = M(t) \quad \text{for } t \in (0,L). \end{aligned} \tag{4.13}$$

Moreover, η is convex. To see this, notice that, by (4.11),

$$\omega_{n-1}^{\frac{1}{n-1}}\eta'(r) = \left(I(M(r))^{\frac{1}{n-1}}\right)' = \frac{1}{n-1}I'(M(r))I(M(r))^{\frac{1}{n-1}-1}M'(r) \tag{4.14}$$
$$= -\frac{1}{n-1}I'(M(r))I(M(r))^{\frac{1}{n-1}} \quad \text{for } r \in (0,L).$$

Thus, since $M(r)$ is decreasing, $\eta'(r)$ is increasing if and only if $I'(s)I(s)^{\frac{1}{n-1}}$ is increasing, and this is in turn equivalent to the convexity of $I(s)^{n'}$. Equation (4.14) also tells us that

$$\eta'(0^+) = -\frac{1}{n-1}\omega_{n-1}^{-\frac{1}{n-1}}I'(1^-)I(1^-)^{\frac{1}{n-1}} > -\infty.$$

By (4.13), with $t=0$, the set Ω_I as in (4.7), with η given by (4.12), satisfies $|\Omega_I| = 1$. Furthermore, owing to the properties of η, from either [47, Example 5.3.3.1] or [47, Example 5.3.3.2], according to whether $L<\infty$ or $L=\infty$, we infer that there exist positive constants C_1 and C_2 such that

$$C_1\,\eta(M^{-1}(s))^{n-1} \le I_{\Omega_I}(s) \le C_2\,\eta(M^{-1}(s))^{n-1} \qquad \text{for } s \in (0,\tfrac{1}{2}]. \tag{4.15}$$

Hence, by (4.12),

$$\frac{C_1}{\omega_{n-1}}\,I(s) \le I_{\Omega_I}(s) \le \frac{C_2}{\omega_{n-1}}\,I(s) \qquad \text{for } s \in (0,\tfrac{1}{2}], \tag{4.16}$$

and (4.8) follows. □

Lemma 4.3. *Let $n \in \mathbb{N}$, $n \ge 2$. Let I be a positive non-decreasing function on $(0,1)$ such that $I \in C^1(0,1)$ and $I^{n'}$ is convex in $(0,1)$. Let L, M, η and Ω_I be as in Lemma 4.2. Given $h \in \mathcal{M}_+(0,1)$, let $F : \Omega_I \to [0,\infty)$ be the function defined by*

$$F(x) = h(M(x_n)) \quad \text{for } x \in \Omega_I.$$

Then

$$F^*(t) = h^*(t) \quad \text{for } t \in (0,1). \tag{4.17}$$

Proof. On making use of (4.12), of a change of variables and of (4.11), we obtain that

$$\begin{aligned}|\{x \in \Omega_I \colon F(x) > \lambda\}| &= |\{x \in \Omega_I \colon h(M(x_n)) > \lambda\}| = \omega_{n-1}\int_{\{r\in(0,L)\colon h(M(r))>\lambda\}} \eta(r)^{n-1}\,dr \\ &= \int_{\{r\in(0,L)\colon h(M(r))>\lambda\}} I(M(r))\,dr = \int_{\{t\in(0,1)\colon h(t)>\lambda\}} -\frac{I(t)}{M'(M^{-1}(t))}\,dt \\ &= |\{t\in(0,1)\colon h(t)>\lambda\}| \qquad \text{for } \lambda > 0.\end{aligned}$$

Equation (4.17) hence follows, via the definition of the decreasing rearrangement. □

Proposition 4.4. *Assume that $m,n \in \mathbb{N}$, $n \ge 2$, and let I be a positive non-decreasing function satisfying (1.11). Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Assume that for every domain $\Omega \in \mathcal{J}_I$, the Sobolev space $\mathcal{V}^m X(\Omega)$ is a Banach algebra. Then condition (1.9) holds. Moreover,*

$$\mathcal{V}^m X(\Omega) \to L^\infty(\Omega) \tag{4.18}$$

for every such domain Ω.

Proof. By Lemma 4.1, we can assume, without loss of generality, that $I \in C^1(0,1)$ and $I^{n'}$ is convex in $(0,1)$. Let L, M, η and Ω_I be as in Lemma 4.2. Let $f, g \in \mathcal{M}_+(0,1)$. We define the functions $u, v : \Omega_I \to [0,\infty]$ by

$$u(x) = \int_{M(x_n)}^1 \frac{1}{I(r_1)} \int_{r_1}^1 \frac{1}{I(r_2)} \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_1 \quad \text{for } x \in \Omega_I, \tag{4.19}$$

and

$$v(x) = \int_{M(x_n)}^1 \frac{1}{I(r_1)} \int_{r_1}^1 \frac{1}{I(r_2)} \cdots \int_{r_{m-1}}^1 \frac{g(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_1 \quad \text{for } x \in \Omega_I. \tag{4.20}$$

Then the functions u and v are m times weakly differentiable in Ω_I. Since u is a non-decreasing function of the variable x_n,

$$\begin{aligned} |\nabla u(x)| = \frac{\partial u}{\partial x_n}(x) &= -\frac{M'(x_n)}{I(M(x_n))} \int_{M(x_n)}^1 \frac{1}{I(r_2)} \int_{r_2}^1 \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_2 \\ &= \int_{M(x_n)}^1 \frac{1}{I(r_2)} \int_{r_2}^1 \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_2 \quad \text{for a.e. } x \in \Omega_I, \end{aligned}$$

where the last equality holds by (4.11). Similarly, if $1 \le k \le m-1$,

$$|\nabla^k u(x)| = \frac{\partial^k u}{\partial x_n^k}(x) = \int_{M(x_n)}^1 \frac{1}{I(r_{k+1})} \int_{r_{k+1}}^1 \cdots \int_{r_{m-1}}^1 \frac{f(r_m)}{I(r_m)} dr_m\, dr_{m-1} \ldots dr_{k+1} \quad \text{for a.e. } x \in \Omega_I, \tag{4.21}$$

and

$$|\nabla^m u(x)| = \frac{\partial^m u}{\partial x_n^m}(x) = f(M(x_n)) \quad \text{for a.e. } x \in \Omega_I. \tag{4.22}$$

Thus, if $0 \le k \le m$, then

$$|\nabla^k u(x)| = \frac{\partial^k u}{\partial x_n^k}(x) = H_I^{m-k} f(M(x_n)) \quad \text{for a.e. } x \in \Omega_I, \tag{4.23}$$

where, as agreed, $H_I^0 f = f$. Analogously, if $0 \le k \le m$,

$$|\nabla^k v(x)| = \frac{\partial^k v}{\partial x_n^k}(x) = H_I^{m-k} g(M(x_n)) \quad \text{for a.e. } x \in \Omega_I. \tag{4.24}$$

By the Leibniz Rule,

$$\begin{aligned} |\nabla^m (uv)(x)| = \frac{\partial^m (uv)}{\partial x_n^m}(x) &= \sum_{k=0}^m \binom{m}{k} \frac{\partial^k u}{\partial x_n^k}(x) \frac{\partial^{m-k} v}{\partial x_n^{m-k}}(x) \\ &= \sum_{k=0}^m \binom{m}{k} H_I^{m-k} f(M(x_n)) H_I^k g(M(x_n)) \quad \text{for a.e. } x \in \Omega_I. \end{aligned}$$

Since $\Omega_I \in \mathcal{J}_I$, the space $\mathcal{V}^m X(\Omega_I)$ is a Banach algebra by our assumption. Therefore, in particular, there exists a positive constant C such that

$$\|\nabla^m (uv)\|_{X(\Omega_I)} \le C \|u\|_{\mathcal{V}^m X(\Omega_I)} \|v\|_{\mathcal{V}^m X(\Omega_I)},$$

namely,

$$\left\| \sum_{k=0}^m \binom{m}{k} H_I^{m-k} f(M(x_n)) H_I^k g(M(x_n)) \right\|_{X(\Omega_I)} \le C \|u\|_{\mathcal{V}^m X(\Omega_I)} \|v\|_{\mathcal{V}^m X(\Omega_I)}. \tag{4.25}$$

Now, if $0 \leq k \leq m$, then

$$H_I^{m-k} f(M(x_n)) H_I^k g(M(x_n)) \geq 0 \quad \text{for a.e. } x \in \Omega_I.$$

Therefore, we can disregard the terms with $k \neq 0$ in (4.25), and obtain

$$\|H_I^m f(M(x_n))\, g(M(x_n))\|_{X(\Omega_I)} \leq C \|u\|_{\mathcal{V}^m X(\Omega_I)} \|v\|_{\mathcal{V}^m X(\Omega_I)}. \tag{4.26}$$

By Lemma 4.3, the function $F_{m,I} : \Omega_I \to [0,\infty)$, defined as

$$F_{m,I}(x) = H_I^m f(M(x_n)) g(M(x_n)) \qquad \text{for } x \in \Omega_I,$$

is such that

$$F_{m,I}^* = (H_I^m fg)^* .$$

Hence,

$$\|H_I^m f(M(x_n)) g(M(x_n))\|_{X(\Omega_I)} = \|H_I^m fg\|_{X(0,1)}. \tag{4.27}$$

As for the terms on the right-hand side, note that, by (1.2), (4.22) and Lemma 4.3,

$$\|u\|_{\mathcal{V}^m X(\Omega)} = \|f\|_{X(0,1)} + \sum_{k=0}^{m-1} \|\nabla^k u\|_{L^1(B)}, \tag{4.28}$$

where B is any ball in Ω_I. It is readily verified from (4.13) that there exists a constant $c > 0$ such that $M(x_n) \geq c$ for every $x \in B$. Thus,

$$\frac{1}{I(r)} \leq \frac{1}{I(c)} \quad \text{if } x \in B \text{ and } r \in [M(x_n), 1].$$

Hence, by (4.19) and (4.21), if $0 \leq k \leq m-1$,

$$\begin{aligned}
|\nabla^k u(x)| &\leq \frac{1}{I(c)^{m-k}} \int_{M(x_n)}^1 \int_{r_{k+1}}^1 \dots \int_{r_{m-1}}^1 f(r_m) dr_m\, dr_{m-1} \dots dr_{k+1} \\
&\leq \frac{C_1}{I(c)^{m-k}} \int_{M(x_n)}^1 f(r_m) dr_m \leq C_2 \int_0^1 f(r)\, dr \qquad \text{for a.e. } x \in B,
\end{aligned}$$

for suitable positive constants C_1 and C_2. Consequently, by (2.5), if $0 \leq k \leq m-1$

$$\|\nabla^k u\|_{L^1(B)} \leq C_2 |B| \|f\|_{L^1(0,1)} \leq C_3 \|f\|_{X(0,1)}$$

for a suitable constant C_3. Hence, via (4.28),

$$\|u\|_{\mathcal{V}^m X(\Omega)} \leq C \|f\|_{X(0,1)} \tag{4.29}$$

for some constant C. Analogously,

$$\|v\|_{\mathcal{V}^m X(\Omega)} \leq C \|g\|_{X(0,1)}. \tag{4.30}$$

From (4.26), (4.27), (4.29) and (4.30) we infer that

$$\|g H_I^m f\|_{X(0,1)} \leq C \|f\|_{X(0,1)} \|g\|_{X(0,1)} \tag{4.31}$$

for some positive constant C.
We now claim that

$$\|H_I^m f\|_{L^\infty(0,1)} \leq \sup_{g \geq 0, \|g\|_{X(0,1)} \leq 1} \|g H_I^m f\|_{X(0,1)}. \tag{4.32}$$

Indeed, if $\lambda < \|H_I^m f\|_{L^\infty(0,1)}$, then there exists a set $E \subset (0,1)$ of positive measure such that $H_I^m f \geq \lambda \chi_E$. Thus,

$$\sup_{g \geq 0, \|g\|_{X(0,1)} \leq 1} \|g H_I^m f\|_{X(0,1)} \geq \lambda \sup_{g \geq 0, \|g\|_{X(0,1)} \leq 1} \|g \chi_E\|_{X(0,1)} = \lambda\,,$$

whence (4.32) follows. Coupling inequality (4.32) with (4.31) tells us that

$$\|H_I^m f\|_{L^\infty(0,1)} \le C\|f\|_{X(0,1)}$$

for every $f \in X(0,1)$. The last inequality can be rewritten in the form

$$\int_0^1 \frac{f(r)}{I(r)} \left(\int_0^r \frac{dt}{I(t)} \right)^{m-1} dr \le C\|f\|_{X(0,1)},$$

and hence, by (2.1),

$$\left\| \frac{1}{I(t)} \left(\int_0^t \frac{ds}{I(s)} \right)^{m-1} \right\|_{X'(0,1)} \le C,$$

namely (1.9).

In particular, we have established (1.20), and hence also (1.17). Therefore, by Remark 1.8, the three spaces $V^m X(\Omega)$, $\mathcal{V}^m X(\Omega)$ and $W^m X(\Omega)$ coincide. Moreover, thanks to (1.9), we may apply [25, Corollary 5.5] and obtain thereby that $V^m X(\Omega) \to L^\infty(\Omega)$ for every $\Omega \in \mathcal{J}_I$. This establishes (4.18). Since $V^m X(\Omega) = \mathcal{V}^m X(\Omega)$, the proof is complete. □

The following theorem shows that the embedding into the space of essentially bounded functions is necessary for a Banach space to be a Banach algebra, in a quite general framework.

Theorem 4.5. *Assume that $Z(\Omega)$ is a Banach algebra such that $Z(\Omega) \subset \mathcal{M}(\Omega)$ and*

$$Z(\Omega) \to L^{1,\infty}(\Omega). \tag{4.33}$$

Then

$$Z(\Omega) \to L^\infty(\Omega). \tag{4.34}$$

Proof. Since $Z(\Omega)$ is a Banach algebra, there exists a constant $C \ge 1$ such that

$$\|uv\|_{Z(\Omega)} \le C\|u\|_{Z(\Omega)}\|v\|_{Z(\Omega)} \tag{4.35}$$

for every $u, v \in Z(\Omega)$. Suppose, by contradiction, that (4.34) fails. Then there exists a function $w \in Z(\Omega)$ such that

$$\|w\|_{L^\infty(\Omega)} > 2C\|w\|_{Z(\Omega)},$$

where C is the constant from (4.35). In other words, the set

$$E = \{x \in \Omega;\ |w(x)| > 2C\|w\|_{Z(\Omega)}\}$$

has positive Lebesgue measure. Fix $j \in \mathbb{N}$. Applying (4.35) $(j-1)$-times, we obtain

$$\|w^j\|_{Z(\Omega)} \le C^{j-1}\|w\|_{Z(\Omega)}^j.$$

Combining this inequality with (4.33) yields

$$\lambda|\{x \in \Omega : |w(x)|^j > \lambda\}| \le C'C^{j-1}\|w\|_{Z(\Omega)}^j.$$

for some constant C', and for every $\lambda > 0$. In particular, the choice $\lambda = (2C\|w\|_{Z(\Omega)})^j$ yields

$$(2C\|w\|_{Z(\Omega)})^j|E| \le C'C^{j-1}\|w\|_{Z(\Omega)}^j,$$

namely

$$2^j|E| \le CC^{-1}.$$

However, this is impossible, since $|E| > 0$ and j is arbitrary. □

Given a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$, with $\gamma_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \dots, n$, set $|\gamma| = \gamma_1 + \dots + \gamma_n$, and $D^\gamma u = \frac{\partial^{|\gamma|} u}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}$ for $u : \Omega \to \mathbb{R}$. Moreover, given two multi-indices γ and δ, we write $\gamma \le \delta$ to denote that $\gamma_i \le \delta_i$ for $i = 1, \dots, n$. Accordingly, by $\gamma < \delta$ we mean that $\gamma \le \delta$ and $\gamma_i < \delta_i$ for at least one $i \in \{1, \dots, n\}$.

Proof of Theorem 1.9. Let $1 \le k \le m$. Fix any multi-index γ satisfying $|\gamma| \le m - k$. Assumption (1.22) implies that

$$\int_0^1 \frac{ds}{I(s)} < \infty. \tag{4.36}$$

From (1.22) and (4.36), we obtain that there exists a positive constant C such that

$$\left(\int_0^t \frac{ds}{I(s)}\right)^{2m-|\gamma|} \le \left(\int_0^1 \frac{ds}{I(s)}\right)^{m-k-|\gamma|} \left(\int_0^t \frac{ds}{I(s)}\right)^{m+k} \le C\varphi_X(t). \tag{4.37}$$

Suppose that δ is any multi-index fulfilling $\delta \le \gamma$. Since (1.22) holds, Lemma 3.2, implication (vii) $\Rightarrow$ (iii), combined with (4.1), tells us that

$$V^{m-|\delta|} X(\Omega) \to X^{\frac{2m-|\gamma|}{m-|\gamma|+|\delta|}}(\Omega) \quad \text{and} \quad V^{m-|\gamma|+|\delta|}(\Omega) \to X^{\frac{2m-|\gamma|}{m-|\delta|}}(\Omega). \tag{4.38}$$

On applying (3.6) with $p = \frac{2m-|\gamma|}{m-|\gamma|+|\delta|}$ and the two embeddings in (4.38), we deduce that

$$\begin{aligned} \|D^\delta u D^{\gamma-\delta} v\|_{X(\Omega)} &\le \|D^\delta u\|_{X^{\frac{2m-|\gamma|}{m-|\gamma|+|\delta|}}(\Omega)} \|D^{\gamma-\delta} v\|_{X^{\frac{2m-|\gamma|}{m-|\delta|}}(\Omega)} \\ &\le C\|D^\delta u\|_{V^{m-|\delta|}X(\Omega)} \|D^{\gamma-\delta} v\|_{V^{m-|\gamma|+|\delta|}X(\Omega)} \le C'\|u\|_{V^m X(\Omega)} \|v\|_{V^m X(\Omega)}, \end{aligned} \tag{4.39}$$

for some constants C and C', and for every $u, v \in V^m X(\Omega)$. In particular, inequality (4.39) implies that $\sum_{\delta \le \gamma} |D^\delta u D^{\gamma-\delta} v| \in L^1(\Omega)$. Hence, via [3, Ex. 3.17], we deduce that the function uv is $(m-k)$-times weakly differentiable and

$$D^\gamma(uv) = \sum_{\delta \le \gamma} \frac{\gamma!}{\delta!(\gamma-\delta)!} D^\delta u D^{\gamma-\delta} v.$$

It follows from (4.39) that

$$\|D^\gamma(uv)\|_{X(\Omega)} \le \sum_{\delta \le \gamma} \frac{\gamma!}{\delta!(\gamma-\delta)!} \|D^\delta u D^{\gamma-\delta} v\|_{X(\Omega)} \le C\|u\|_{V^m X(\Omega)} \|v\|_{V^m X(\Omega)},$$

for some constant C. Thus,

$$\|uv\|_{W^{m-k}X(\Omega)} = \sum_{|\gamma| \le m-k} \|D^\gamma(uv)\|_{X(\Omega)} \le C\|u\|_{V^m X(\Omega)} \|v\|_{V^m X(\Omega)} \tag{4.40}$$

for some constant C, and for every $u, v \in V^m X(\Omega)$. Since (4.36) is in force, $W^{m-k}X(\Omega) = \mathcal{V}^{m-k}X(\Omega)$ and $V^m X(\Omega) = \mathcal{V}^m X(\Omega)$, up to equivalent norms. As a consequence of (4.40), inequality (1.23) follows.

In order to prove the converse assertion, observe that, by Lemma 4.1, we can assume with no loss of generality that $I \in C^1(0,1)$ and $I^{n'}$ is convex. Let L, M, η and Ω_I be as in Lemma 4.2. Since, by (4.8), $\Omega_I \in \mathcal{J}_I$, condition (1.23) is fulfilled. Thus, there exists a positive constant C such that

$$\|\nabla^{m-k}(uv)\|_{X(\Omega_I)} \le C\|u\|_{\mathcal{V}^m X(\Omega_I)} \|v\|_{\mathcal{V}^m X(\Omega_I)} \tag{4.41}$$

for every $u,\ v \in \mathcal{V}^m X(\Omega_I)$. Given $f, g \in \mathcal{M}_+(0,1)$, define $u, v : \Omega_I \to [0,\infty]$ as in (4.19) and (4.20), respectively. The functions u and v are m-times weakly differentiable in Ω_I. Furthermore, by (4.23) and (4.24),

$$
\begin{aligned}
|\nabla^{m-k}(uv)(x)| &= \frac{\partial^{m-k}(uv)}{\partial x_n^{m-k}}(x) = \sum_{\ell=0}^{m-k} \binom{m-k}{\ell} \frac{\partial^\ell u}{\partial x_n^\ell}(x) \frac{\partial^{m-k-\ell} v}{\partial x_n^{m-k-\ell}}(x) \\
&= \sum_{\ell=0}^{m-k} \binom{m-k}{\ell} H_I^{m-\ell} f(M(x_n)) H_I^{k+\ell} g(M(x_n)) \\
&\geq H_I^m f(M(x_n)) H_I^k g(M(x_n)) \quad \text{for a.e. } x \in \Omega_I.
\end{aligned} \tag{4.42}
$$

From (4.41), (4.42), (4.29) and (4.30) we infer that

$$
\|H_I^m f\, H_I^k g\|_{X(0,1)} = \|H_I^m f(M(x_n)) H_I^k g(M(x_n))\|_{X(\Omega)} \leq C \|f\|_{X(0,1)} \|g\|_{X(0,1)}
$$

for some constant C, and for every $f, g \in \mathcal{M}_+(0,1)$. Since $1 \leq k \leq m \leq m+k-1$, it follows from Lemma 3.2, implication (vi) $\Rightarrow$ (vii), that (1.22) holds. $\square$

Proof of Theorem 1.11. Let $\Omega \in \mathcal{J}_I$, $\|\cdot\|_{X(0,1)}$ and $1 \leq k \leq m$ be such that (1.22) is fulfilled. By Theorem 1.9, there exists a positive constant C such that

$$
\sum_{l=0}^{m-k} \| |\nabla^l u| |\nabla^{m-k-l} v| \|_{X(\Omega)} \leq C \|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)} \tag{4.43}
$$

for every $u, v \in \mathcal{V}^m X(\Omega)$.
In order to prove (1.25), it suffices to show that there exists $C > 0$ such that

$$
\|D^\gamma u\, D^\delta v\|_{X(\Omega)} \leq C \|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)} \tag{4.44}
$$

for every $u, v \in \mathcal{V}^m X(\Omega)$ and every multi-indices γ and δ satisfying $|\gamma| + |\delta| = m$, $|\gamma| \geq 1$ and $|\delta| \geq 1$. Fix such u, v, γ and δ. Without loss of generality we may assume that $|\gamma| \leq |\delta|$. Let σ be an arbitrary multi-index such that $\sigma \leq \delta$ and $|\sigma| = |\gamma|$. Assumption (1.24) ensures that condition (1.22) is fulfilled, with m and k replaced with $m - |\gamma|$ and $|\gamma|$. Thus, owing to (4.43), applied with u, v, m and k replaced by $D^\gamma u$, $D^{\delta_0} v$, $m - |\gamma|$ and $|\gamma|$, respectively, (and disregarding the terms with $\ell > 0$ on the left-hand side) we obtain that

$$
\begin{aligned}
\|D^\gamma u\, D^\delta v\|_{X(\Omega)} &\leq \| D^\gamma u |\nabla^{m-2|\gamma|} D^\sigma v| \|_{X(\Omega)} \\
&\leq C \|D^\gamma u\|_{\mathcal{V}^{m-|\gamma|} X(\Omega)} \|D^\sigma v\|_{\mathcal{V}^{m-|\gamma|} X(\Omega)} \\
&\leq C \|u\|_{\mathcal{V}^m X(\Omega)} \|v\|_{\mathcal{V}^m X(\Omega)},
\end{aligned}
$$

whence (4.44) follows.
A proof of the necessity of condition (1.24), under (1.25) and (1.11) follows along the same lines as in the proof of (1.22) in Theorem 1.9, and will be omitted for brevity. $\square$

We shall now prove a general sufficient condition for the space $W^m X(\Omega)$ to be a Banach algebra.

Proposition 4.6. *Let $m, n \in \mathbb{N}$, $n \geq 2$, and let I be a positive non-decreasing function in $(0,1)$. Assume that $\Omega \in \mathcal{J}_I$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement invariant function norm. If (1.9) holds, then the Sobolev space $W^m X(\Omega)$ is a Banach algebra.*

Proof. It suffices to show that, for each pair of multi-indices γ and δ such that $|\gamma| \leq m$ and $\delta \leq \gamma$, there exists a positive constant C such that inequality

$$\|D^\delta u D^{\gamma-\delta} v\|_{X(\Omega)} \leq C\|u\|_{W^m X(\Omega)}\|v\|_{W^m X(\Omega)}. \tag{4.45}$$

holds for all u, $v \in W^m X(\Omega)$. Indeed, such inequality implies, in particular, that

$$\sum_{\delta\leq\gamma} |D^\delta u D^{\gamma-\delta} v| \in L^1(\Omega).$$

Hence, once again, one can use [3, Ex. 3.17] to deduce that the function uv is m-times weakly differentiable in Ω, and that, for each γ with $1 \leq |\gamma| \leq m$,

$$D^\gamma(uv) = \sum_{\delta\leq\gamma} \frac{\gamma!}{\delta!(\gamma-\delta)!} D^\delta u D^{\gamma-\delta} v. \tag{4.46}$$

In order to prove (4.45), let us begin by noting that, by (1.9) and (2.3),

$$\int_0^1 \frac{g(t)}{I(t)} \left(\int_0^t \frac{dr}{I(r)}\right)^{m-1} dt \leq C\|g\|_{X(0,1)}, \tag{4.47}$$

for some constant C and every function $g \in \mathcal{M}_+(0,1)$. Since (4.47) can be rewritten in the form

$$\left\| \int_t^1 \frac{g(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)}\right)^{m-1} ds \right\|_{L^\infty(0,1)} \leq C\|g\|_{X(0,1)},$$

it follows from [25, Theorem 5.1] that $V^m X(\Omega) \to L^\infty(\Omega)$. Thus, by (1.14), $W^m X(\Omega) \to L^\infty(\Omega)$ as well.

Assume now that γ is an arbitrary multi-index such that $0 \leq |\gamma| \leq m$. Then, for every $u, v \in W^m X(\Omega)$,

$$\|uD^\gamma v\|_{X(\Omega)} \leq \|u\|_{L^\infty(\Omega)}\|D^\gamma v\|_{X(\Omega)} \leq C\|u\|_{W^m X(\Omega)}\|v\|_{W^m X(\Omega)},$$

and, analogously,

$$\|(D^\gamma u)v\|_{X(\Omega)} \leq C\|u\|_{W^m X(\Omega)}\|v\|_{W^m X(\Omega)}.$$

This establishes (4.45) whenever $|\gamma| \leq m$ and either $\delta = 0$ or $\delta = \gamma$.

Assume now that $|\delta| \geq 1$ and $\delta < \gamma$. It follows from Proposition 3.3 that (1.9) implies (1.24), and hence also (1.22) for every $k \in \mathbb{N}$. Clearly,

$$\|D^\delta u\, D^{\gamma-\delta} v\|_{X(\Omega)} \leq \|\, |\nabla^{|\delta|} u|\, |\nabla^{|\gamma-\delta|} v|\, \|_{X(\Omega)}. \tag{4.48}$$

On the other hand, we claim that

$$\|\, |\nabla^{|\delta|} u|\, |\nabla^{|\gamma-\delta|} v|\, \|_{X(\Omega)} \leq C\|u\|_{\mathcal{V}^m X(\Omega)}\|v\|_{\mathcal{V}^m X(\Omega)} \tag{4.49}$$

for some positive C and for every $u, v \in \mathcal{V}^m X(\Omega)$. Indeed, if $|\gamma| = m$, then the claim is a straightforward consequence of Theorem 1.11. If $|\gamma| < m$, then the claim follows from inequality (4.43), applied with the choice $k = m - |\gamma|$. Combining inequalities (4.48), (4.49) and the first embedding in (1.14) completes the proof. □

Proof of Theorem 1.3. Assume that (1.9) is satisfied. Then, by Proposition 4.6, the space $W^m X(\Omega)$ is a Banach algebra for all $\Omega \in \mathcal{J}_\alpha$. Moreover, as we have already observed, condition (1.9) implies (1.17). Therefore, by Remark 1.8, the spaces $W^m X(\Omega)$ and $\mathcal{V}^m X(\Omega)$ coincide. Consequently, $\mathcal{V}^m X(\Omega)$ is a Banach algebra.

The second part of the theorem is a straightforward consequence of Proposition 4.4. □

Proof of Corollary 1.6. Assume that the embedding $\mathcal{V}^m X(\Omega) \to L^\infty(\Omega)$ holds for every $\Omega \in \mathcal{J}_I$. Let Ω_I be the domain defined by (4.7). Given $f \in \mathcal{M}_+(0,1)$, define u by (4.19). It follows from (4.23) with $k=0$ that $u(x) = H_I^m f(M(x_n))$ for a.e. $x \in \Omega_I$ and M defined by (4.10). Therefore, $\|u\|_{L^\infty(\Omega)} = \|H_I^m f\|_{L^\infty(0,1)}$. Furthermore, by (4.29), we get $\|u\|_{\mathcal{V}^m X(\Omega)} \le C\|f\|_{X(0,1)}$ for some constant $C > 0$. Hence, our assumptions imply that there exists a positive constant C such that

$$\|H_I^m f\|_{L^\infty(0,1)} \le C\|f\|_{X(0,1)} \tag{4.50}$$

for every nonnegative function $f \in X(0,1)$. Inequality (4.50) implies (1.9), as was again observed in the course of proof of Proposition 4.4. By Theorem 1.3, this tells us that $\mathcal{V}^m X(\Omega)$ is a Banach algebra for every $\Omega \in \mathcal{J}_I$.
The converse implication follows at once from Proposition 4.4. □

Proof of Corollary 1.7. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then, by the assumption and (2.5), condition (1.9) is satisfied. Hence, owing to Theorem 1.3, the space $\mathcal{V}^m X(\Omega)$ is a Banach algebra. □

Proof of Theorem 1.1. Assume first that $\mathcal{V}^m X(\Omega)$ is a Banach algebra. Due to (1.6), condition (1.17) is satisfied. Hence, by Remark 1.8, the spaces $\mathcal{V}^m X(\Omega)$ and $W^m X(\Omega)$ coincide. In particular, $W^m X(\Omega)$ is a Banach algebra. Furthermore, by (2.5) and trivial inclusions, we clearly have

$$W^m X(\Omega) \to X(\Omega) \to L^1(\Omega) \to L^{1,\infty}(\Omega),$$

hence the assumption (4.33) of Theorem 4.5 is satisfied. Thus, as a special case of Theorem 4.5 we obtain that $W^m X(\Omega) \to L^\infty(\Omega)$. Therefore, $W^m X(\Omega) \to L^\infty(\Omega)$. On the other hand, by [25, Theorem 6.1], this embedding is equivalent to the inequality

$$\int_0^1 g(s) s^{\frac{m}{n}-1}\, ds \le C\|g\|_{X(0,1)}$$

for some constant C, and for every nonnegative function $g \in X(0,1)$. Hence, via the very definition of associate function norm, we obtain (1.3).

Conversely, assume that (1.3) holds. Since Ω is a John domain, inequality (1.8) is satisfied with $I(t) = t^{\frac{1}{n'}}$, $t \in (0,1)$. Hence, it follows from Theorem 1.3 that the space $\mathcal{V}^m X(\Omega)$ is a Banach algebra. □

Proof of Proposition 1.12. If $I(t) = t^\alpha$, with $\alpha \in [\frac{1}{n'}, \infty)$, then condition (1.11) is clearly satisfied. Hence, by Theorem 1.3, the space $\mathcal{V}^m L^{p,q;\beta}(\Omega)$ is a Banach algebra for every $\Omega \in \mathcal{J}_\alpha$ if and only if (1.9) holds. Owing to Remark 1.5, condition (1.9) entails that $\alpha < 1$. Thus,

$$\frac{1}{I(t)}\left(\int_0^t \frac{ds}{I(s)}\right)^{m-1} = (\tfrac{1}{1-\alpha})^{m-1} t^{m(1-\alpha)-1} \quad \text{for } t \in (0,1).$$

Therefore, it only remains to analyze under which conditions the power function $t^{m(1-\alpha)-1}$ belongs to $(L^{p,q;\beta})'(0,1)$. It is easily verified, via (2.9), that this is the case if and only if one of the conditions in (1.27) is satisfied. □

Proof of Proposition 1.13. By the same argument as in the proof of Proposition 1.12 we deduce that for $\mathcal{V}^m L^A(\Omega)$ to be a Banach algebra it is necessary that $\alpha < 1$. If $m \ge \frac{1}{1-\alpha}$, then (1.12) holds with $I(t) = t^\alpha$, $t \in (0,1)$, hence, by Corollary 1.7, $\mathcal{V}^m L^A(\Omega)$ is a Banach algebra whatever A is. If $m < \frac{1}{1-\alpha}$, then assumption (1.9) of Theorem 1.3 is equivalent to

$$\|r^{m(1-\alpha)-1}\|_{L^{\tilde{A}}(0,1)} < \infty. \tag{4.51}$$

The latter condition is, in turn, equivalent to

$$\int_0 \widetilde{A}\big(r^{m(1-\alpha)-1}\big)\,dr < \infty\,, \tag{4.52}$$

namely to

$$\int^\infty \frac{\widetilde{A}(t)}{t^{1+\frac{1}{1-m(1-\alpha)}}}\,dt < \infty. \tag{4.53}$$

By [20, Lemma 2.3], equation (4.53) is equivalent to

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{(1-\alpha)m}{1-(1-\alpha)m}} dt < \infty.$$

□

Proof of Proposition 1.14. Condition (1.11) is satisfied if $I(t) = t^\alpha$, with $\alpha \in [\frac{1}{n'}, \infty)$. By Theorem 1.9, condition (1.23) for $X(\Omega) = L^{p,q;\beta}(\Omega)$ holds for every $\Omega \in \mathcal{J}_\alpha$ if and only if (1.22) is in force. In turn, inequality (1.22) entails (1.20), whence $\alpha < 1$. Now,

$$\varphi_{L^{p,q;\beta}}(t) \approx \begin{cases} t^{\frac{1}{p}}(\log\frac{2}{t})^\beta & \text{if } 1 < p < \infty,\ 1 \le q \le \infty,\ \beta \in \mathbb{R};\\ t(\log\frac{2}{t})^\beta & \text{if } p = 1,\ q = 1,\ \beta \ge 0;\\ 1 & \text{if } p = \infty,\ q = \infty,\ \beta = 0;\\ (\log\frac{2}{t})^{\beta+\frac{1}{q}} & \text{if } p = \infty,\ 1 \le q < \infty,\ \beta + \frac{1}{q} < 0, \end{cases} \tag{4.54}$$

for $t \in (0,1)$ (see e.g. [52, Lemma 9.4.1, page 318]). On the other hand,

$$\left(\int_0^t s^{-\alpha}\,ds\right)^{m+k} = (1-\alpha)^{-m-k} t^{(1-\alpha)(m+k)} \quad \text{for } t \in (0,1). \tag{4.55}$$

Thus, inequality (1.22) holds, with $X(0,1) = L^{p,q;\beta}(0,1)$, if and only if one of the conditions in (1.29) is satisfied. □

Proof of Proposition 1.15. As observed in the above proof, we may assume that $\alpha < 1$, and, by Theorem 1.9, reduce (1.23), with $I(t) = t^\alpha$, $t \in (0,1)$, and $X(\Omega) = L^A(\Omega)$, to the validity of (1.22). It is easily seen that

$$\varphi_{L^A}(t) = \frac{1}{A^{-1}(1/t)} \quad \text{for } t \in (0,1).$$

Thus, the conclusion follows via (4.55). □

References

[1] R.A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[2] A. Alvino, V. Ferone and G. Trombetti, *Moser-type inequalities in Lorentz spaces*, Potential Anal. **5** (1996), 273–299.

[3] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Clarendon Press, Oxford, 2000.

[4] T. Aubin, *Problèmes isopérimetriques et espaces de Sobolev*, J. Diff. Geom. **11** (1976), 573–598.

[5] F. Barthe, P. Cattiaux and C. Roberto, *Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry*, Rev. Mat. Iberoam. **22** (2006), 993–1067.

[6] T. Bartsch, T. Weth and M. Willem, *A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator*, Calc. Var. Partial Differential Equations **18** (2003), 253–268.

[7] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics Vol. 129, Academic Press, Boston 1988.

[8] C. Bennett and K. Rudnick, *On Lorentz–Zygmund spaces*, Dissertationes Math. **175** (1980), 1–67.
[9] S.G. Bobkov and C. Houdré, *Some connections between isoperimetric and Sobolev-type inequalities*, Mem. Am. Math. Soc. **129** (1997), viii+111.
[10] S.G. Bobkov and M. Ledoux, *From Brunn-Minkowski to sharp Sobolev inequalities*, Ann. Mat. Pura Appl. **187** (2008), 389–384.
[11] S. Buckley and P. Koskela, *Sobolev-Poincaré implies John*, Math. Res. Lett. **2** (1995), 577–593.
[12] S. Buckley and P. Koskela, *Criteria for embeddings of Sobolev-Poincaré type*, Int. Math. Res. Not. **18** (1996), 881–902.
[13] Yu.D. Burago and V.A. Zalgaller, *Geometric inequalities*, Springer, Berlin, 1988.
[14] L. Capogna, D. Danielli, S.D. Pauls and J.T. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Birkhauser, Basel, 2007.
[15] E.A. Carlen and C. Kerce, *On the cases of equality in Bobkov's inequality and Gaussian rearrangement*, Calc. Var. Partial Differential Equations **13** (2001), 1–18.
[16] I. Chavel, *Isoperimetric inequalities: differential geometric aspects and analytic perspectives*, Cambridge University Press, Cambridge, 2001.
[17] J. Cheeger, *A lower bound for the smallest eigevalue of the Laplacian, in Problems in analysis, 195–199*, Princeton Univ. Press, Princeton, 1970.
[18] A. Cianchi, *A sharp embedding theorem for Orlicz–Sobolev spaces*, Indiana Univ. Math. J. **45** (1996), 39–65.
[19] A. Cianchi, *Symmetrization and second-order Sobolev inequalities*, Ann. Mat. Pura Appl. **183** (2004), 45–77.
[20] A. Cianchi, *Optimal Orlicz-Sobolev embeddings*, Rev. Mat. Iberoamericana **20** (2004), 427–474.
[21] A. Cianchi, *Orlicz–Sobolev algebras*, Potential Anal. **28** (2008), 379–388.
[22] A. Cianchi, N. Fusco, F. Maggi and A. Pratelli, *The sharp Sobolev inequality in quantitative form*, J. Eur. Math. Soc. **11** (2009), 1105–1139.
[23] A. Cianchi and L. Pick, *Sobolev embeddings into BMO, VMO and L_∞*, Ark. Mat. **36** (1998), 317–340.
[24] A. Cianchi and L. Pick, *Optimal Gaussian Sobolev embeddings*, J. Funct. Anal. **256** (2009), 3588–3642.
[25] A. Cianchi, L. Pick and L. Slavíková, *Higher-order Sobolev embeddings and isoperimetric inequalities*, Adv. Math, to appear.
[26] E. De Giorgi, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*, Atti. Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. **5** (1958), 33–44.
[27] D. E. Edmunds, R. Kerman and L. Pick, *Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms*, J. Funct. Anal. **170** (2000), 307–355.
[28] L. Esposito, V. Ferone, B. Kawohl, C. Nitsch, and C. Trombetti, *The longest shortest fence and sharp Poincaré-Sobolev inequalities*, Arch. Ration. Mech. Anal. **206** (2012), 821–851.
[29] H. Federer and W. Fleming, *Normal and integral currents*, Annals of Math. **72** (1960), 458–520.
[30] F. Gazzola and H.-C. Grunau, *Critical dimensions and higher order Sobolev inequalities with remainder terms*, NoDEA Nonlinear Differential Equations Appl. **8** (2001), 35–44.
[31] F. Gazzola, H.-C. Grunau and G. Sweers, *Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, 2010.
[32] A. Grygor'yan, *Isoperimetric inequalities and capacities on Riemannian manifolds*, The Maz'ya Anniversary Collection, vol. 1, Operator Theory, Advances and Applications **110**, Birkhäuser, Basel, 1999, 139–153.
[33] P. Hajłasz and P. Koskela, *Isoperimetric inequalites and imbedding theorems in irregular domains*, J. London Math. Soc. **58** (1998), 425–450.
[34] E. Hebey, *Analysis on manifolds: Sobolev spaces and inequalities*, Courant Lecture Notes in Mathematics **5**, AMS, Providence, 1999.
[35] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715–727.
[36] D. Hoffmann and J. Spruck, *A correction to: "Sobolev and isoperimetric inequalities for Riemannian submanifolds" (Comm. Pure Appl. Math. 27 (1974), 715–725)*, Comm. Pure Appl. Math. **28** (1975), 765–766.
[37] T. Kilpeläinen and J. Malý, *Sobolev inequalities on sets with irregular boundaries*, Z. Anal. Anwend. **19** (2000), 369–380.
[38] V.S. Klimov, *Imbedding theorems and geometric inequalities*, Izv. Akad. Nauk SSSR **40** (1976), 645–671 (Russian); English translation: Math. USSR Izv. **10** (1976), 615–638.
[39] R. Kerman and L. Pick, *Optimal Sobolev imbeddings*, Forum Math. **18, 4** (2006), 535–570.
[40] V.I. Kolyada, *Estimates on rearrangements and embedding theorems*, Mt. Sb. **136** (1988), 3–23 (Russian); English translation: Math. USSR Sb. **64** (1989), 1–21.

[41] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. II.*, Rev. Mat. Iberoamericana **1** (1985), 45–121.

[42] P.-L. Lions F. Pacella and M. Tricarico, *Best constants in Sobolev inequalities for functions vanishing on some part of the boundary and related questions*, Indiana Univ. Math. J. **37** (1988), 301–324.

[43] E. Lutwak D. Yang and G. Zhang, *Sharp affine L_p Sobolev inequalities*, J. Diff. Geom. **62** (2002), 17–38.

[44] L. Maligranda and L.E. Persson, *Generalized duality of some Banach function spaces*, Nederl. Akad. Wetensch. Indag. Math. **51** (1989), 323–338.

[45] V.G. Maz'ya, *Classes of regions and imbedding theorems for function spaces*, Dokl. Akad. Nauk. SSSR **133** (1960), 527–530 (Russian); English translation: Soviet Math. Dokl. **1** (1960), 882–885.

[46] V.G. Maz'ya, *On p-conductivity and theorems on embedding certain functional spaces into a C-space*, Dokl. Akad. Nauk. SSSR **140** (1961), 299–302 (Russian); English translation: Soviet Math. Dokl. **3** (1962).

[47] V.G. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, Springer, Berlin, 2011.

[48] V.G. Maz'ya and T.O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Pitman, Boston, 1985.

[49] E. Milman, *On the role of convexity in functional and isoperimetric inequalities*, Proc. London Math. Soc. **99** (2009), 32–66.

[50] J. Moser, *A sharp form of an inequality by Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[51] B. Opic and L. Pick, *On generalized Lorentz-Zygmund spaces*, Math. Inequal. Appl. **2** (1999), 391–467.

[52] L. Pick, A. Kufner, O. John and S. Fučík, *Function Spaces, Volume 1*, 2nd Revised and Extended Edition, De Gruyter Series in Nonlinear Analysis and Applications 14, Walter De Gruyter GmbH, Berlin/Boston 2013, ISBN 978-3-11-025041-1, xv+479 pp.

[53] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, Cambridge University Press, Cambridge, 2002.

[54] C.A. Swanson, *The best Sobolev constant*, Apl.Anal. **47** (1992), 227–239.

[55] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

[56] G. Zhang, *The affine Sobolev inequality*, J. Diff. Geom. **53** (1999), 183–202.

[57] W.P. Ziemer, *Weakly differentiable functions*, Graduate texts in Math. **120**, Springer, Berlin, 1989.

Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Piazza Ghiberti 27, 50122 Firenze, Italy

E-mail address: `cianchi@unifi.it`

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: `pick@karlin.mff.cuni.cz`

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: `slavikova@karlin.mff.cuni.cz`

Note to Paper II

L. Slavíková. A Sobolev space embedded to L^∞ does not need to be a Banach algebra.

A SOBOLEV SPACE EMBEDDED TO L^∞ DOES NOT NEED TO BE A BANACH ALGEBRA

LENKA SLAVÍKOVÁ

In [1, 2, 3] it was shown that a quite general Sobolev-type space is a Banach algebra with respect to a pointwise multiplication of functions if and only if it is continuously embedded into the space of essentially bounded functions L^∞. In this note we prove that such an equivalence is not true in full generality. Namely, we present an example of a domain $\Omega \subseteq \mathbb{R}^2$ for which the Sobolev space $W^{2,2}(\Omega)$ is embedded into $L^\infty(\Omega)$ but is not a Banach algebra. The domain Ω from our example is very much in the spirit of the domain appearing in [5, Section 3], where it is shown that the space $W^{2,2} \cap L^\infty$ over a general domain in $\mathbb{R}^2$ is not necessarily a Banach algebra.

Let us start by introducing some notation. Given a set $M \subseteq \mathbb{R}^2$, we denote by M^0, $\overline{M}$ and ∂M the interior, the closure and the boundary of M, respectively. If $G \subseteq \mathbb{R}^2$ is a bounded open set then $C^2(\overline{G})$ stands for the space of all continuous functions on $\overline{G}$ whose partial derivatives up to the second order exist everywhere in G and can be extended to a continuous function on $\overline{G}$.

For any measurable set $E \subseteq \mathbb{R}^2$ and any $q \in [1,\infty]$, we denote by $L^q(E)$ the Lebesgue space consisting of all measurable functions f on E for which $\|f\|_{L^q(E)} < \infty$, where

$$\|f\|_{L^q(E)} = \begin{cases} \left(\int_E |f|^q\right)^{\frac{1}{q}}, & q \in [1,\infty); \\ \mathrm{esssup_E}\, |f|, & q = \infty. \end{cases}$$

Further, given a domain $\Omega \subseteq \mathbb{R}^2$, $p \in [1,\infty]$ and $k \in \mathbb{N}$, $W^{k,p}(\Omega)$ stands for the Sobolev space of all k-times weakly differentiable functions u on Ω for which

$$\|u\|_{W^{k,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{l=1}^{k} \||\nabla^l u|\|_{L^p(\Omega)} < \infty,$$

where $\nabla^l u$ denotes the vector of all l-th order weak derivatives of the function u and $|\nabla^l u|$ its Euclidean length.

We are going to make use of the following well known facts about Sobolev spaces (see, e.g., [4] for more details and proofs).

Given an open square S in $\mathbb{R}^2$, we have the first-order Sobolev embeddings

$$W^{1,2}(S) \hookrightarrow L^p(S) \quad \text{for any } p \in [1,\infty)$$

and

$$W^{1,q}(S) \hookrightarrow L^\infty(S) \quad \text{for any } q \in (2,\infty).$$

The latter embedding is equivalent to the Poincaré inequality

$$\mathrm{esssup}_{x\in S} \left| v(x) - \frac{1}{|S|}\int_S v(z)\,dz \right| \leq C\||\nabla v|\|_{L^q(S)}. \tag{1}$$

1

In particular, if $v \in C^2(\overline{S})$ then inequality (1) implies that for any x, $y \in \overline{S}$,

$$|v(x) - v(y)| \leq \left| v(x) - \frac{1}{|S|} \int_S v(z)\,dz \right| + \left| \frac{1}{|S|} \int_S v(z)\,dz - v(y) \right| \leq 2C \| |\nabla v| \|_{L^q(S)}. \tag{2}$$

We shall also make use of the first-order trace embedding

$$W^{1,2}(S) \hookrightarrow L^p(\partial S) \quad \text{for any } p \in [1, \infty)$$

and of the second-order embedding

$$W^{2,r}(S) \hookrightarrow L^\infty(S) \quad \text{for any } r \in (1, \infty).$$

The final result that will be needed tells us that if $\Omega \subseteq \mathbb{R}^2$ is a bounded domain having a continuous boundary, then the space $C^2(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for any $k \in \mathbb{N}$ and any $p \in [1, \infty)$.

Let us now present two auxiliary results.

Lemma 1. *There is a positive constant C such that for every closed square S in $\mathbb{R}^2$ whose sides are parallel to coordinate axes and of length at most 1, for every $u \in C^2(S)$ and for all x, $y \in S$, we have*

$$\frac{1}{C} |u(x) - u(y)| \leq \left(\int_S |\nabla u|^2 \right)^{\frac{1}{2}} + \left(\int_S |\nabla^2 u|^2 \right)^{\frac{1}{2}}. \tag{3}$$

Proof. Without loss of generality we may assume that $S = [0, b]^2$ for some $b \in (0, 1]$. Set $T = [0, 1]^2$. A combination of (2) with $q = 3$ and of the embedding $W^{1,2}(T^0) \hookrightarrow L^3(T^0)$ yields that whenever x, $y \in T$ and $v \in C^2(T)$ then

$$|v(x) - v(y)| \leq C_1 \| |\nabla v| \|_{L^3(T)} \leq C \left(\| |\nabla v| \|_{L^2(T)} + \| |\nabla^2 v| \|_{L^2(T)} \right) \tag{4}$$

for some constants C_1, C independent of x, y and v. Given a function $u \in C^2(S)$, define the function $v \in C^2(T)$ by $v(z) = u(bz)$, $z \in T$. Then, applying (4) to this choice of v and using the change of variables formula, we obtain that for any x, $y \in S$,

$$\begin{aligned} \frac{1}{C} |u(x) - u(y)| &= \frac{1}{C} \left| v\left(\frac{x}{b}\right) - v\left(\frac{y}{b}\right) \right| \leq \left(\int_T |\nabla v|^2 \right)^{\frac{1}{2}} + \left(\int_T |\nabla^2 v|^2 \right)^{\frac{1}{2}} \\ &= \left(\int_S |\nabla u|^2 \right)^{\frac{1}{2}} + \left(b^2 \int_S |\nabla^2 u|^2 \right)^{\frac{1}{2}} \leq \left(\int_S |\nabla u|^2 \right)^{\frac{1}{2}} + \left(\int_S |\nabla^2 u|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

□

Remark 2. Let us now show that the statement of Lemma 1 cannot be extended to rectangles in $\mathbb{R}^2$. Namely, suppose that $p > 1$ and for every $a \in (0, 1]$ denote $R_a = [0, a^p] \times [0, a]$. Let C_a be the least constant in the inequality (3) with S replaced by R_a. Then we claim that $\lim_{a \to 0_+} C_a = \infty$.

To prove the claim, we set

$$u_a(t, s) = a^{-\frac{p+1}{2}} s, \quad (t, s) \in R_a.$$

Then

$$\left(\int_{R_a} |\nabla u_a|^2 \right)^{\frac{1}{2}} + \left(\int_{R_a} |\nabla^2 u_a|^2 \right)^{\frac{1}{2}} = \left(\int_0^{a^p} \int_0^a a^{-p-1}\,ds\,dt \right)^{\frac{1}{2}} = 1$$

and

$$|u_a(0, a) - u_a(0, 0)| = a^{-\frac{p-1}{2}}.$$

Thus,

$$\lim_{a\to 0_+} C_a \geq \lim_{a\to 0_+} a^{-\frac{p-1}{2}} = \infty,$$

as required.

Despite of Remark 2, we can still find a version of inequality (3) which holds for certain class of rectangles in $\mathbb{R}^2$ with a constant independent of the size of the rectangle. The inequality is the object of the following lemma.

Lemma 3. *Let $0 < b \leq a \leq b^{\frac{1}{3}}$. Set $R = [0,b] \times [0,a]$. Suppose that $u \in C^2(R)$ and $x,\, y \in R$. Then*

$$\frac{1}{4}|u(x) - u(y)| \leq \left(\int_0^b |\nabla u(t,0)|^3\, dt\right)^{\frac{1}{3}} + \left(\int_R |\nabla^2 u|^2\right)^{\frac{1}{2}}. \tag{5}$$

Proof. Let $z = (z_1, z_2) \in R$. Then

$$\begin{aligned}
u(z_1,z_2) - \frac{1}{ab}\int_R u &= \frac{1}{ab}\int_R (u(z_1,z_2) - u(t,s))\, dtds \\
&= \frac{1}{ab}\int_R (u(z_1,z_2) - u(t,z_2))\, dtds + \frac{1}{ab}\int_R (u(t,z_2) - u(t,s))\, dtds \\
&= \frac{1}{ab}\int_R \int_t^{z_1} \frac{\partial u}{\partial x_1}(r,z_2)\, drdtds + \frac{1}{ab}\int_R \int_s^{z_2} \frac{\partial u}{\partial x_2}(t,r)\, drdtds \\
&= \frac{1}{ab}\int_R \int_t^{z_1} \left(\frac{\partial u}{\partial x_1}(r,0) + \int_0^{z_2} \frac{\partial^2 u}{\partial x_2 \partial x_1}(r,q)\, dq\right) drdtds \\
&\quad + \frac{1}{ab}\int_R \int_s^{z_2} \left(\frac{\partial u}{\partial x_2}(t,0) + \int_0^{r} \frac{\partial^2 u}{\partial x_2^2}(t,q)\, dq\right) drdtds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|u(z_1,z_2) - \frac{1}{ab}\int_R u\right| &\leq \frac{1}{ab}\int_R \int_0^b |\nabla u(r,0)|\, drdtds + \frac{1}{ab}\int_R \int_0^b \int_0^a |\nabla^2 u(r,q)|\, dqdrdtds \\
&\quad + \frac{1}{ab}\int_R \int_0^a |\nabla u(t,0)|\, drdtds + \frac{1}{ab}\int_R \int_0^a \int_0^a |\nabla^2 u(t,q)|\, dqdrdtds \\
&\leq \int_0^b |\nabla u(r,0)|\, dr + \int_R |\nabla^2 u| + \frac{a}{b}\int_0^b |\nabla u(t,0)|\, dt + \frac{a}{b}\int_R |\nabla^2 u| \\
&\leq \frac{2a}{b}\left(\int_0^b |\nabla u(t,0)|\, dt + \int_R |\nabla^2 u|\right) \\
&\leq \frac{2a}{b^{\frac{1}{3}}}\left(\int_0^b |\nabla u(t,0)|^3\, dt\right)^{\frac{1}{3}} + \frac{2a^{\frac{3}{2}}}{b^{\frac{1}{2}}}\left(\int_R |\nabla^2 u|^2\right)^{\frac{1}{2}} \\
&\leq 2\left(\left(\int_0^b |\nabla u(t,0)|^3\, dt\right)^{\frac{1}{3}} + \left(\int_R |\nabla^2 u|^2\right)^{\frac{1}{2}}\right),
\end{aligned}$$

where the last but one inequality is a consequence of the Hölder inequality. We thus conclude that, whenever x, $y \in R$, then

$$\begin{aligned}|u(x)-u(y)| &\le \left|u(x)-\frac{1}{ab}\int_R u\right| + \left|u(y)-\frac{1}{ab}\int_R u\right| \\ &\le 4\left(\left(\int_0^b |\nabla u(t,0)|^3\,dt\right)^{\frac{1}{3}} + \left(\int_R |\nabla^2 u|^2\right)^{\frac{1}{2}}\right).\end{aligned}$$

This completes the proof. □

We are now in a position to construct the set Ω mentioned in the first paragraph of this note and to prove that it has the required properties.

Suppose that $(a_k)_{k=1}^{\infty}$ is a nonincreasing sequence of positive real numbers such that $\sum_{k=1}^{\infty} a_k \le \frac{1}{2}$. Set $a_0 = 0$. Given $k \in \mathbb{N}$, we consider the rectangle

$$R_k = [2\sum_{i=0}^{k-1} a_i, a_k^3 + 2\sum_{i=0}^{k-1} a_i] \times [0, a_k]$$

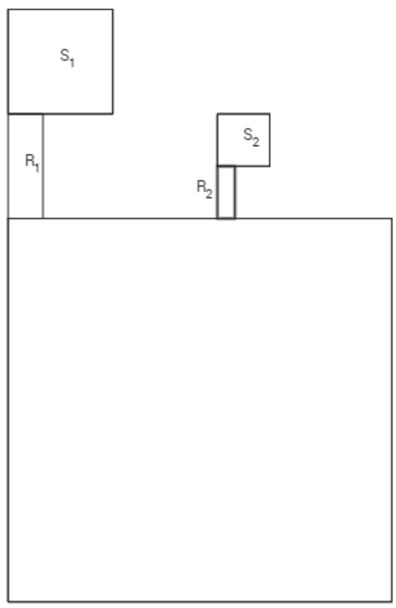
and the square

$$S_k = [2\sum_{i=0}^{k-1} a_i, a_k + 2\sum_{i=0}^{k-1} a_i] \times [a_k, 2a_k].$$

Further, we set

$$M = [0,1]\times[-1,0] \cup \bigcup_{k=1}^{\infty} R_k \cup \bigcup_{k=1}^{\infty} S_k$$

and $\Omega = M^0$.



Theorem 4. *The Sobolev space $W^{2,2}(\Omega)$ satisfies $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, but it is not a Banach algebra.*

Proof. We shall first prove the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$. Fix a function $u \in W^{2,2}(\Omega)$. For every $k \in \mathbb{N}$ we denote $M_k = [0,1] \times [-1,0] \cup R_k \cup S_k$ and $\Omega_k = M_k^0$. Then, in particular, $u \in W^{2,2}(\Omega_k)$. Since Ω_k is a domain having a continuous boundary, there is a sequence $(v_{k,\ell})_{\ell=1}^\infty$ of functions belonging to $C^2(M_k)$ such that $v_{k,\ell} \to u$ in $W^{2,2}(\Omega_k)$. Then $v_{k,\ell} \to u$ in $L^2(\Omega_k)$, and thus, in particular (passing, if necessary, to a subsequence) $v_{k,\ell} \to u$ a.e. in Ω_k.

We recall that $W^{2,2}((0,1) \times (-1,0)) \hookrightarrow L^\infty((0,1) \times (-1,0))$ and denote by K the constant of the embedding. Thus, whenever $x \in [0,1] \times [-1,0]$, we have

$$|v_{k,\ell}(x)| \le K\|v_{k,\ell}\|_{W^{2,2}((0,1)\times(-1,0))} \le K\|v_{k,\ell}\|_{W^{2,2}(\Omega_k)}. \tag{6}$$

Further, if $x \in R_k$, then, by Lemma 3 and by (6), we obtain that

$$\begin{aligned} |v_{k,\ell}(x)| &\le |v_{k,\ell}(x) - v_{k,\ell}(\frac{a_k^3}{2} + 2\sum_{i=0}^{k-1} a_i, 0)| + |v_{k,\ell}(\frac{a_k^3}{2} + 2\sum_{i=0}^{k-1} a_i, 0)| \\ &\le 4\left(\left(\int_{2\sum_{i=0}^{k-1} a_i}^{a_k^3+2\sum_{i=0}^{k-1} a_i} |\nabla v_{k,\ell}(t,0)|^3\,dt\right)^{\frac{1}{3}} + \left(\int_{R_k} |\nabla^2 v_{k,\ell}|^2\right)^{\frac{1}{2}}\right) + K\|v_{k,\ell}\|_{W^{2,2}(\Omega_k)} \\ &\le 4\left(\left(\int_{\partial((0,1)\times(-1,0))} |\nabla v_{k,\ell}|^3\right)^{\frac{1}{3}} + \left(\int_{\Omega_k} |\nabla^2 v_{k,\ell}|^2\right)^{\frac{1}{2}}\right) + K\|v_{k,\ell}\|_{W^{2,2}(\Omega_k)} \\ &\le D\|v_{k,\ell}\|_{W^{2,2}(\Omega_k)}, \end{aligned} \tag{7}$$

where the last inequality also makes use of the embeddings $W^{1,2}(\Omega_k) \hookrightarrow W^{1,2}((0,1)\times(-1,0)) \hookrightarrow L^3(\partial((0,1) \times (-1,0)))$ applied to first-order derivatives of the function $v_{k,\ell}$. Note that the first of these embeddings holds with a constant independent of k (namely, with constant 1). Finally, if $x \in S_k$ then, by Lemma 1 and by (7), we have

$$\begin{aligned} |v_{k,\ell}(x)| &\le |v_{k,\ell}(x) - v_{k,\ell}(\frac{a_k^3}{2} + 2\sum_{i=0}^{k-1} a_i, a_k)| + |v_{k,\ell}(\frac{a_k^3}{2} + 2\sum_{i=0}^{k-1} a_i, a_k)| \\ &\le C\|v_{k,\ell}\|_{W^{2,2}(S_k)} + D\|v_{k,\ell}\|_{W^{2,2}(\Omega_k)} \\ &\le (C+D)\|v_{k,\ell}\|_{W^{2,2}(\Omega_k)}. \end{aligned} \tag{8}$$

Combining estimates (6), (7) and (8), we obtain that for every $x \in \Omega_k$,

$$|v_{k,\ell}(x)| \le (C+D)\|v_{k,\ell}\|_{W^{2,2}(\Omega_k)}.$$

Passing to limit when ℓ tends to infinity, this yields

$$|u(x)| \le (C+D)\|u\|_{W^{2,2}(\Omega_k)} \le (C+D)\|u\|_{W^{2,2}(\Omega)}$$

for a.e. $x \in \Omega_k$. Therefore, since $\Omega = \cup_{k=1}^\infty \Omega_k$,

$$\|u\|_{L^\infty(\Omega)} \le (C+D)\|u\|_{W^{2,2}(\Omega)},$$

which implies the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$.

We shall now construct a function $u \in W^{2,2}(\Omega)$ such that u^2 does not belong to $W^{2,2}(\Omega)$. This will prove that $W^{2,2}(\Omega)$ is not a Banach algebra.

We set

$$u(t,s)=\begin{cases}0, & (t,s)\in(0,1)\times(-1,0),\\ \frac{s^2}{2k^{\frac{3}{4}}a_k^2}, & (t,s)\in R_k,\\ \frac{2s-a_k}{2k^{\frac{3}{4}}a_k}, & (t,s)\in S_k.\end{cases}$$

It is not hard to verify that u is twice weakly differentiable on Ω,

$$|\nabla u(t,s)|=\frac{\partial u}{\partial x_2}(t,s)=\begin{cases}0, & (t,s)\in(0,1)\times(-1,0),\\ \frac{s}{k^{\frac{3}{4}}a_k^2}, & (t,s)\in R_k,\\ \frac{1}{k^{\frac{3}{4}}a_k}, & (t,s)\in S_k,\end{cases}$$

and

$$|\nabla^2 u(t,s)|=\frac{\partial^2 u}{\partial x_2^2}(t,s)=\begin{cases}0, & (t,s)\in(0,1)\times(-1,0),\\ \frac{1}{k^{\frac{3}{4}}a_k^2}, & (t,s)\in R_k,\\ 0, & (t,s)\in S_k.\end{cases}$$

Therefore, using also the estimate $a_k\leq\frac{1}{2}$, we obtain

$$\begin{aligned}\|u\|_{W^{2,2}(\Omega)}&=\|u\|_{L^2(\Omega)}+\||\nabla u|\|_{L^2(\Omega)}+\||\nabla^2 u|\|_{L^2(\Omega)}\\ &=\left(\sum_{k=1}^\infty\frac{a_k^4}{20k^{\frac{3}{2}}}+\sum_{k=1}^\infty\frac{13a_k^2}{12k^{\frac{3}{2}}}\right)^{\frac{1}{2}}+\left(\sum_{k=1}^\infty\frac{a_k^2}{3k^{\frac{3}{2}}}+\sum_{k=1}^\infty\frac{1}{k^{\frac{3}{2}}}\right)^{\frac{1}{2}}+\left(\sum_{k=1}^\infty\frac{1}{k^{\frac{3}{2}}}\right)^{\frac{1}{2}}\\ &\leq C'\left(\sum_{k=1}^\infty\frac{1}{k^{\frac{3}{2}}}\right)^{\frac{1}{2}}<\infty.\end{aligned}$$

Consequently, $u\in W^{2,2}(\Omega)$. Further,

$$\begin{aligned}\|u^2\|_{W^{2,2}(\Omega)}&\geq\left\|\frac{\partial^2 u^2}{\partial x_2^2}\right\|_{L^2(\Omega)}=\left\|2\left(\frac{\partial u}{\partial x_2}\right)^2+2u\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(\Omega)}\\ &\geq 2\left\|\left(\frac{\partial u}{\partial x_2}\right)^2\right\|_{L^2(\Omega)}\geq 2\left\|\left(\frac{\partial u}{\partial x_2}\right)^2\right\|_{L^2(\cup_{k=1}^\infty S_k)}\\ &\geq 2\left(\sum_{k=1}^\infty\frac{1}{k^3a_k^2}\right)^{\frac{1}{2}}\geq 2\left(\sum_{k=1}^\infty\frac{4}{k}\right)^{\frac{1}{2}}=\infty.\end{aligned}$$

Notice that the last inequality follows from the estimate

$$ka_k\leq\sum_{i=1}^k a_i\leq\sum_{i=1}^\infty a_i\leq\frac{1}{2}.$$

We have thus shown that u^2 does not belong to $W^{2,2}(\Omega)$. This completes the proof. □

Acknowledgment

I would like to thank Jan Malý for suggesting to me the problem of finding a Sobolev space embedded into L^∞ which is not a Banach algebra.

References

[1] R. A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.
[2] A. Cianchi. Orlicz-Sobolev algebras. *Potential Anal.*, 28(4):379–388, 2008.
[3] A. Cianchi, L. Pick, and L. Slavíková. Banach algebras of weakly differentiable functions. Preprint 2015.
[4] A. Kufner, O. John, and S. Fučík. *Function spaces.* Noordhoff International Publishing, Leyden; Academia, Prague, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.
[5] V. Maz'ya and Y. Netrusov. Some counterexamples for the theory of Sobolev spaces on bad domains. *Potential Anal.*, 4(1):47–65, 1995.

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: slavikova@karlin.mff.cuni.cz

Paper III

P. Cavaliere, A. Cianchi, L. Pick, and L. Slavíková. Norms supporting the Lebesgue differentiation theorem. Preprint, 2015.

NORMS SUPPORTING THE LEBESGUE DIFFERENTIATION THEOREM

PAOLA CAVALIERE, ANDREA CIANCHI, LUBOŠ PICK AND LENKA SLAVÍKOVÁ

Abstract. A version of the Lebesgue differentiation theorem is offered, where the L^p norm is replaced with any rearrangement-invariant norm. Necessary and sufficient conditions for a norm of this kind to support the Lebesgue differentiation theorem are established. In particular, Lorentz, Orlicz and other customary norms for which Lebesgue's theorem holds are characterized.

1. Introduction and main results

A standard formulation of the classical Lebesgue differentiation theorem asserts that, if $u \in L^1_{\rm loc}(\mathbb{R}^n)$, $n \geq 1$, then

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^n(B_r(x))} \int_{B_r(x)} u(y)\, d\mathcal{L}^n(y) \qquad \text{exists and is finite for a.e. } x \in \mathbb{R}^n, \tag{1.1}$$

where $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$, and $B_r(x)$ the ball, centered at x, with radius r. Here, and in what follows, "a.e." means "almost every" with respect to Lebesgue measure. In addition to (1.1), one has that

$$\lim_{r\to 0^+} \|u - u(x)\|^{\oslash}_{L^1(B_r(x))} = 0 \qquad \text{for a.e. } x \in \mathbb{R}^n, \tag{1.2}$$

where $\|\cdot\|^{\oslash}_{L^1(B_r(x))}$ stands for the averaged norm in $L^1(B_r(x))$ with respect to the normalized Lebesgue measure $\frac{1}{\mathcal{L}^n(B_r(x))}\mathcal{L}^n$. Namely,

$$\|u\|^{\oslash}_{L^1(B_r(x))} = \tfrac{1}{\mathcal{L}^n(B_r(x))} \int_{B_r(x)} |u(y)|\, d\mathcal{L}^n(y)$$

for $u \in L^1_{\rm loc}(\mathbb{R}^n)$.

A slight extension of this property ensures that an analogous conclusion holds if the L^1-norm in (1.2) is replaced with any L^p-norm, with $p \in [1,\infty)$. Indeed, if $u \in L^p_{\rm loc}(\mathbb{R}^n)$, then

$$\lim_{r\to 0^+} \|u - u(x)\|^{\oslash}_{L^p(B_r(x))} = 0 \qquad \text{for a.e. } x \in \mathbb{R}^n, \tag{1.3}$$

the averaged norm $\|\cdot\|^{\oslash}_{L^p(B_r(x))}$ being defined accordingly. By contrast, property (1.3) fails when $p = \infty$.

2000 *Mathematics Subject Classification.* 46E35, 46E30.

Key words and phrases. Lebesgue differentiation theorem, rearrangement-invariant spaces, Lorentz spaces, Orlicz spaces, Marcinkiewicz spaces.

This research was partly supported by the research project of MIUR (Italian Ministry of Education, University and Research) Prin 2012, n. 2012TC7588, "Elliptic and parabolic partial differential equations: geometric aspects, related inequalities, and applications", by GNAMPA of the Italian INdAM (National Institute of High Mathematics, by the grant P201/13/14743S of the Grant Agency of the Czech Republic, and by the research project GA UK No. 62315 of the Charles University in Prague.

The question thus arises of a characterization of those norms, defined on the space $L^0(\mathbb{R}^n)$ of measurable functions on $\mathbb{R}^n$, for which a version of the Lebesgue differentiation theorem continues to hold.

In the present paper we address this issue in the class of all rearrangement-invariant norms, i.e. norms which only depend on the "size" of functions, or, more precisely, on the measure of their level sets. A precise definition of this class of norms, as well as other notions employed hereafter, can be found in Section 2 below, where the necessary background material is collected.

Let us just recall here that, if $\|\cdot\|_{X(\mathbb{R}^n)}$ is a rearrangement-invariant norm, then

$$\|u\|_{X(\mathbb{R}^n)} = \|v\|_{X(\mathbb{R}^n)} \qquad \text{whenever } u^* = v^*, \tag{1.4}$$

where u^* and v^* denote the decreasing rearrangements of the functions $u, v \in L^0(\mathbb{R}^n)$. Moreover, given any norm of this kind, there exists another rearrangement-invariant function norm $\|\cdot\|_{\overline{X}(0,\infty)}$ on $L^0(0,\infty)$, called the representation norm of $\|\cdot\|_{X(\mathbb{R}^n)}$, such that

$$\|u\|_{X(\mathbb{R}^n)} = \|u^*\|_{\overline{X}(0,\infty)} \tag{1.5}$$

for every $u \in L^0(\mathbb{R}^n)$. By $X(\mathbb{R}^n)$ we denote the Banach function space, in the sense of Luxemburg, of all functions $u \in L^0(\mathbb{R}^n)$ such that $\|u\|_{X(\mathbb{R}^n)} < \infty$. Classical instances of rearrangement-invariant function norms are Lebesgue, Lorentz, Orlicz, and Marcinkiewicz norms.

In analogy with (1.3), a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ will be said to satisfy the Lebesgue point property if, for every $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\lim_{r\to 0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} = 0 \qquad \text{for a.e. } x \in \mathbb{R}^n. \tag{1.6}$$

Here, $\|\cdot\|^{\oslash}_{X(B_r(x))}$ denotes the norm on $X(B_r(x))$ with respect to the normalized Lebesgue measure $\frac{1}{\mathcal{L}^n(B_r(x))}\mathcal{L}^n$ – see (2.15), Section 2.

We shall exhibit necessary and sufficient conditions for $\|\cdot\|_{X(\mathbb{R}^n)}$ to enjoy the Lebesgue point property. To begin with, a necessary condition for $\|\cdot\|_{X(\mathbb{R}^n)}$ to satisfy the Lebesgue point property is to be locally absolutely continuous (Proposition 3.1, Section 3). This means that, for each function $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$, one has $\lim\limits_{j\to\infty} \|u\chi_{K_j}\|_{X(\mathbb{R}^n)} = 0$ for every non-increasing sequence $\{K_j\}$ of bounded measurable sets in $\mathbb{R}^n$ such that $\cap_{j\in\mathbb{N}} K_j = \emptyset$.
The local absolute continuity of $\|\cdot\|_{X(\mathbb{R}^n)}$ is in turn equivalent to the local separability of $X(\mathbb{R}^n)$, namely to the separability of each subspace of $X(\mathbb{R}^n)$ consisting of all functions which are supported in any given bounded measurable subset of $\mathbb{R}^n$.

As will be clear from applications of our results to special instances, this necessary assumption is not yet sufficient. In order to ensure the Lebesgue point property for $\|\cdot\|_{X(\mathbb{R}^n)}$, it has to be complemented with an additional assumption on the functional $\mathcal{G}_X$, associated with the representation norm $\|\cdot\|_{\overline{X}(0,\infty)}$, and defined as

$$\mathcal{G}_X(f) = \|f^{-1}\|_{\overline{X}(0,\infty)} \tag{1.7}$$

for every non-increasing function $f : [0,\infty) \to [0,\infty]$. Here, $f^{-1} : [0,\infty) \to [0,\infty]$ denotes the (generalized) right-continuous inverse of f. Such an assumption amounts to requiring that $\mathcal{G}_X$ be "almost concave". By this expression, we mean that the functional $\mathcal{G}_X$, restricted to the convex set $\mathcal{C}$ of all non-increasing functions from $[0,\infty)$ into $[0,1]$, fulfils the inequality in the definition of concavity possibly up to a multiplicative positive constant c. Namely,

$$c\sum_{i=1}^{k} \lambda_i \mathcal{G}_X(f_i) \le \mathcal{G}_X\bigg(\sum_{i=1}^{k} \lambda_i f_i\bigg) \tag{1.8}$$

for any numbers $\lambda_i \in (0,1)$, $i = 1,\dots,k$, $k \in \mathbb{N}$, such that $\sum_{i=1}^k \lambda_i = 1$, and any functions $f_i \in \mathcal{C}$, $i = 1,\dots,k$. Clearly, the functional $\mathcal{G}_X$ is concave on $\mathcal{C}$, in the usual sense, if inequality (1.8) holds with $c = 1$.

Theorem 1.1. *A rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, it is locally absolutely continuous and the functional $\mathcal{G}_X$ is almost concave.*

Remark 1.2. In order to give an idea of how the functional $\mathcal{G}_X$ looks like in classical instances, consider the case when $\|\cdot\|_{X(\mathbb{R}^n)} = \|\cdot\|_{L^p(\mathbb{R}^n)}$. One has that

$$\mathcal{G}_{L^p}(f) = \begin{cases} \left(p\int_0^\infty s^{p-1} f(s)\, d\mathcal{L}^1(s)\right)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \\ \mathcal{L}^1(\{s \in [0,\infty) : f(s) > 0\}) & \text{if } p = \infty\,, \end{cases}$$

for every non-increasing function $f : [0,\infty) \to [0,\infty]$. The functional $\mathcal{G}_{L^p}$ is concave for every $p \in [1,\infty]$. However, $\|\cdot\|_{L^p(\mathbb{R}^n)}$ is locally absolutely continuous only for $p < \infty$.

Remark 1.3. The local absolute continuity of a rearrangement invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ and the almost concavity of the functional $\mathcal{G}_X$ are independent properties. For instance, as noticed in the previous remark, the norm $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$ is not locally absolutely continuous, although the functional $\mathcal{G}_{L^\infty}$ is concave. On the other hand, whenever $q < \infty$, the Lorentz norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ is locally absolutely continuous , but $\mathcal{G}_{L^{p,q}}$ is almost concave if and only if $q \le p$. The Luxemburg norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ in the Orlicz space $L^A(\mathbb{R}^n)$ is almost concave for every N-function A, but is locally absolutely continuous if and only if A satisfies the Δ_2-condition near infinity. These properties are established in Section 6 below, where the validity of the Lebesgue point property for various classes of norms is discussed.

An alternative characterization of the rearrangement-invariant norms satisfying the Lebesgue point property involves a maximal function operator associated with the norms in question. The relevant operator, denoted by $\mathcal{M}_X$, is defined, at each $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$, as

$$\mathcal{M}_X u(x) = \sup_{B \ni x} \|u\|^{\oslash}_{X(B)} \qquad \text{for } x \in \mathbb{R}^n, \tag{1.9}$$

where B stands for any ball in $\mathbb{R}^n$.

In the case when $X(\mathbb{R}^n) = L^1(\mathbb{R}^n)$, the operator $\mathcal{M}_X$ coincides with the classical Hardy-Littlewood maximal operator $\mathcal{M}$. It is well known that $\mathcal{M}$ is of weak type from $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Moreover, since

$$\|u^*\|^{\oslash}_{L^1(0,s)} = \frac{1}{s}\int_0^s u^*(t)\, d\mathcal{L}^1(t) \qquad \text{for } s \in (0,\infty),$$

for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the celebrated Riesz-Wiener inequality takes the form

$$(\mathcal{M}u)^*(s) \le C\|u^*\|^{\oslash}_{L^1(0,s)} \qquad \text{for } s \in (0,\infty),$$

for some constant $C = C(n)$ [4, Theorem 3.8, Chapter 3].

The validity of the Lebesgue point property for a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ turns out to be intimately connected to a suitable version of these two results for the maximal operator $\mathcal{M}_X$ defined by (1.9). This is the content of our next result, whose statement makes use of a notion of weak-type operators between local rearrangement-invariant spaces. We say that $\mathcal{M}_X$ is of weak type from $X_{\mathrm{loc}}(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if for every bounded measurable set $K \subseteq \mathbb{R}^n$, there exists a constant $C = C(K)$ such that

$$\mathcal{L}^n(\{x \in K : \mathcal{M}_X u(x) > t\}) \le \frac{C}{t}\|u\|_{X(\mathbb{R}^n)} \qquad \text{for } t \in (0,\infty), \tag{1.10}$$

for every function $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$ whose support is contained in K.

Theorem 1.4. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Then the following statements are equivalent:*

(i) $\|\cdot\|_{X(\mathbb{R}^n)}$ *satisfies the Lebesgue point property;*

(ii) $\|\cdot\|_{X(\mathbb{R}^n)}$ *is locally absolutely continuous, and the Riesz-Wiener type inequality*

$$(\mathcal{M}_X u)^*(s) \leq C\|u^*\|^{\oslash}_{\overline{X}(0,s)} \qquad \text{for } s \in (0,\infty), \tag{1.11}$$

holds for some positive constant C, and for every $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$;

(iii) $\|\cdot\|_{X(\mathbb{R}^n)}$ *is locally absolutely continuous, and the operator $\mathcal{M}_X$ is of weak type from $X_{\mathrm{loc}}(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Remark 1.5. The local absolute continuity of the norm $\|\cdot\|_{X(\mathbb{R}^n)}$ is an indispensable hypothesis in both conditions (ii) and (iii) of Theorem 1.4. Indeed, its necessity is already known from Theorem 1.1, and, on the other hand, it does not follow from the other assumptions in (ii) or (iii). For instance, both these assumptions are fulfilled by the rearrangement-invariant norm $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$, which, however, is not locally absolutely continuous, and, in fact, does not satisfy the Lebesgue point property.

Remark 1.6. Riesz-Wiener type inequalities for special classes of rearrangement-invariant norms have been investigated in the literature – see e.g. [3, 2, 11, 12]. In particular, in [2] inequality (1.11) is shown to hold when $\|\cdot\|_{X(\mathbb{R}^n)}$ is an Orlicz norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ associated with any Young function A. The case of Lorentz norms $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ is treated in [3], where it is proved that (1.11) holds if, and only if, $1 \leq q \leq p$. In fact, a different notion of maximal operator is considered in [3], which, however, is equivalent to (1.9) when $\|\cdot\|_{X(\mathbb{R}^n)}$ is a Lorentz norm, as is easily seen from [7, Equation (3.7)].
A simple sufficient condition for the validity of the Riesz-Wiener type inequality for very general maximal operators is proposed in [12]. In our framework, where maximal operators built upon rearrangement-invariant norms are taken into account, such condition turns out to be also necessary, as will be shown in Proposition 4.2. The approach introduced in [12] leads to alternative proofs of the Riesz-Wiener type inequality for Orlicz and Lorentz norms, and was also used in [14] to prove the validity of (1.11) for further families of rearrangement-invariant norms, including, in particular, all Lorentz endpoint norms $\|\cdot\|_{\Lambda_\varphi(\mathbb{R}^n)}$. A kind of rearrangement inequality for the maximal operator built upon these Lorentz norms already appears in [11].
Results on weak type boundedness of the maximal operator $\mathcal{M}_X$ are available in the literature as well [1, 8, 13, 15, 22]. For instance, in [22] it is pointed out that the operator $\mathcal{M}_{L^{p,q}}$ is of weak type from $L^{p,q}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, if $1 \leq q \leq p$, and hence, in particular, it is of weak type from $L^{p,q}_{\mathrm{loc}}(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Our last main result provides us with necessary and sufficient conditions for the Lebesgue point property of a rearrangement-invariant norm which do not make explicit reference to the local absolute continuity of the relevant norm.

Theorem 1.7. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Then the following statements are equivalent:*

(i) $\|\cdot\|_{X(\mathbb{R}^n)}$ *satisfies the Lebesgue point property;*

(ii) *For every function $u \in X(\mathbb{R}^n)$, supported in a set of finite measure,*

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon \mathcal{M}_X u(x) > 1\}) < \infty;$$

(iii) *For every function $u \in X(\mathbb{R}^n)$, supported in a set of finite measure,*

$$\lim_{s\to\infty} (\mathcal{M}_X u)^*(s) = 0.$$

Theorems 1.1, 1.4 and 1.7 enable us to characterize the validity of the Lebesgue point property in customary classes of rearrangement-invariant norms.

The following proposition deals with the case of standard Lorentz norms $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$.

Proposition 1.8. *The Lorentz norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, $1 \le q \le p < \infty$.*

Since $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, Proposition 1.8 recovers, in particular, the standard result, mentioned above, that the norm $\|\cdot\|_{L^p(\mathbb{R}^n)}$ enjoys the Lebesgue point property if, and only if, $1 \le p < \infty$.

This fact is also reproduced by the following proposition, which concerns Orlicz norms $\|\cdot\|_{L^A(\mathbb{R}^n)}$ built upon a Young function A.

Proposition 1.9. *The Orlicz norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, the Young function A satisfies the Δ_2-condition near infinity.*

The last two results concern the so called Lorentz and Marcinkiewicz endpoint norms $\|\cdot\|_{\Lambda_\varphi(\mathbb{R}^n)}$ and $\|\cdot\|_{M_\varphi(\mathbb{R}^n)}$, respectively, associated with a (non identically vanishing) concave function $\varphi : [0,\infty) \to [0,\infty)$.

Proposition 1.10. *The Lorentz norm $\|\cdot\|_{\Lambda_\varphi(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, $\lim\limits_{s\to 0^+} \varphi(s) = 0$.*

Proposition 1.11. *The Marcinkiewicz norm $\|\cdot\|_{M_\varphi(\mathbb{R}^n)}$ satisfies the Lebesgue point property if, and only if, $\lim\limits_{s\to 0^+} \frac{s}{\varphi(s)} > 0$, namely, if and only if, $(M_\varphi)_{\mathrm{loc}}(\mathbb{R}^n) = L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

When the present paper was almost in final form, it was pointed out to us by A. Gogatishvili that the Lebesgue point property of rearrangement-invariant spaces has also been investigated in [5, 6, 17, 19]. The analysis of those papers is however limited to the case of functions of one variable. Moreover, the characterizations of those norms having Lebesgue point property that are proved there are less explicit, and have a somewhat more technical nature.

2. Background

In this section we recall some definitions and basic properties of decreasing rearrangements and rearrangement-invariant function norms. For more details and proofs, we refer to [4, 16].

Let E be a Lebesgue-measurable subset of $\mathbb{R}^n$, $n \ge 1$. The Riesz space of measurable functions from E into $[-\infty,\infty]$ is denoted by $L^0(E)$. We also set $L^0_+(E) = \{u \in L^0(E) : u \ge 0 \text{ a.e. in } E\}$, and $L^0_0(E) = \{u \in L^0(E) : u \text{ is finite a.e. in } E\}$. The *distribution function* $u_* : [0,\infty) \to [0,\infty]$ and the *decreasing rearrangement* $u^* : [0,\infty) \to [0,\infty]$ of a function $u \in L^0(E)$ are defined by

$$u_*(t) = \mathcal{L}^n(\{y \in E : |u(y)| > t\}) \qquad \text{for } t \in [0,\infty), \tag{2.1}$$

and by

$$u^*(s) = \inf\{t \ge 0 : u_*(t) \le s\} \qquad \text{for } s \in [0,\infty), \tag{2.2}$$

respectively.

The Hardy-Littlewood inequality tells us that

$$\int_E |u(y)v(y)|\, d\mathcal{L}^n(y) \le \int_0^\infty u^*(s)v^*(s)\, d\mathcal{L}^1(s) \tag{2.3}$$

for every $u, v \in L^0(E)$. The function $u^{**} : (0,\infty) \to [0,\infty]$, given by

$$u^{**}(s) = \frac{1}{s}\int_0^s u^*(t)d\mathcal{L}^1(t) \qquad \text{for } s \in (0,\infty), \tag{2.4}$$

is non-increasing and satisfies $u^* \le u^{**}$. Moreover,

$$(u+v)^{**} \le u^{**} + v^{**} \tag{2.5}$$

for every $u, v \in L^0_+(E)$.

A *rearrangement-invariant norm* is a functional $\|\cdot\|_{X(E)} : L^0(E) \to [0,\infty]$ such that

(N1): $\|u+v\|_{X(E)} \le \|u\|_{X(E)} + \|v\|_{X(E)}$ for all $u, v \in L^0_+(E)$;
$\|\lambda u\|_{X(E)} = |\lambda| \, \|u\|_{X(E)}$ for all $\lambda \in \mathbb{R}$, $u \in L^0(E)$;
$\|u\|_{X(E)} > 0$ if u does not vanish a.e. in E;
(N2): $\|u\|_{X(E)} \le \|v\|_{X(E)}$ whenever $0 \le u \le v$ a.e. in E;
(N3): $\sup_k \|u_k\|_{X(E)} = \|u\|_{X(E)}$ if $\{u_k\} \subset L^0_+(E)$ with $u_k \nearrow u$ a.e. in E;
(N4): $\|\chi_G\|_{X(E)} < \infty$ for every measurable set $G \subseteq E$, such that $\mathcal{L}^n(G) < \infty$;
(N5): for every measurable set $G \subseteq E$, with $\mathcal{L}^n(G) < \infty$, there exists a positive constant $C(G)$ such that $\|u\|_{L^1(G)} \le C(G) \, \|u\chi_G\|_{X(E)}$ for all $u \in L^0(E)$;
(N6): $\|u\|_{X(E)} = \|v\|_{X(E)}$ for all $u, v \in L^0(E)$ such that $u^* = v^*$.

The functional $\|\cdot\|_{X(E)}$ is a norm in the standard sense when restricted to the set

$$X(E) = \{u \in L^0(E) : \|u\|_{X(E)} < \infty\}. \tag{2.6}$$

The latter is a Banach space endowed with such norm, and is called a rearrangement-invariant Banach function space, briefly, a *rearrangement-invariant space*.

Given a measurable subset E' of E and a function $u \in L^0(E')$, define the function $\widehat{u} \in L^0(E)$ as

$$\widehat{u}(x) = \begin{cases} u(x) & \text{if } x \in E', \\ 0 & \text{if } x \in E \setminus E'. \end{cases}$$

Then the functional $\|\cdot\|_{X(E')}$ given by

$$\|u\|_{X(E')} = \|\widehat{u}\|_{X(E)}$$

for $u \in L^0(E')$ is a rearrangement-invariant norm.

If $\mathcal{L}^n(E) < \infty$, then

$$L^\infty(E) \to X(E) \to L^1(E), \tag{2.7}$$

where $\to$ stands for a continuous embedding.
The local r.i. space $X_{\mathrm{loc}}(E)$ is defined as

$$X_{\mathrm{loc}}(E) = \{u \in L^0(E) : \ u\chi_K \in X(E) \ \text{for every bounded measurable set } K \subset E\}.$$

The *fundamental function* of $X(E)$ is defined by

$$\varphi_{X(E)}(s) = \|\chi_G\|_{X(E)} \qquad \text{for } s \in [0, \mathcal{L}^n(E)), \tag{2.8}$$

where G is any measurable subset of E such that $\mathcal{L}^n(G) = s$. It is non-decreasing on $[0, \mathcal{L}^n(E))$, $\varphi_{X(E)}(0) = 0$ and $\varphi_{X(E)}(s)/s$ is non-increasing for $s \in (0, \mathcal{L}^n(E))$.
Hardy's Lemma tells us that, given $u, v \in L^0(E)$ and any rearrangement-invariant norm $\|\cdot\|_{X(E)}$,

$$\text{if } u^{**} \le v^{**}, \ \text{then } \|u\|_{X(E)} \le \|v\|_{X(E)}. \tag{2.9}$$

The *associate rearrangement-invariant norm* of $\|\cdot\|_{X(E)}$ is the rearrangement-invariant norm $\|\cdot\|_{X'(E)}$ defined by

$$\|v\|_{X'(E)} = \sup\Big\{ \int_E |u(y)v(y)| \, d\mathcal{L}^n(y) : \ u \in L^0(E), \ \|u\|_{X(E)} \le 1 \Big\}. \tag{2.10}$$

The corresponding rearrangement-invariant space $X'(E)$ is called the *associate space* of $X(E)$. The Hölder type inequality

$$\int_E |u(y)v(y)|\,d\mathcal{L}^n(y) \leq \|u\|_{X(E)}\|v\|_{X'(E)} \tag{2.11}$$

holds for every $u \in X(E)$ and $v \in X'(E)$. One has that $X(E) \ = X''(E)$.

The rearrangement-invariant norm, defined as

$$\|f\|_{\overline{X}(0,\mathcal{L}^n(E))} = \sup_{\|u\|_{X'(E)}\leq 1} \int_0^\infty f^*(s)u^*(s)\,d\mathcal{L}^1(s)$$

for $f \in L^0(0,\mathcal{L}^n(E))$, is a *representation norm* for $\|\cdot\|_{X(E)}$. It has the property that

$$\|u\|_{X(E)} = \|u^*\|_{\overline{X}(0,\mathcal{L}^n(E))} \tag{2.12}$$

for every $u \in X(E)$. For customary rearrangement-invariant norms, an expression for $\|\cdot\|_{\overline{X}(0,\mathcal{L}^n(E))}$ is immediately derived from that of $\|\cdot\|_{X(E)}$.

The *dilation operator* $D_\delta : \overline{X}(0,\mathcal{L}^n(E)) \to \overline{X}(0,\mathcal{L}^n(E))$ is defined for $\delta > 0$ and $f \in \overline{X}(0,\mathcal{L}^n(E))$ as

$$(D_\delta f)(s) = \begin{cases} f(s\delta) & \text{if } s\delta \in (0,\mathcal{L}^n(E)), \\ 0 & \text{otherwise,} \end{cases} \tag{2.13}$$

and is bounded [4, Chap. 3, Proposition 5.11].

We shall make use of the subspace $\overline{X}_1(0,\infty)$ of $\overline{X}(0,\infty)$ defined as

$$\overline{X}_1(0,\infty) = \{f \in \overline{X}(0,\infty) :\, f(s) = 0 \text{ for a.e. } s > 1\}. \tag{2.14}$$

Now, assume that E is a measurable positive cone in $\mathbb{R}^n$ with vertex at 0, namely, a measurable set which is closed under multiplication by positive scalars. In what follows, we shall focus the nontrivial case when $\mathcal{L}^n(E)$ does not vanish, and hence $\mathcal{L}^n(E) = \infty$. Let $\|\cdot\|_{X(E)}$ be a rearrangement-invariant norm, and let G be a measurable subset of E such that $0 < \mathcal{L}^n(G) < \infty$. We define the functional $\|\cdot\|^{\oslash}_{X(G)}$ as

$$\|u\|^{\oslash}_{X(G)} = \big\|(u\chi_G)(\sqrt[n]{\mathcal{L}^n(G)}\ \cdot)\big\|_{X(E)} \tag{2.15}$$

for $u \in L^0(E)$. We call it the *averaged norm of* $\|\cdot\|_{X(E)}$ *on* G, since

$$\|u\|^{\oslash}_{X(G)} = \|u\chi_G\|_{X\left(G,\frac{\mathcal{L}^n}{\mathcal{L}^n(G)}\right)} \tag{2.16}$$

for $u \in L^0(E)$, where $\|\cdot\|_{X\left(G,\frac{\mathcal{L}^n}{\mathcal{L}^n(G)}\right)}$ denotes the rearrangement-invariant norm, defined as $\|\cdot\|_{X(G)}$, save that the Lebesgue measure $\mathcal{L}^n$ is replaced with the normalized Lebesgue measure $\frac{\mathcal{L}^n}{\mathcal{L}^n(G)}$. Notice that

$$\|u\|^{\oslash}_{X(G)} = \|(u\chi_G)^*(\mathcal{L}^n(G)\,\cdot)\|_{\overline{X}(0,\infty)} \tag{2.17}$$

for $u \in L^0(E)$. Moreover,

$$\|1\|^{\oslash}_{X(G)} \quad \text{is independent of } G. \tag{2.18}$$

The Hölder type inequality for averaged norms takes the form:

$$\frac{1}{\mathcal{L}^n(G)}\int_G |u(y)v(y)|\,d\mathcal{L}^n(y) \leq \|u\|^{\oslash}_{X(G)}\ \|v\|^{\oslash}_{X'(G)} \tag{2.19}$$

for $u, v \in L^0(E)$.

We conclude this section by recalling the definition of some customary, and less standard, instances of rearrangement-invariant function norms of use in our applications. In what follows, we set $p' = \frac{p}{p-1}$ for $p \in (1,\infty)$, with the usual modifications when $p = 1$ and $p = \infty$. We also adopt the convention that $1/\infty = 0$.
Prototypal examples of rearrangement-invariant function norms are the classical Lebesgue norms. Indeed, $\|u\|_{L^p(\mathbb{R}^n)} = \|u^*\|_{L^p(0,\infty)}$, if $p \in [1,\infty)$, and $\|u\|_{L^\infty(\mathbb{R}^n)} = u^*(0)$.

Let $p, q \in [1,\infty]$. Assume that either $1 < p < \infty$ and $1 \le q \le \infty$, or $p = q = 1$, or $p = q = \infty$. Then the functional defined as

$$\|u\|_{L^{p,q}(\mathbb{R}^n)} = \|s^{\frac{1}{p}-\frac{1}{q}}\, u^*(s)\|_{L^q(0,\infty)} \tag{2.20}$$

for $u \in L^0(\mathbb{R}^n)$, is equivalent (up to multiplicative constants) to a rearrangement-invariant norm. The corresponding rearrangement-invariant space is called a *Lorentz space*. Note that $\|\cdot\|_{L^{p,q}(0,\infty)}$ is the representation norm for $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$, and $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Moreover, $L^{p,q}(\mathbb{R}^n) \to L^{p,r}(\mathbb{R}^n)$ if $1 \le q < r \le \infty$.

Let A be a Young function, namely a left-continuous convex function from $[0,\infty)$ into $[0,\infty]$, which is neither identically equal to 0, nor to ∞. The *Luxemburg rearrangement-invariant norm* associated with A is defined as

$$\|u\|_{L^A(\mathbb{R}^n)} = \inf\left\{\lambda > 0 \ : \ \int_{\mathbb{R}^n} A\left(\frac{|u(x)|}{\lambda}\right) d\mathcal{L}^n(x) \ \le \ 1\right\} \tag{2.21}$$

for $u \in L^0(\mathbb{R}^n)$. Its representation norm is $\|u\|_{L^A(0,\infty)}$. The space $L^A(\mathbb{R}^n)$ is called an *Orlicz space*. In particular, $L^A(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $A(t) = t^p$ for $p \in [1,\infty)$, and $L^A(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ if $A(t) = \infty\chi_{(1,\infty)}(t)$.
Recall that A is said to satisfy the Δ_2*–condition near infinity* if it is finite valued and there exist constants $C > 0$ and $t_0 \ge 0$ such that

$$A(2t) \le CA(t) \qquad \text{for } t \in [t_0,\infty)\,. \tag{2.22}$$

If A satisfies the Δ_2–condition near infinity, and $u \in L^A(\mathbb{R}^n)$ has support of finite measure, then

$$\int_{\mathbb{R}^n} A(c|u(x)|)d\mathcal{L}^n(x) < \infty$$

for every positive number c.
A subclass of Young functions which is often considered in the literature is that of the so called N-functions. A Young function A is said to be an N-function if it is finite-valued, and

$$\lim_{t\to 0^+}\frac{A(t)}{t} = 0 \qquad \lim_{t\to\infty}\frac{A(t)}{t} = \infty\,.$$

Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave function which does not vanish identically. Hence, in particular, φ is non-decreasing, and $\varphi(t) > 0$ for $t \in (0,\infty)$. The *Marcinkiewicz* and *Lorentz endpoint norm* associated with φ are defined as

$$\|u\|_{M_\varphi(\mathbb{R}^n)} = \sup_{s\in(0,\infty)} u^{**}(s)\varphi(s), \tag{2.23}$$

$$\|u\|_{\Lambda_\varphi(\mathbb{R}^n)} = \int_0^\infty u^*(s)d\varphi(s), \tag{2.24}$$

for $u \in L^0(\mathbb{R}^n)$, respectively. The representation norms are $\|\cdot\|_{M_\varphi(0,\infty)}$ and $\|\cdot\|_{\Lambda_\varphi(0,\infty)}$, respectively. The spaces $M_\varphi(\mathbb{R}^n)$ and $\Lambda_\varphi(\mathbb{R}^n)$ are called *Marcinkiewicz endpoint space* and *Lorentz* *endpoint space* associated with φ. The fundamental functions of $M_\varphi(\mathbb{R}^n)$ and $\Lambda_\varphi(\mathbb{R}^n)$ coincide with φ. In fact, $M_\varphi(\mathbb{R}^n)$ and $\Lambda_\varphi(\mathbb{R}^n)$ are respectively the largest and the smallest rearrangement-invariant space whose fundamental function is φ, and this accounts for the expression "endpoint" which is usually attached to their names. Note the alternative expression

$$\|f\|_{\Lambda_\varphi(0,\infty)} = f^*(0)\varphi(0^+) + \int_0^\infty f^*(s)\varphi'(s)d\mathcal{L}^1(s), \tag{2.25}$$

for $f \in L^0(0,\infty)$, where $\varphi(0^+) = \lim_{s\to 0^+} \varphi(s)$.

3. A necessary condition: local absolute continuity

In the present section we are mainly concerned with a proof of the following necessary conditions for a rearrangement-invariant norm to satisfy the Lebesgue point property.

Proposition 3.1. *If $\|\cdot\|_{X(\mathbb{R}^n)}$ is a rearrangement-invariant norm satisfying the Lebesgue point property, then:*

(i) *$\|\cdot\|_{X(\mathbb{R}^n)}$ is locally absolutely continuous;*
(ii) *$X(\mathbb{R}^n)$ is locally separable.*

The proof of Proposition 3.1 is split in two steps, which are the content of the next two lemmas.

Lemma 3.2. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm which satisfies the Lebesgue point property. Then:*

(H) *Given any function $f \in \overline{X}_1(0,\infty)$, any sequence $\{I_k\}$ of pairwise disjoint intervals in $(0,1)$, and any sequence $\{a_k\}$ of positive numbers such that $a_k \geq \mathcal{L}^1(I_k)$ and*

$$\|(f\chi_{I_k})^*\|^{\oslash}_{\overline{X}(0,a_k)} > 1, \tag{3.1}$$

one has that

$$\sum_{k=1}^\infty a_k < \infty.$$

Let us stress in advance that condition (H) is not only necessary, but also sufficient for a rearrangement-invariant norm to satisfy the Lebesgue point property. This is a consequence of Proposition 5.1, Section 5, and of the following lemma.

Lemma 3.3. *If a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$ fulfills condition (H) of Lemma 3.2, then $\|\cdot\|_{X(\mathbb{R}^n)}$ is locally absolutely continuous.*

The proof of Lemma 3.2 in turn exploits the following property, which will also be of later use.

Lemma 3.4. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Given any function $f \in \overline{X}(0,\infty)$, the function $F : (0,\infty) \to [0,\infty)$, defined as*

$$F(r) = r\,\|f^*\|^{\oslash}_{\overline{X}(0,r)} \qquad \text{for } r \in (0,\infty), \tag{3.2}$$

is non-decreasing on $(0,\infty)$, and the function $\frac{F(r)}{r}$ is non-increasing on $(0,\infty)$. In particular, the function F is continuous on $(0,\infty)$.

Proof. Let $0 < r_1 < r_2$. An application of (2.10) tells us that

$$\begin{aligned} F(r_1) =\ & r_1 \|(f^*\chi_{(0,r_1)})(r_1 \cdot)\|_{\overline{X}(0,\infty)} = r_1 \sup_{\|g\|_{\overline{X}'(0,\infty)}\le 1} \int_0^1 g^*(s) f^*(r_1 s)\, d\mathcal{L}^1(s) \\ =\ & \sup_{\|g\|_{\overline{X}'(0,\infty)}\le 1} \int_0^{r_1} g^*\left(\tfrac{t}{r_1}\right) f^*(t)\, d\mathcal{L}^1(t) \le \sup_{\|g\|_{\overline{X}'(0,\infty)}\le 1} \int_0^{r_2} g^*\left(\tfrac{t}{r_1}\right) f^*(t)\, d\mathcal{L}^1(t) \\ \le\ & \sup_{\|g\|_{\overline{X}'(0,\infty)}\le 1} \int_0^{r_2} g^*\left(\tfrac{t}{r_2}\right) f^*(t)\, d\mathcal{L}^1(t) = r_2 \sup_{\|g\|_{\overline{X}'(0,\infty)}\le 1} \int_0^1 g^*(s) f^*(r_2 s)\, d\mathcal{L}^1(s) \\ =\ & r_2 \|(f^*\chi_{(0,r_2)})(r_2 \cdot)\|_{\overline{X}(0,\infty)} = F(r_2). \end{aligned}$$

Namely, F is non-decreasing on $(0,\infty)$. The fact that the function $\frac{F(r)}{r}$ is non-increasing on $(0,\infty)$ is a consequence of property (N2) and of the inequality

$$f^*(r_1 \cdot)\chi_{(0,r_1)}(r_1 \cdot) \ge f^*(r_2 \cdot)\chi_{(0,r_2)}(r_2 \cdot)$$

if $0 < r_1 < r_2$. Hence, in particular, the function F is continuous on $(0,\infty)$ (see e.g. [10, Chapter 2, p. 49]). $\square$

Proof of Lemma 3.2. Assume that $\|\cdot\|_{X(\mathbb{R}^n)}$ satisfies the Lebesgue point property. Suppose, by contradiction, that condition (H) fails, namely, there exist a function $f \in \overline{X}_1(0,\infty)$, a sequence $\{I_k\}$ of pairwise disjoint intervals in $(0,1)$ and a sequence $\{a_k\}$ of positive numbers, with $a_k \ge \mathcal{L}^1(I_k)$, fulfilling (3.1) and such that $\sum_{k=1}^\infty a_k = \infty$.

We may assume, without loss of generality, that

$$\lim_{k\to\infty} a_k = 0. \tag{3.3}$$

Indeed, if (3.3) fails, then the sequence $\{a_k\}$ can be replaced with another sequence, enjoying the same properties, and also (3.3). To verify this assertion, note that, if (3.3) does not hold, then there exist $\varepsilon > 0$ and a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that $a_{k_j} \ge \varepsilon$ for all j. Consider the sequence $\{b_j\}$, defined as

$$b_j = \max\left\{\tfrac{\varepsilon}{j},\ \mathcal{L}^1(I_{k_j})\right\} \quad \text{for } j \in \mathbb{N}. \tag{3.4}$$

Equation (3.4) immediately tells us that $b_j \ge \mathcal{L}^1(I_{k_j})$, and $\sum_{j=1}^\infty b_j = \infty$. Moreover, Lemma 3.4 and the inequality $b_j \le a_{k_j}$ for $j \in \mathbb{N}$ ensure that (3.1) holds with a_k and I_k replaced by b_j and I_{k_j}, respectively. Finally, $\lim\limits_{j\to\infty} b_j = 0$, since $\sum_{j=1}^\infty \mathcal{L}^1(I_{k_j}) \le 1$, and hence $\lim\limits_{j\to\infty} \mathcal{L}^1(I_{k_j}) = 0$.

Moreover, by skipping, if necessary, a finite number of terms in the relevant sequences, we may also assume that

$$\sum_{k=1}^\infty \mathcal{L}^1(I_k) < 1\,. \tag{3.5}$$

Now, set $a_0 = 0$, and $J_k = (\sum_{j=0}^{k-1} a_j, \sum_{j=0}^{k} a_j)$ for each $k \in \mathbb{N}$. We define the function $g : (0,\infty) \to [0,\infty)$ as

$$g(s) = \sum_{k=1}^\infty (f\chi_{I_k})^*\Big(s - \sum_{j=0}^{k-1} a_j\Big)\chi_{J_k}(s) \qquad \text{for } s \in (0,\infty),$$

and the function $u : \mathbb{R}^n \to [0,\infty)$ as

$$u(y) = \sup_{k\in\mathbb{N}} g(y_1 + k - 1)\, \chi_{(0,1)^n}(y) \qquad \text{for } y = (y_1,\dots,y_n) \in \mathbb{R}^n.$$

The function u belongs to $X(\mathbb{R}^n)$. To verify this fact, note that

$$\mathcal{L}^n(\{y \in \mathbb{R}^n : u(y) > t\}) \le \mathcal{L}^1(\{s \in (0,\infty) : |f\chi_{\cup_{k\in\mathbb{N}} I_k}|(s) > t\}) \tag{3.6}$$

for every $t \ge 0$. Indeed, thanks to the equimeasurability of g and $f\chi_{\cup_{k\in\mathbb{N}} I_k}$,

$$\begin{aligned}
\mathcal{L}^n(\{y \in \mathbb{R}^n : u(y) > t\}) &= \mathcal{L}^1(\{s \in (0,1) : \sup_{k\in\mathbb{N}} g(s+k-1) > t\}) \\
&= \mathcal{L}^1(\cup_{k\in\mathbb{N}}\{s \in (0,1) : g(s+k-1) > t\}) \le \sum_{k=1}^{\infty} \mathcal{L}^1(\{s \in (0,1) : g(s+k-1) > t\}) \\
&= \mathcal{L}^1(\{s \in (0,\infty) : g(s) > t\}) = \mathcal{L}^1(\{s \in (0,\infty) : |f\chi_{\cup_{k\in\mathbb{N}} I_k}|(s) > t\}).
\end{aligned}$$

From (3.6) it follows that

$$\|u\|_{X(\mathbb{R}^n)} \le \|f\chi_{\cup_{k\in\mathbb{N}} I_k}\|_{\overline{X}(0,\infty)} \le \|f\|_{\overline{X}(0,\infty)} < \infty\,,$$

whence $u \in X(\mathbb{R}^n)$. Next, one has that

$$\limsup_{r\to 0^+} \|u\|^{\oslash}_{X(B_r(x))} > 0 \qquad \text{for a.e. } x \in (0,1)^n. \tag{3.7}$$

To prove (3.7), set

$$\Lambda_k = \{l \in \mathbb{N} : J_l \subseteq [k-1,k]\} \qquad \text{for } k \in \mathbb{N}.$$

Since $\sum_{l=0}^{\infty} a_l = \infty$ and $\lim_{l\to\infty} |J_l| = \lim_{l\to\infty} a_l = 0$, we have that

$$\lim_{k\to\infty} \bigcup_{l\in\Lambda_k} (\overline{J_l} - k + 1) = (0,1), \tag{3.8}$$

where $\overline{J_l}$ denotes the closure of the open interval J_l. Equation (3.8) has to be interpreted in the following set-theoretic sense: fixed any $x \in (0,1)^n$, there exist k_0 and an increasing sequence $\{l_k\}_{k=k_0}^{\infty}$ in $\mathbb{N}$ such that $l_k \in \Lambda_k$ and $x_1 \in (\overline{J_{l_k}} - k + 1)$ for all $k \in \mathbb{N}$ greater than k_0. Such a k_0 can be chosen so that $B_{\sqrt{n}a_{l_k}}(x) \subseteq (0,1)^n$ for all $k \ge k_0$, since $\lim_{k\to\infty} a_{l_k} = 0$.

On the other hand, for every $k \ge k_0$, one also has

$$B_{\sqrt{n}a_{l_k}}(x) \;\supseteq\; \prod_{i=1}^{n} [x_i - a_{l_k}, x_i + a_{l_k}] \;\supseteq\; (J_{l_k} - k + 1) \times \prod_{i=2}^{n} [x_i - a_{l_k}, x_i + a_{l_k}].$$

Consequently, for every $y \in \mathbb{R}^n$,

$$\begin{aligned}
u\chi_{B_{\sqrt{n}a_{l_k}}(x)}(y) \;&\ge\; g(y_1 + k - 1)\; \chi_{(J_{l_k}-k+1)}(y_1) \prod_{i=2}^{n} \chi_{[x_i - a_{l_k}, x_i + a_{l_k}]}(y_i) \\
&=\; (f\chi_{I_{l_k}})^*\Big(y_1 + k - 1 - \sum_{j=0}^{k-1} a_j\Big)\; \chi_{(J_{l_k}-k+1)}(y_1) \prod_{i=2}^{n} \chi_{[x_i - a_{l_k}, x_i + a_{l_k}]}(y_i).
\end{aligned}$$

Hence,

$$(u\chi_{B_{\sqrt{n}a_{l_k}}(x)})^*(s) \;\ge\; (f\chi_{I_{l_k}})^*\left((2a_{l_k})^{1-n}s\right) \qquad \text{for } s \in (0,\infty). \tag{3.9}$$

Therefore, thanks to the boundedness on rearrangement-invariant spaces of the dilation operator, defined as in (2.13), one gets

$$\begin{aligned}
\|u\|^{\oslash}_{X(B_{\sqrt{n}a_{l_k}}(x))} &= \|(u\chi_{B_{\sqrt{n}a_{l_k}}})^*(\mathcal{L}^n(B_{\sqrt{n}a_{l_k}}(x))\,\cdot)\|_{\overline{X}(0,\infty)} \\
&\ge C\,\|(f\chi_{I_{l_k}})^*(a_{l_k}\,\cdot)\|_{\overline{X}(0,\infty)} = \|(f\chi_{I_{l_k}})^*\|^{\oslash}_{\overline{X}(0,a_{l_k})} > 1
\end{aligned}$$

for some positive constant $C = C(n)$. Hence inequality (3.7) follows, since $\lim_{k\to\infty} a_{l_k} = 0$.

To conclude, consider the set $M = \{y \in (0,1)^n : u(y) = 0\}$. This set M has positive measure. Indeed, (3.6) with $t = 0$ and (3.5) imply

$$\mathcal{L}^n(M) = 1 - \mathcal{L}^n(\{y \in (0,1)^n : u(y) > 0\}) \geq 1 - \mathcal{L}^1(\{s \in (0,\infty) : |f\chi_{\cup_{k\in\mathbb{N}} I_k}|(s) > 0\})$$
$$\geq 1 - \sum_{k=1}^{\infty} \mathcal{L}^1(I_k) > 0.$$

Then, estimate (3.7) tells us that

$$\limsup_{r\to 0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} = \limsup_{r\to 0^+} \|u\|^{\oslash}_{X(B_r(x))} > 0 \qquad \text{for a.e. } x \in M.$$

This contradicts the Lebesgue point property for $\|\cdot\|_{X(\mathbb{R}^n)}$. $\square$

Proof of Lemma 3.3. Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm satisfying condition (H). We first prove that, if (H) is in force, then

$$\lim_{t\to 0^+} \|g^*\chi_{(0,t)}\|_{\overline{X}(0,\infty)} = 0 \tag{3.10}$$

for every $g \in \overline{X}_1(0,\infty)$.

Arguing by contradiction, assume the existence of some $g \in \overline{X}_1(0,\infty)$ for which (3.10) fails. From property (N2) of rearrangement-invariant norms, this means that some $\varepsilon > 0$ exists such that $\|g^*\chi_{(0,t)}\|_{\overline{X}(0,\infty)} \geq \varepsilon$ for every $t \in (0,1)$. Thanks to (N1), we may assume, without loss of generality, that $\varepsilon = 2$.

Then, by induction, construct a decreasing sequence $\{b_k\}$, with $0 < b_k \leq 1$, such that

$$\|g^*\chi_{(b_{k+1},b_k)}\|_{\overline{X}(0,\infty)} > 1 \tag{3.11}$$

for every $k \in \mathbb{N}$. To this purpose, set $b_1 = 1$, and assume that b_k is given for some $k \in \mathbb{N}$. Then define

$$h_l = g^*\chi_{\left(\frac{b_k}{l},b_k\right)} \qquad \text{for } l \in \mathbb{N}, \text{ with } l \geq 2.$$

Since $0 \leq h_l \nearrow g^*\chi_{(0,b_k)}$, property (N3) tells us that $\|h_l\|_{\overline{X}(0,\infty)} \nearrow \|g^*\chi_{(0,b_k)}\|_{\overline{X}(0,\infty)}$. Inasmuch as $\|g^*\chi_{(0,b_k)}\|_{\overline{X}(0,\infty)} \geq 2$, then there exists an l_0, with $l_0 \geq 2$, such that $\|h_{l_0}\|_{\overline{X}(0,\infty)} > 1$. Defining $b_{k+1} = \frac{b_k}{l_0}$ entails that $0 < b_{k+1} < b_k$ and $\|g^*\chi_{(b_{k+1},b_k)}\|_{\overline{X}(0,\infty)} = \|h_{l_0}\|_{\overline{X}(0,\infty)} > 1$, as desired.

Observe that choosing $f = g^*\chi_{(0,1)}$, and $a_k = 1$, $I_k = (b_{k+1}, b_k)$ for each $k \in \mathbb{N}$ provides a contradiction to assumption (H). Indeed, inequality (3.1), which agrees with (3.11) in this case, holds for every k, whereas $\sum_{k=1}^{\infty} a_k = \infty$. Consequently, (3.10) does hold.

Now, take any $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$ and any non-increasing sequence $\{K_j\}$ of measurable bounded sets in $\mathbb{R}^n$ such that $\cap_{j\in\mathbb{N}} K_j = \emptyset$. Clearly, $u\chi_{K_1} \in X(\mathbb{R}^n)$ and $\lim_{j\to\infty} \mathcal{L}^n(K_j) = 0$. We may assume that $\mathcal{L}^n(K_1) < 1$, whence $(u\chi_{K_j})^* \in \overline{X}_1(0,\infty)$ for each $j \in \mathbb{N}$.

From (3.10) it follows that

$$\lim_{j\to\infty} \|u\chi_{K_j}\|_{X(\mathbb{R}^n)} = \lim_{j\to\infty} \|(u\chi_{K_j})^*\|_{\overline{X}(0,\infty)} \leq \lim_{j\to\infty} \|(u\chi_{K_1})^*\chi_{(0,\mathcal{L}^n(K_j))}\|_{\overline{X}(0,\infty)} = 0,$$

namely the local absolute continuity of $\|\cdot\|_{X(\mathbb{R}^n)}$. $\square$

Proof of Proposition 3.1. Owing to [4, Corollary 5.6, Chap. 1], assertions (i) and (ii) are equivalent. Assertion (i) follows from Lemmas 3.2 and 3.3. $\square$

4. The functional $\mathcal{G}_X$ and the operator $\mathcal{M}_X$

This section is devoted to a closer analysis of the functional $\mathcal{G}_X$ and the operator $\mathcal{M}_X$ associated with a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$.

We begin with alternate characterizations of the almost concavity of the functional $\mathcal{G}_X$. In what follows, we shall make use of the fact that

$$\|h\|_{\overline{X}(0,\infty)} = \|h^*\|_{\overline{X}(0,\infty)} = \|(h_*)_*\|_{\overline{X}(0,\infty)} = \mathcal{G}_X(h_*) \tag{4.1}$$

for every $h \in L^0(0,\infty)$.
Moreover, by a *partition* of the interval $(0,1)$ we shall mean a finite collection $\{I_k : k = 1, \dots, m\}$, where $I_k = (\tau_{k-1}, \tau_k)$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_m = 1$.

Proposition 4.1. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Then the following conditions are equivalent:*

(i) *the functional $\mathcal{G}_X$ is almost concave;*

(ii) *a positive constant C exists such that*

$$\sum_{k=1}^{m} \mathcal{L}^1(I_k)\, \|f\|^{\oslash}_{\overline{X}(I_k)} \le C \|f\|_{\overline{X}(0,\infty)} \tag{4.2}$$

for every $f \in \overline{X}_1(0,\infty)$, and for every partition $\{I_k : k = 1, \dots, m\}$ of $(0,1)$;

(iii) *a positive constant C exists such that*

$$\sum_{k=1}^{m} \mathcal{L}^n(B_k)\, \|u\|^{\oslash}_{X(B_k)} \le C\ \mathcal{L}^n\big(\cup_{k=1}^m B_k\big)\ \|u\|^{\oslash}_{X(\cup_{k=1}^m B_k)} \tag{4.3}$$

for every $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$, and for every finite collection $\{B_k : k = 1, \dots, m\}$ of pairwise disjoint balls in $\mathbb{R}^n$.

Proof. (i) $\Rightarrow$ (ii) Assume that $\mathcal{G}_X$ is almost concave. Fix any function $f \in \overline{X}_1(0,\infty)$, and any partition $\{I_k : k = 1, \dots, m\}$ of $(0,1)$. It is easily verified that

$$\Big((f\chi_{I_k})^*(\mathcal{L}^1(I_k)\cdot)\Big)_* = \frac{(f\chi_{I_k})_*}{\mathcal{L}^1(I_k)} \quad \text{and} \quad \big(f\chi_{\cup_{k=1}^m I_k}\big)_* = \sum_{k=1}^{m} (f\chi_{I_k})_*\,.$$

Hence, by (4.1) and the almost concavity of $\mathcal{G}_X$, there exists a constant C such that

$$\begin{aligned}\sum_{k=1}^{m} \mathcal{L}^1(I_k)\, \|f\|^{\oslash}_{\overline{X}(I_k)} &= \sum_{k=1}^{m} \mathcal{L}^1(I_k)\, \|(f\chi_{I_k})^*(\mathcal{L}^1(I_k)\cdot)\|_{\overline{X}(0,\infty)} = \sum_{k=1}^{m} \mathcal{L}^1(I_k)\, \mathcal{G}_X\Big(\frac{(f\chi_{I_k})_*}{\mathcal{L}^1(I_k)}\Big) \\ &\le C\, \mathcal{G}_X\Big(\sum_{k=1}^{m} (f\chi_{I_k})_*\Big) = C\, \mathcal{G}_X\Big((f\chi_{\cup_{k=1}^m I_k})_*\Big) \le C\, \mathcal{G}_X(f_*) = C\, \|f\|_{\overline{X}(0,\infty)}.\end{aligned}$$

This yields inequality (4.2).
(ii) $\Rightarrow$ (i) Take any finite collections $\{g_k : k = 1, \dots, m\}$ in $\mathcal{C}$ and $\{\lambda_k : k = 1, \dots, m\}$ in $(0,1)$, respectively, with $\sum_{k=1}^m \lambda_k = 1$. For each $k = 1, \dots, m$, write $f_k = (g_k)_*$, $a_k = \sum_{i=1}^k \lambda_i$, and $I_k = (a_{k-1}, a_k)$ with $a_0 = 0$. Then define

$$f(t) = \sum_{k=1}^{m} f_k\big(\tfrac{t - a_{k-1}}{\lambda_k}\big)\chi_{I_k}(t) \qquad \text{for } t \in (0,\infty).$$

Observe that $f_* = \sum_{k=1}^m \lambda_k (f_k)_* = \sum_{k=1}^m \lambda_k g_k$. Owing to (4.1) and (4.2), one thus obtains

$$\begin{aligned}\mathcal{G}_X\Big(\sum_{k=1}^m \lambda_k g_k\Big) &= \mathcal{G}_X(f_*) = \|f\|_{\overline{X}(0,\infty)} \geq \frac{1}{C}\sum_{k=1}^m \lambda_k \|f\|^{\oslash}_{\overline{X}(I_k)} = \frac{1}{C}\sum_{k=1}^m \lambda_k \|f_k^*(\tfrac{1}{\lambda_k}\,\cdot)\|^{\oslash}_{\overline{X}(0,\lambda_k)}\\ &= \frac{1}{C}\sum_{k=1}^m \lambda_k \|f_k^*\|_{\overline{X}(0,\infty)} = \frac{1}{C}\sum_{k=1}^m \lambda_k \mathcal{G}_X\big((f_k)_*\big) = \frac{1}{C}\sum_{k=1}^m \lambda_k \mathcal{G}_X(g_k),\end{aligned}$$

whence the almost concavity of $\mathcal{G}_X$ follows.
(ii) $\Rightarrow$ (iii) Fix any function $u \in X_{\rm loc}(\mathbb{R}^n)$, and any finite collection $\{B_k : k = 1,\dots,m\}$ of pairwise disjoint balls in $\mathbb{R}^n$. Set $a_k = \mathcal{L}^n(B_k)$, for $k = 1,\dots,m$, and $a_0 = 0$. Define

$$I_k = \left(\frac{\sum_{j=0}^{k-1} a_j}{\sum_{j=1}^m a_j}, \frac{\sum_{j=0}^{k} a_j}{\sum_{j=1}^m a_j}\right) \qquad \text{for } k = 1,\dots,m.$$

Thanks to rearrangement-invariance of $\overline{X}(0,\infty)$, assumption (ii) ensures that

$$\begin{aligned}\|u\|^{\oslash}_{X(\cup_{k=1}^m B_k)} &= \Big\|\big(u\chi_{\cup_{k=1}^m B_k}\big)^*\big(\sum_{k=1}^m a_k\ \cdot\big)\Big\|_{\overline{X}(0,\infty)} = \Big\|\sum_{k=1}^m (u\chi_{B_k})^*\big(\sum_{j=1}^m a_j\ \cdot - \sum_{j=0}^{k-1} a_j\big)\ \chi_{I_k}\Big\|_{\overline{X}(0,\infty)}\\ &\geq \frac{1}{C}\ \sum_{k=1}^m \frac{a_k}{\sum_{j=1}^m a_j}\Big\|(u\chi_{B_k})^*\big(\sum_{k=1}^m a_k\ \cdot - \sum_{j=0}^{k-1} a_j\big)\ \chi_{I_k}\Big\|^{\oslash}_{\overline{X}(I_k)}\\ &= \frac{1}{C\sum_{j=1}^m a_j}\ \sum_{k=1}^m a_k\,\|(u\chi_{B_k})^*(a_k\,\cdot)\|_{\overline{X}(0,\infty)} = \frac{1}{C\,\mathcal{L}^n(\cup_{k=1}^m B_k)}\sum_{k=1}^m \mathcal{L}^n(B_k)\|u\|^{\oslash}_{X(B_k)}.\end{aligned}$$

Inequality (4.3) is thus established.
(iii) $\Rightarrow$ (ii) Assume that $f \in \overline{X}_1(0,\infty)$, and that $\{I_k : k = 1,\dots,m\}$ is a partition of $(0,1)$. Let $\{B_k : k = 1,\dots,m\}$ be a family of pairwise disjoint balls in $\mathbb{R}^n$ such that $\mathcal{L}^n(B_k) = \mathcal{L}^1(I_k)$, and let u be a measurable function on $\mathbb{R}^n$ vanishing outside of $\cup_{k=1}^m B_k$ and fulfilling $(u\chi_{B_k})^* = (f\chi_{I_k})^*$, for $k = 1,2,\dots,m$. Assumption (iii) then tells us that

$$\sum_{k=1}^m \mathcal{L}^1(I_k)\|f\|^{\oslash}_{\overline{X}(I_k)} \leq C\,\mathcal{L}^1(\cup_{k=1}^m I_k)\|f\|^{\oslash}_{\overline{X}(\cup_{k=1}^m I_k)} = C\,\|f\|^{\oslash}_{\overline{X}(0,1)} = C\,\|f\|_{\overline{X}(0,\infty)},$$

namely (4.2). $\square$

We next focus on the maximal operator $\mathcal{M}_X$. Criteria for the validity of the Riesz-Wiener type inequality (1.11) are the content of the following result.

Proposition 4.2. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm. Then the following conditions are equivalent:*

(i) *the Riesz-Wiener type inequality* (1.11) *holds for some positive constant C, and for every $u \in X_{\rm loc}(\mathbb{R}^n)$;*
(ii) *a positive constant C_1 exists such that*

$$\min_{k=1,\dots,m}\|u\|^{\oslash}_{X(B_k)} \leq C_1\|u\|^{\oslash}_{X(\cup_{k=1}^m B_k)} \tag{4.4}$$

for every $u \in X_{\rm loc}(\mathbb{R}^n)$, and for every finite collection $\{B_k : k = 1,\dots,m\}$ of pairwise disjoint balls in $\mathbb{R}^n$;

(iii) *a positive constant* C_2 *exists such that*

$$\min_{k=1,\dots,m} \|f\|^{\oslash}_{\overline{X}(I_k)} \leq C_2 \|f\|_{\overline{X}(0,\infty)}$$

for every $f \in \overline{X}_1(0,\infty)$, *and for every partition* $\{I_k : k = 1,\dots,m\}$ *of* $(0,1)$.

Proof. (i) $\Rightarrow$ (ii) Let $u \in X_{\rm loc}(\mathbb{R}^n)$, and let $\{B_k : k = 1,\dots,m\}$ be a collection of pairwise disjoint balls in $\mathbb{R}^n$. When $\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} = 0$, then (4.4) trivially holds. Assume that $\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} > 0$. Fix any $s \in \big(0, \mathcal{L}^n(\cup_{k=1}^m B_k)\big)$, and any $t \in \big(0, \min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)}\big)$. If $x \in B_j$ for some $j = 1,\dots,m$, then

$$\mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})(x) \geq \|u\chi_{\cup_{k=1}^m B_k}\|^{\oslash}_{X(B_j)} \geq \min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} > t.$$

Thus,

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : \mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})(x) > t\}) \geq \mathcal{L}^n(\cup_{k=1}^m B_k) > s,$$

and, consequently,

$$\big(\mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})\big)^*(s) \geq t.$$

Since the last inequality holds for every $t < \min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)}$, one infers that

$$\big(\mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})\big)^*(s) \;\geq\; \min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)}. \tag{4.5}$$

On the other hand, an application of assumption (i) with u replaced by $u\chi_{\cup_{k=1}^m B_k}$ tells us that

$$\big(\mathcal{M}_X(u\chi_{\cup_{k=1}^m B_k})\big)^*(s) \leq C\, \|(u\chi_{\cup_{k=1}^m B_k})^*\|^{\oslash}_{\overline{X}(0,s)} \qquad \text{for } s \in \big(0, \mathcal{L}^n(\cup_{k=1}^m B_k)\big). \tag{4.6}$$

Coupling (4.5) with (4.6) implies that

$$\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} \leq C\, \|(u\chi_{\cup_{k=1}^m B_k})^*\|^{\oslash}_{\overline{X}(0,s)} \qquad \text{for } s \in \big(0, \mathcal{L}^n(\cup_{k=1}^m B_k)\big).$$

Thus, owing to the continuity of the function $s \mapsto \|(u\chi_{\cup_{k=1}^m B_k})^*\|^{\oslash}_{\overline{X}(0,s)}$, which is guaranteed by Lemma 3.4, we deduce that

$$\min_{k=1,\dots,m} \|u\|^{\oslash}_{X(B_k)} \leq C \|(u\chi_{\cup_{k=1}^m B_k})^*\|^{\oslash}_{\overline{X}(0,\mathcal{L}^n(\cup_{k=1}^m B_k))} = C\|u\|^{\oslash}_{X(\cup_{k=1}^m B_k)},$$

namely (4.4).
(ii) $\Rightarrow$ (i) By [14, Proposition 3.2], condition (ii) implies the existence of a constant C' such that

$$(\mathcal{M}_X u)^*(s) \leq C' \|u^*\big(3^{-n}s\ \cdot\big)\chi_{(0,1)}(\,\cdot\,)\|_{\overline{X}(0,\infty)} \qquad \text{for } s \in (0,\infty), \tag{4.7}$$

for every $u \in X_{\rm loc}(\mathbb{R}^n)$. By the boundedness of the dilation operator on rearrangement-invariant spaces, there exists a constant C'', independent of u, such that

$$\begin{aligned} \|u^*\big(3^{-n}s\ \cdot\big)\chi_{(0,1)}(\,\cdot\,)\|_{\overline{X}(0,\infty)} &\leq C'' \|u^*(s\,\cdot)\chi_{(0,1)}(3^n\,\cdot)\|_{\overline{X}(0,\infty)} \\ &\leq C'' \|u^*(s\,\cdot)\chi_{(0,1)}(\,\cdot\,)\|_{\overline{X}(0,\infty)} = C''\|u^*\|^{\oslash}_{\overline{X}(0,s)} \qquad \text{for } s \in (0,\infty). \end{aligned} \tag{4.8}$$

Inequality (1.11) follows from (4.7) and (4.8).
(ii) $\Leftrightarrow$ (iii) The proof is completely analogous to that of the equivalence between conditions (ii) and (iii) in Proposition 4.1. We omit the details for brevity. $\square$

Condition (H) introduced in Lemma 3.2 can be characterized in terms of the maximal operator $\mathcal{M}_X$ as follows.

Proposition 4.3. *Let* $\|\cdot\|_{X(\mathbb{R}^n)}$ *be a rearrangement-invariant norm. Then the following assertions are equivalent:*

(i) $\|\cdot\|_{X(\mathbb{R}^n)}$ *fulfils condition* (H) *in Lemma 3.2;*
(ii) *For every function* $u \in X(\mathbb{R}^n)$*, supported in a set of finite measure,*

$$\mathcal{L}^n(\{x \in \mathbb{R}^n \colon \mathcal{M}_X u(x) > 1\}) \ < \infty.$$

Proof. (i) $\Rightarrow$ (ii) Let $u \in X(\mathbb{R}^n)$ be supported in a set of finite measure. Set $E = \{x \in \mathbb{R}^n : \mathcal{M}_X u(x) > 1\}$. According to (1.9), for any $y \in E$, there exists a ball B_y in $\mathbb{R}^n$ such that $y \in B_y$ and $\|u\|^{\oslash}_{X(B_y)} > 1$. Define

$$E_1 = \Big\{y \in E : \mathcal{L}^n(B_y) > \max\{1, \mathcal{L}^n(\{|u| > 0\})\}\Big\}. \tag{4.9}$$

We claim that, if $y \in E_1$, then

$$(u\chi_{B_y})^*\big(\mathcal{L}^n(B_y)s\big) \le (u\chi_{B_y})^*(s)\chi_{\big(0,\frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}\big)}(s) \qquad \text{for } s \in (0,\infty). \tag{4.10}$$

Indeed, since $\mathcal{L}^n(B_y) \ge 1$, by the monotonicity of the decreasing rearrangement

$$(u\chi_{B_y})^*(s) \ge (u\chi_{B_y})^*\big(\mathcal{L}^n(B_y)s\big) \qquad \text{for } s \in (0,\infty).$$

Thus, inequality (4.10) certainly holds if $s \in \big(0, \frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}\big]$. On the other hand, $\mathcal{L}^n(B_y) \ge \mathcal{L}^n(\{|u| > 0\})$, and since $(u\chi_{B_y})^*(s) = 0$ for $s \ge \mathcal{L}^n(\{|u| > 0\})$, we have that $(u\chi_{B_y})^*\big(\mathcal{L}^n(B_y)s\big) = 0$ if $s \in \big(\frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}, \infty\big)$. Thereby, inequality (4.10) also holds for these values of s. Owing to (4.10),

$$\begin{aligned} 1 < \|u\|^{\oslash}_{X(B_y)} &= \|(u\chi_{B_y})^*(\mathcal{L}^n(B_y)\ \cdot)\|_{\overline{X}(0,\infty)} \\ &\le \Big\|(u\chi_{B_y})^*\chi_{\big(0,\frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}\big)}\Big\|_{\overline{X}(0,\infty)} \le \Big\|u^*\chi_{\big(0,\frac{\mathcal{L}^n(\{|u|>0\})}{\mathcal{L}^n(B_y)}\big)}\Big\|_{\overline{X}(0,\infty)}. \end{aligned} \tag{4.11}$$

Since (i) is in force, equation (3.10) holds with $g = u^*\chi_{(0,1)} \in \overline{X}_1(0,\infty)$, namely,

$$\lim_{t\to 0^+} \|u^*\chi_{(0,t)}\|_{\overline{X}(0,\infty)} = 0.$$

This implies the existence of some $t_0 \in (0,1)$ such that $\|u^*\chi_{(0,t)}\|_{\overline{X}(0,\infty)} < 1$ for every $t \in (0,t_0)$. Thus, (4.11) entails that

$$\frac{\mathcal{L}^n(\{|u| > 0\})}{\mathcal{L}^n(B_y)} \ge t_0$$

for every $y \in E_1$. Hence, by (4.9),

$$\sup_{y\in E} \mathcal{L}^n(B_y) \le \max\Big\{1, \frac{\mathcal{L}^n(\{|u| > 0\})}{t_0}\Big\}. \tag{4.12}$$

An application of Vitali's covering lemma, in the form of [21, Lemma 1.6, Chap. 1], ensures that there exists a countable set $\mathcal{I} \subseteq E$ such that the family $\{B_y : y \in \mathcal{I}\}$ consists of pairwise disjoint balls, such that $E \subseteq \cup_{y\in\mathcal{I}} 5B_y$. Here, $5B_y$ denotes the ball, with the same center as B_y, whose radius is 5 times the radius of B_y. If $\mathcal{I}$ is finite, then trivially $\mathcal{L}^n(E) \le 5^n \sum_{y\in\mathcal{I}} \mathcal{L}^n(B_y) < \infty$. Assume that, instead, $\mathcal{I}$ is infinite, and let $\{y_k\}$ be the sequence of its elements. For each $k \in \mathbb{N}$, set, for simplicity, $B_k = B_{y_k}$, and

$$\alpha_k = \mathcal{L}^n(\{y \in B_k : u(y) \neq 0\}), \qquad I_k = \left(\frac{\sum_{i=0}^{k-1}\alpha_i}{\alpha}, \frac{\sum_{i=1}^{k}\alpha_i}{\alpha}\right), \qquad a_k = \frac{\mathcal{L}^n(B_k)}{\alpha}, \tag{4.13}$$

where $\alpha = \mathcal{L}^n(\{y \in \mathbb{R}^n : u(y) \neq 0\})$ and $\alpha_0 = 0$. Note that $\{I_k\}$ is a sequence of pairwise disjoint intervals in $(0,1)$, and $a_k \geq \mathcal{L}^1(I_k)$ for each $k \in \mathbb{N}$. The function $f : (0,\infty) \to [0,\infty)$, defined by

$$f(s) = \sum_{k=1}^{\infty} \left(u\chi_{\{x \in B_k : u(x) \neq 0\}}\right)^* \Big(\alpha s - \sum_{i=1}^{k-1} \alpha_i\Big) \chi_{I_k}(s) \quad \text{for } s \in (0,\infty),$$

belongs to $\overline{X}_1(0,\infty)$, and

$$\begin{aligned} \|(f\chi_{I_k})^*\|^{\oslash}_{\overline{X}(0,a_k)} &= \|(u\chi_{\{x \in B_k : u(x) \neq 0\}})^*(\alpha\, \cdot)\|^{\oslash}_{\overline{X}(0,a_k)} \\ &= \|(u\chi_{B_k})^*(\mathcal{L}^n(B_k)\, \cdot)\|_{\overline{X}(0,\infty)} = \|u\|^{\oslash}_{X(B_k)} > 1. \end{aligned}$$

By (i), one thus obtains that $\sum_{k=1}^{\infty} \mathcal{L}^n(B_k) = \alpha \sum_{k=1}^{\infty} a_k < \infty$. Hence $\mathcal{L}^n(E) \leq 5^n \sum_{k=1}^{\infty} \mathcal{L}^n(B_k) < \infty$, also in this case.
(ii) $\Rightarrow$ (i) Let $f \in \overline{X}_1(0,\infty)$, let $\{I_k\}$ be any sequence of pairwise disjoint intervals in $(0,1)$, and let $\{a_k\}$ be a sequence of real numbers, such that $a_k \geq \mathcal{L}^1(I_k)$, fulfilling (3.1).
Consider any sequence $\{B_k\}$ of pairwise disjoint balls in $\mathbb{R}^n$, such that $\mathcal{L}^n(B_k) = a_k$ for $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, choose a function $g_k : \mathbb{R}^n \to [0,\infty)$, supported in B_k, and such that g_k is equimeasurable with $f\chi_{I_k}$. Then, define $u = \sum_{k=1}^{\infty} g_k$. Note that $u \in X(\mathbb{R}^n)$, since $u^* = (f\chi_{\bigcup_{k=1}^{\infty} I_k})^* \leq f^*$. Furthermore, u is supported in a set of finite measure. Thus, assumption (ii) implies that

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : \mathcal{M}_X u(x) > 1\}) < \infty. \tag{4.14}$$

If $x \in B_k$ for some $k \in \mathbb{N}$, then

$$\mathcal{M}_X u(x) \geq \|u\|^{\oslash}_{X(B_k)} = \|g_k\|^{\oslash}_{X(B_k)} = \|g_k^*\|^{\oslash}_{\overline{X}(0,\mathcal{L}^n(B_k))} = \|(f\chi_{I_k})^*\|^{\oslash}_{\overline{X}(0,a_k)} > 1. \tag{4.15}$$

Consequently,

$$\cup_{k=1}^{\infty} B_k \subseteq \{x \in \mathbb{R}^n : \mathcal{M}_X u(x) > 1\}$$

and

$$\sum_{k=1}^{\infty} a_k = \mathcal{L}^n(\cup_{k=1}^{\infty} B_k) \leq \mathcal{L}^n(\{x \in \mathbb{R}^n : \mathcal{M}_X u(x) > 1\}) < \infty.$$

Condition (i) is thus fulfilled. □

5. Proofs of Theorems 1.1, 1.4 and 1.7

The core of Theorems 1.1, 1.4 and 1.7 is contained in the following statement.

Proposition 5.1. *Given a rearrangement-invariant norm $\|\cdot\|_{X(\mathbb{R}^n)}$, consider the following properties:*

(i) $\|\cdot\|_{X(\mathbb{R}^n)}$ *satisfies the Lebesgue point property;*
(ii) $\|\cdot\|_{X(\mathbb{R}^n)}$ *fulfills condition* (H) *of Lemma 3.2;*
(iii) *The functional $\mathcal{G}_X$ is almost concave;*
(iv) *The Riesz-Wiener type inequality* (1.11) *holds for some positive constant C, and for every $u \in X_{\mathrm{loc}}(\mathbb{R}^n)$;*
(v) *The operator $\mathcal{M}_X$ is of weak type from $X_{\mathrm{loc}}(\mathbb{R}^n)$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Then:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)}.$$

If, in addition, $\|\cdot\|_{X(\mathbb{R}^n)}$ is locally absolutely continuous, then

$$\text{(v)} \Rightarrow \text{(i)}.$$

A proof of Proposition 5.1 requires the next lemma.

Lemma 5.2. *Let $\|\cdot\|_{X(\mathbb{R}^n)}$ be a rearrangement-invariant norm such that*

$$\lim_{s\to 0^+} \varphi_{X(\mathbb{R}^n)}(s) = 0. \tag{5.1}$$

If $u : \mathbb{R}^n \to \mathbb{R}$ is any simple function, then

$$\lim_{r\to 0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} = 0 \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

Proof. Let E be any measurable subset of $\mathbb{R}^n$. By the Lebesgue density theorem,

$$\begin{aligned} &\lim_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\setminus E)}{\mathcal{L}^n(B_r(x))} = 0 \qquad \text{for a.e. } x \in E \\ &\lim_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))} = 0 \qquad \text{for a.e. } x \in \mathbb{R}^n \setminus E. \end{aligned} \tag{5.2}$$

Since

$$\lim_{r\to 0^+} \|\chi_E - \chi_E(x)\|^{\oslash}_{X(B_r(x))} = \begin{cases} \lim_{r\to 0^+} \left\|\chi_{\left(0,\frac{\mathcal{L}^n(B_r(x)\setminus E)}{\mathcal{L}^n(B_r(x))}\right)}\right\|_{\overline{X}(0,\infty)} & \text{for a.e. } x \in E, \\ \lim_{r\to 0^+} \left\|\chi_{\left(0,\frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))}\right)}\right\|_{\overline{X}(0,\infty)} & \text{for a.e. } x \in \mathbb{R}^n \setminus E, \end{cases} \tag{5.3}$$

it follows from (5.1) that

$$\lim_{r\to 0^+} \|\chi_E - \chi_E(x)\|^{\oslash}_{X(B_r(x))} = 0 \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{5.4}$$

Hence, if u is any simple function having the form $u = \sum_{i=1}^k a_i\chi_{E_i}$, where $E_1,\dots,E_k$ are pairwise disjoint measurable subsets of $\mathbb{R}^n$, and $a_1,\dots,a_k \in \mathbb{R}$, then

$$\lim_{r\to 0^+} \|u - u(x)\|^{\oslash}_{X(B_r(x))} \le \lim_{r\to 0^+} \sum_{i=1}^k |a_i|\, \|\chi_{E_i} - \chi_{E_i}(x)\|^{\oslash}_{X(B_r(x))} = 0 \tag{5.5}$$

for a.e. $x \in \mathbb{R}^n$. □

Proof of Proposition 5.1. (i) ⇒ (ii) This is just the content of Lemma 3.2 above.
(ii) ⇒ (iii) We prove this implication by contradiction. Assume that the functional $\mathcal{G}_X$ is not almost concave. Owing to Proposition 4.1, this amounts to assuming that, for every $k \in \mathbb{N}$, there exist a function $f_k \in \overline{X}_1(0,\infty)$ and a partition $\{J_{k,l} : l = 1,\dots,m_k\}$ of $(0,1)$ such that

$$\sum_{l=1}^{m_k} \mathcal{L}^1(J_{k,l})\, \|f_k\|^{\oslash}_{\overline{X}(J_{k,l})} \;>\; 4^k \|f_k\|_{\overline{X}(0,\infty)}. \tag{5.6}$$

Define the function $f : (0,\infty) \to \mathbb{R}$ as

$$f(t) = \sum_{k=1}^{\infty} \frac{\chi_{(2^{-k},2^{-k+1})}(t)\, f_k(2^k t - 1)}{2^k \left\|\chi_{(2^{-k},2^{-k+1})}\, f_k(2^k\cdot -1)\right\|_{\overline{X}(0,\infty)}} \qquad \text{for } t \in (0,\infty). \tag{5.7}$$

Since $f \in L^0(0,\infty)$, $f = 0$ on $(1,\infty)$ and $\|f\|_{\overline{X}(0,\infty)} \le \sum_{k=1}^{\infty} 2^{-k}$, we have that $f \in \overline{X}_1(0,\infty)$. Let us denote by Λ the set $\{(k,l) \in \mathbb{N}^2 : l \le m_k\}$, ordered according to the lexicographic order, and define the sequence $\{I_{k,l}\}$ as

$$I_{k,l} = \frac{1}{2^k} J_{k,l} + \frac{1}{2^k} \qquad \text{for } (k,l) \in \Lambda. \tag{5.8}$$

Each element $I_{k,l}$ is an open subinterval of $(0,1)$. Moreover, the intervals $I_{k,l}$ and $I_{h,j}$ are disjoint if $(k,l) \neq (h,j)$. Actually, if $k \neq h$, then

$$I_{k,l} \cap I_{h,j} \;\subseteq\; (2^{-k}, 2^{1-k}) \cap (2^{-h}, 2^{1-h}) \;=\; \emptyset;$$

if, instead, $k = h$ but $l \neq j$, then the same conclusion immediately follows from the fact that the intervals $J_{k,l}$ and $J_{k,j}$ are disjoint. Owing to (5.6) and (5.7),

$$\begin{aligned}
\sum_{(k,l)\in\Lambda} \mathcal{L}^1(I_{k,l})\, \|f\|^{\oslash}_{\overline{X}(I_{k,l})} &= \sum_{(k,l)\in\Lambda} \mathcal{L}^1(I_{k,l})\, \|(f\chi_{I_{k,l}})^*(\mathcal{L}^1(I_{k,l})\,\cdot)\|_{\overline{X}(0,\infty)} \\
&= \sum_{(k,l)\in\Lambda} \frac{\mathcal{L}^1(J_{k,l})}{2^k}\, \frac{\|(f_k\chi_{J_{k,l}})^*(\mathcal{L}^1(J_{k,l})\,\cdot)\|_{\overline{X}(0,\infty)}}{2^k\|\chi_{(2^{-k},2^{1-k})} f_k(2^k\,\cdot -1)\|_{\overline{X}(0,\infty)}} \\
&= \sum_{k=1}^{\infty} \frac{1}{4^k\|f_k^*(2^k\,\cdot)\|_{\overline{X}(0,\infty)}} \sum_{l=1}^{m_k} \mathcal{L}^1(J_{k,l})\, \|f_k\|^{\oslash}_{\overline{X}(J_{k,l})} \\
&\geq \sum_{k=1}^{\infty} \frac{4^k\|f_k\|_{\overline{X}(0,\infty)}}{4^k\|f_k^*(2^k\,\cdot)\|_{\overline{X}(0,\infty)}} \geq \sum_{k=1}^{\infty} \frac{4^k\|f_k\|_{\overline{X}(0,\infty)}}{4^k\|f_k^*\|_{\overline{X}(0,\infty)}} = \sum_{k=1}^{\infty} 1 = \infty.
\end{aligned} \tag{5.9}$$

Set $M = \{(k,l) \in \Lambda : \|f\|^{\oslash}_{\overline{X}(I_{k,l})} \leq 2\}$, and observe that

$$\sum_{(k,l)\in M} \mathcal{L}^1(I_{k,l})\, \|f\|^{\oslash}_{\overline{X}(I_{k,l})} \leq 2 \sum_{(k,l)\in M} \mathcal{L}^1(I_{k,l}) \leq 2. \tag{5.10}$$

From (5.9) and (5.10) we thus infer that

$$\sum_{(k,l)\in\Lambda\setminus M} \mathcal{L}^1(I_{k,l})\, \|f\|^{\oslash}_{\overline{X}(I_{k,l})} = \infty.$$

On the other hand, assumption (ii) implies property (3.10). This property, applied with $g = f\chi_{I_{k,l}}$, in turn ensures that, for every $(k,l) \in \Lambda$,

$$\lim_{t\to+\infty} \|(f\chi_{I_{k,l}})^*\|^{\oslash}_{\overline{X}(0,t)} = \lim_{t\to+\infty} \|(f\chi_{I_{k,l}})^*(t\,\cdot)\|_{\overline{X}(0,\infty)} \leq \lim_{t\to+\infty} \|(f\chi_{I_{k,l}})^*\, \chi_{(0,\frac{1}{t})}\|_{\overline{X}(0,\infty)} = 0.$$

Note that the inequality holds since the function $(f\chi_{I_{k,l}})^*$ belongs to $\overline{X}_1(0,\infty)$, and is non-increasing, and hence $(f\chi_{I_{k,l}})^*(ts) \leq (f\chi_{I_{k,l}})^*(s)\, \chi_{(0,\frac{1}{t})}(s)$ for $s \in (0,\infty)$.
Thus, owing to Lemma 3.4, if $(k,l) \in \Lambda \setminus M$, there exists a number $a_{k,l} \geq \mathcal{L}^1(I_{k,l})$ such that

$$\|(f\chi_{I_{k,l}})^*\|^{\oslash}_{\overline{X}(0,a_{k,l})} = 2\,. \tag{5.11}$$

Furthermore, by (5.11),

(5.12)

$$\begin{aligned}
\sum_{(k,l)\in\Lambda\setminus M} a_{k,l} &= \frac{1}{2} \sum_{(k,l)\in\Lambda\setminus M} a_{k,l}\, \|(f\chi_{I_{k,l}})^*\|^{\oslash}_{\overline{X}(0,a_{k,l})} \geq \frac{1}{2} \sum_{(k,l)\in\Lambda\setminus M} \mathcal{L}^1(I_{k,l})\, \|(f\chi_{I_{k,l}})^*\|^{\oslash}_{\overline{X}(0,\mathcal{L}^1(I_{k,l}))} \\
&= \frac{1}{2} \sum_{(k,l)\in\Lambda\setminus M} \mathcal{L}^1(I_{k,l})\, \|f\|^{\oslash}_{\overline{X}(I_{k,l})} = \infty.
\end{aligned}$$

Thanks to (5.12), the function $f \in \overline{X}_1(0,\infty)$, defined by (5.7), the sequence $\{I_{k,l}\}$, defined by (5.8), and the sequence $\{a_{k,l}\}$ contradict condition (H) in Lemma 3.2, and, thus, assumption (ii).
(iii) $\Rightarrow$ (iv) This implication follows from Propositions 4.1 and 4.2, since condition (ii) of Proposition 4.1 trivially implies condition (iii) of Proposition 4.2.

(iv) $\Rightarrow$ (v) Let K be a bounded subset of $\mathbb{R}^n$. Fix any function $u \in X_{\rm loc}(\mathbb{R}^n)$ whose support is contained in K. Clearly, $u = u\chi_K$. From (iv), we infer that

$$\begin{aligned}
\sup_{t>0} t\,\mathcal{L}^n(\{x \in K : \mathcal{M}_X u(x) > t\}) &= \sup_{t>0} t\,\mathcal{L}^n(\{x \in \mathbb{R}^n : \chi_K \mathcal{M}_X u(x) > t\}) \\
&= \sup_{s>0} s\big(\chi_K\,\mathcal{M}_X u\big)^*(s) \le \sup_{s\in(0,\mathcal{L}^n(K))} s\big(\mathcal{M}_X u\big)^*(s) \le C \sup_{s\in(0,\mathcal{L}^n(K))} s\|u^*\|^{\oslash}_{\overline{X}(0,s)} \\
&\le C\,\mathcal{L}^n(K)\,\|u^*\|^{\oslash}_{\overline{X}(0,\mathcal{L}^n(K))} \le C'\|u^*\|_{\overline{X}(0,\infty)} = C'\|u\|_{X(\mathbb{R}^n)},
\end{aligned}$$

for some constants C and C', where the last but one inequality follows from Lemma 3.4, and the last one from the boundedness of the dilation operator on rearrangement-invariant spaces. Property (v) is thus established.

Finally, assume that $\|\cdot\|_{X(\mathbb{R}^n)}$ is locally absolutely continuous and satisfies condition (v). Since $\mathbb{R}^n$ is the countable union of balls, in order to prove (i) it suffices to show that, given any $u \in X_{\rm loc}(\mathbb{R}^n)$ and any ball B in $\mathbb{R}^n$,

$$\lim_{r\to 0^+} \|u-u(x)\|^{\oslash}_{X(B_r(x))} = 0 \qquad \text{for a.e. } x \in B. \tag{5.13}$$

Equation (5.13) will in turn follow if we show that, for every $t>0$, the set

$$A_t = \{x \in B : \limsup_{r\to 0^+} \|u-u(x)\|^{\oslash}_{X(B_r(x))} > 2t\} \tag{5.14}$$

has measure zero. To prove this, we begin by observing that, since $\|\cdot\|_{X(\mathbb{R}^n)}$ is locally absolutely continuous, [4, Theorem 3.11, Chap. 1] ensures that for any $\varepsilon>0$ there exists a simple function v_ε supported on B such that $u\chi_B = v_\varepsilon + w_\varepsilon$ and $\|w_\varepsilon\|_{X(B)} < \varepsilon$. Clearly, w_ε is supported on B as well. Moreover, [4, Theorem 5.5, Part (b), Chap. 2] and Lemma 5.2 imply that

$$\lim_{r\to 0^+} \|v_\varepsilon - v_\varepsilon(x)\|^{\oslash}_{X(B_r(x))} = 0 \qquad \text{for a.e. } x \in B.$$

Fix any $\varepsilon > 0$. Then

$$\begin{aligned}
\limsup_{r\to 0^+} \|u-u(x)\|^{\oslash}_{X(B_r(x))} &\le \limsup_{r\to 0^+} \|v_\varepsilon - v_\varepsilon(x)\|^{\oslash}_{X(B_r(x))} + \limsup_{r\to 0^+} \|w_\varepsilon - w_\varepsilon(x)\|^{\oslash}_{X(B_r(x))} \\
&= \limsup_{r\to 0^+} \|w_\varepsilon - w_\varepsilon(x)\|^{\oslash}_{X(B_r(x))} \le \mathcal{M}_X w_\varepsilon(x) + |w_\varepsilon(x)|\,\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)}.
\end{aligned}$$

Therefore,

$$A_t \subseteq \{x \in B : \mathcal{M}_X w_\varepsilon(x) > t\} \cup \{y \in B : |w_\varepsilon(y)|\,\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)} > t\} \qquad \text{for } t \in (0,\infty). \tag{5.15}$$

Owing to (v),

$$\mathcal{L}^n(\{x \in B : \mathcal{M}_X w_\varepsilon(x) > t\}) \le \frac{C}{t}\,\|w_\varepsilon\|_{X(B)} \qquad \text{for } t \in (0,\infty).$$

On the other hand,

$$\mathcal{L}^n(\{y \in B : |w_\varepsilon(y)|\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)} > t\}) \le \frac{1}{t}\,\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)}\,\|w_\varepsilon\|_{L^1(B)} \le \frac{C_0}{t}\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)}\|w_\varepsilon\|_{X(B)}$$

for every $t \in (0,\infty)$, where C_0 is the norm of the embedding $X(B) \to L^1(B)$. Inasmuch as $\|w_\varepsilon\|_{X(B)} < \varepsilon$, the last two inequalities, combined with (5.15) and with the subadditivity of the outer Lebesgue measure, imply that the outer Lebesgue measure of A_t does not exceed

$$\frac{\varepsilon}{t}\big(C + C_0\|\chi_{(0,1)}\|_{\overline{X}(0,\infty)}\big)$$

for every $t \in (0,\infty)$. Hence, $\mathcal{L}^n(A_t) = 0$, thanks to the arbitrariness of $\varepsilon > 0$. $\square$

Proof of Theorem 1.1. This is a consequence of Propositions 3.1 and 5.1. □

Proof of Theorem 1.4. This is a consequence of Propositions 3.1 and 5.1. □

Proof of Theorem 1.7. The equivalence of conditions (i) and (ii) follows from Proposition 4.3, Proposition 5.1 and Lemma 3.3.
In order to verify the equivalence of (ii) and (iii), it suffices to observe that, thanks to the positive homogeneity of the maximal operator $\mathcal{M}_X$, one has that $\mathcal{L}^n(\{x \in \mathbb{R}^n : \mathcal{M}_X u(x) > 1\}) < \infty$ for every $u \in X(\mathbb{R}^n)$ supported in a set of finite measure if, and only if, $\mathcal{L}^n(\{x \in \mathbb{R}^n : \mathcal{M}_X u(x) > t\}) < \infty$ for every $u \in X(\mathbb{R}^n)$ supported in a set of finite measure and for every $t \in (0, \infty)$. The latter condition is equivalent to (iii). □

6. Proofs of Propositions 1.8–1.11

In this last section, we show how our general criteria can be specialized to characterize those rearrangement-invariant norms, from customary families, which satisfy the Lebesgue point property, as stated in Propositions 1.8–1.11. In fact, these propositions admit diverse proofs, based on the different criteria provided by Theorems 1.1, 1.4 and 1.7. For instance, Propositions 1.8–1.10 can be derived via Theorem 1.4, combined with results on the local absolute continuity of the norms in question and on Riesz-Wiener type inequalities contained in [2] (Orlicz norms), [3] (norms in the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$), and [14] (norms in the Lorentz endpoint spaces $\Lambda_\varphi(\mathbb{R}^n)$). Let us also mention that, at least in the one-dimensional case, results from these propositions overlap with those of [5, 6, 19].

Hereafter, we provide alternative, more self-contained proofs of Propositions 1.8–1.11, relying upon our general criteria. Let us begin with Proposition 1.8, whose proof requires the following preliminarily lemmas.

Lemma 6.1. *Let $p, q \in [1, \infty]$ be admissible values in the definition of the Lorentz norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$. Then*

$$\mathcal{G}_{L^{p,q}}(f) = \begin{cases} \left(p \int_0^\infty s^{q-1} (f(s))^{\frac{q}{p}} \, d\mathcal{L}^1(s)\right)^{\frac{1}{q}} & \text{if } 1 < p < \infty \text{ and } 1 \le q < \infty, \text{or } p = q = 1; \\ \sup\limits_{s \in (0,\infty)} s (f(s))^{\frac{1}{p}} & \text{if } 1 < p < \infty \text{ and } q = \infty; \\ \mathcal{L}^1(\{s \in (0,\infty) : f(s) > 0\}) & \text{if } p = q = \infty\,, \end{cases} \tag{6.1}$$

for every non-increasing function $f : [0, \infty) \to [0, \infty]$. Hence, the functional $\mathcal{G}_{L^{p,q}}$ is concave if $1 \le q \le p$.

Proof. Equation (6.1) follows from a well-known expression of Lorentz norms in terms of the distribution function (see, e.g., [9, Proposition 1.4.9]), from equality (4.1) and from the fact that every non-increasing function $f : [0, \infty) \to [0, \infty]$ agrees a.e. with the function $f = (f_*)_*$. The fact that $\mathcal{G}_{L^{p,q}}$ is concave if $1 \le q \le p$ is an easy consequence of the representation formulas (6.1). In particular, the fact that the function $[0, \infty) \ni t \mapsto t^\alpha$ is concave if $0 < \alpha \le 1$ plays a role here. □

Lemma 6.2. *Suppose that $1 \le p < q < \infty$. Then there exists a function $u \in L^{p,q}(\mathbb{R}^n)$, having support of finite measure, such that*

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : \mathcal{M}_{L^{p,q}} u(x) > 1\}) = \infty. \tag{6.2}$$

Proof. We shall prove that the norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ does not satisfy condition (H) from Lemma 3.2, if $1 \le p < q < \infty$. The conclusion will then follow via Proposition 4.3.
To this purpose, define $f : (0,\infty) \to [0,\infty)$ as

$$f(s) = c\sum_{k=1}^{\infty}\left(\frac{3^k}{k}\right)^{\frac{1}{p}}\chi_{\left(\frac{1}{2\cdot 3^k},\frac{1}{2\cdot 3^{k-1}}\right)}(s) \qquad \text{for } s \in (0,\infty), \tag{6.3}$$

where $c > \left(\frac{q}{p}\right)^{\frac{1}{q}}$. Observe that $f = f^*\chi_{(0,1)}$ a.e., since f is a nonnegative decreasing function in $(0,\infty)$ with support in $(0,1)$. Moreover, $f \in L^{p,q}(0,\infty)$, since

$$\|f\|^q_{L^{p,q}(0,\infty)} = c^q\sum_{k=1}^{\infty}\int_{\frac{1}{2\cdot 3^k}}^{\frac{1}{2\cdot 3^{k-1}}}\left(\frac{3^k}{k}\right)^{\frac{q}{p}} s^{\frac{q}{p}-1}\,d\mathcal{L}^1(s) \le c^q\sum_{k=1}^{\infty}\left(\frac{3^k}{k}\right)^{\frac{q}{p}}\int_0^{\frac{1}{2\cdot 3^{k-1}}} s^{\frac{q}{p}-1}\,d\mathcal{L}^1(s) < \infty\,,$$

thanks to the assumption that $q > p$. For each $k \in \mathbb{N}$, set $I_k = \left(\frac{1}{2\cdot 3^k}, \frac{1}{2\cdot 3^{k-1}}\right)$ and $a_k = \frac{1}{k}$. Then

$$\|(f\chi_{I_k})^*\|^{\oslash}_{L^{p,q}(0,a_k)} = c\left(\int_0^{\infty}\left(\frac{3^k}{k}\right)^{\frac{q}{p}}\chi_{(0,\frac{1}{3^k})}\left(\frac{s}{k}\right)s^{\frac{q}{p}-1}\,d\mathcal{L}^1(s)\right)^{\frac{1}{q}} = c\left(\frac{p}{q}\right)^{\frac{1}{q}} > 1,$$

and hence condition (H) of Lemma 3.2 fails for the norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$. □

We are now in a position to accomplish the proof of Proposition 1.8.

Proof of Proposition 1.8. By Lemma 6.1, the functional $\mathcal{G}_{L^{p,q}}$ is concave if $1 \le q \le p$. Moreover, the norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ is locally absolutely continuous if and only if $q < \infty$ – see e.g. [16, Theorem 8.5.1]. Thereby, an application of Theorem 1.1 tells us that, if $1 \le q \le p < \infty$, then the norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ has the Lebesgue point property.
On the other hand, coupling Theorem 1.7 with Lemma 6.2 implies that the norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ does not have the Lebesgue point property if $1 \le p < q < \infty$.
In the remaining case when $q = \infty$, the norm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ is not locally absolutely continuous. Hence, by Theorem 1.1, it does not have the Lebesgue point property. □

One proof of Proposition 1.9, dealing with Orlicz norms, will follow from Theorem 1.7, via the next lemma.

Lemma 6.3. *Let A be a Young function satisfying the Δ_2-condition near infinity. Then*

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : \mathcal{M}_{L^A}u(x) > 1\}) < \infty$$

for every $u \in L^A(\mathbb{R}^n)$, supported in a set of finite measure.

Proof. Owing to Proposition 4.3, it suffices to show that condition (H) from Lemma 3.2 is fulfilled by the Luxemburg norm.
Consider any function $f \in L^A_1(0,\infty)$, any sequence $\{I_k\}$ of pairwise disjoint intervals in $(0,1)$, and any sequence $\{a_k\}$ of positive real numbers such that

$$a_k \ge \mathcal{L}^1(I_k) \quad \text{and} \quad \|(f\chi_{I_k})^*\|^{\oslash}_{L^A(0,a_k)} > 1 \quad \text{for } k \in \mathbb{N}.$$

Since

$$a_k < \int_0^{a_k} A((f\chi_{I_k})^*(s))\,d\mathcal{L}^1(s) = \int_{I_k} A(|f(s)|)\,d\mathcal{L}^1(s)$$

for every $k \in \mathbb{N}$, one has that

$$\sum_{k=1}^{\infty} a_k < \sum_{k=1}^{\infty}\int_{I_k} A(|f(s)|)\,d\mathcal{L}^1(s) \le \int_0^1 A(|f(s)|)\,d\mathcal{L}^1(s) < \infty.$$

Notice that the last inequality holds owing to the assumption that A satisfies the Δ_2-condition near infinity, and f has support of finite measure. Altogether, condition (H) is satisfied by the norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$. □

Proof of Proposition 1.9. If A satisfies the Δ_2-condition near infinity, then the norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ fulfills the Lebesgue point property, by Lemma 6.3 and Theorem 1.7. Conversely, assume that the norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ fulfills the Lebesgue point property. Then it has to be locally absolutely continuous, by either Theorem 1.1 or Theorem 1.4. Owing to [18, Theorem 14 and Corollary 5, Section 3.4], this implies that A satisfies the Δ_2-condition near infinity. □

In the next proposition, we point out the property, of independent interest, that the functional $\mathcal{G}_{L^A}$ is almost concave for *any* N-function A. Such a property, combined with the fact that the norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ is locally absolutely continuous if and only if A satisfies the Δ_2-condition near infinity, leads to an alternative proof of Proposition 1.9, at least when A is an N-function, via Theorem 1.1.

Proposition 6.4. *The functional $\mathcal{G}_{L^A}$ is almost concave for every N-function A.*

Proof. The norm $\|\cdot\|_{L^A(\mathbb{R}^n)}$ is equivalent, up to multiplicative constants, to the norm $\|\cdot\|_{L_A(\mathbb{R}^n)}$ defined as

$$\|u\|_{L_A(\mathbb{R}^n)} = \inf\left\{\frac{1}{k}\left(1+\int_{\mathbb{R}^n} A(k|u(x)|)\,dx\right) : k>0\right\}$$

for $u \in L^0(\mathbb{R}^n)$ – see [18, Section 3.3, Proposition 4 and Theorem 13]. One has that

$$\frac{1}{k}\left(1+\int_{\mathbb{R}^n} A(k|u(x)|)\,dx\right) = \frac{1}{k} + \int_0^\infty A'(kt)u_*(t)\,dt,$$

where A' denotes the left-continuous derivative of A. Altogether, we have that

$$\mathcal{G}_{L_A}(f) = \inf\left\{\frac{1}{k} + \int_0^\infty A'(kt)f(t)\,dt\right\}$$

for every non-increasing function $f:[0,\infty)\to[0,\infty)$. The functional $\mathcal{G}_{L_A}$ is concave, since it is the infimum of a family of linear functionals, and hence the functional $\mathcal{G}_{L^A}$ is almost concave. □

Let us next focus on the case of Lorentz endpoint norms, which is the object of Proposition 1.10.

Lemma 6.5. *Assume that $\varphi:[0,\infty)\to[0,\infty)$ is a (non identically vanishing) concave function. Then*

$$\mathcal{G}_{\Lambda_\varphi}(f) = \int_0^{\mathcal{L}^1(\{f>0\})} \varphi(f(t))\,d\mathcal{L}^1(t) \tag{6.4}$$

for every non-increasing function $f:[0,\infty)\to[0,\infty]$. In particular, the functional $\mathcal{G}_{\Lambda_\varphi}$ is concave.

Proof. Take any non-increasing function $f:[0,\infty)\to[0,\infty]$. Set $h=f_*$, whence $f=f^*=(f_*)_*=h_*$ a.e., and $h^*(0)=f_*(0)=\mathcal{L}^1(\{f>0\})$. From equations (2.25) and (4.1), one has, via Fubini's theorem,

$$
\begin{aligned}
\mathcal{G}_{\Lambda_\varphi}(f) = \mathcal{G}_{\Lambda_\varphi}(h_*) = \|h\|_{\Lambda_\varphi(0,\infty)} &= h^*(0)\varphi(0^+) + \int_0^\infty h^*(s)\varphi'(s)\,d\mathcal{L}^1(s)\\
&= h^*(0)\varphi(0^+) + \int_0^\infty \int_0^{h^*(s)} d\mathcal{L}^1(t)\,\varphi'(s)\,d\mathcal{L}^1(s)\\
&= h^*(0)\varphi(0^+) + \int_0^{h^*(0)} \int_0^{h_*(t)} \varphi'(s)\,d\mathcal{L}^1(s)\,d\mathcal{L}^1(t)\\
&= h^*(0)\varphi(0^+) + \int_0^{h^*(0)} [\varphi(h_*(t)) - \varphi(0^+)]\,d\mathcal{L}^1(t)\\
&= \int_0^{h^*(0)} \varphi(h_*(t))\,d\mathcal{L}^1(t) = \int_0^{\mathcal{L}^1(\{f>0\})} \varphi(f(t))\,d\mathcal{L}^1(t).
\end{aligned}
$$

Hence, formula (6.4) follows.
In order to verify the concavity of $\mathcal{G}_{\Lambda_\varphi}$, fix any pair of non-increasing functions $f, g : [0,\infty) \to [0,\infty]$ and $\lambda \in (0,1)$. Observe that

$$\{t \in [0,\infty) : \lambda f(t) + (1-\lambda)g(t) > 0\} = \{t \in [0,\infty) : f(t) > 0\} \cup \{t \in [0,\infty) : g(t) > 0\}.$$

The monotonicity of f and g ensures that the two sets on the right-hand side of the last equation are intervals whose left endpoint is 0. Consequently,

$$\mathcal{L}^1(\{\lambda f + (1-\lambda)g > 0\}) = \max\{\mathcal{L}^1(\{f > 0\}), \mathcal{L}^1(\{g > 0\})\}. \tag{6.5}$$

On making use of equations (6.4) and (6.5), and of the concavity of φ, one infers that the functional $\mathcal{G}_{\Lambda_\varphi}$ is concave as well. □

Proof of Proposition 1.10. By Lemma 6.5, the functional $\mathcal{G}_{\Lambda_\varphi}$ is concave for every non identically vanishing concave function $\varphi : [0,\infty) \to [0,\infty)$. On the other hand, it is easily verified, via equation (2.25), that the norm $\|\cdot\|_{\Lambda_\varphi(\mathbb{R}^n)}$ is locally absolutely continuous if, and only if, $\varphi(0^+) = 0$. The conclusion thus follows from Theorem 1.1. □

We conclude with a proof of Proposition 1.11.

Proof of Proposition 1.11. Assume first that $\lim_{s\to 0^+} \frac{s}{\varphi(s)} = 0$. Then we claim that the norm $\|\cdot\|_{M_\varphi(\mathbb{R}^n)}$ is not locally absolutely continuous, and hence, by either Theorem 1.1 or Theorem 1.4, it does not have the Lebesgue point property. To verify this claim, observe that the function $(0,\infty) \ni s \mapsto \frac{s}{\varphi(s)}$ is quasiconcave in the sense of [4, Definition 5.6, Chapter 2], and hence, by [4, Chapter 2, Proposition 5.10], there exists a concave function $\psi : (0,\infty) \to [0,\infty)$ such that $\frac{1}{2}\psi(s) \le \frac{s}{\varphi(s)} \le \psi(s)$ for $s \in (0,\infty)$. Let ψ' denote the right-continuous derivative of ψ, and define $u(x) = \psi'(\omega_n|x|^n)$ for $x \in \mathbb{R}^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Then $u^* = \psi'$ in $(0,\infty)$, so that

$$1 \le u^{**}(s)\varphi(s) \le 2 \quad \text{for } s \in (0,\infty).$$

The second inequality in the last equation ensures that $u \in M_\varphi(\mathbb{R}^n)$, whereas the first one tells us that u does not have a locally absolutely continuous norm in $M_\varphi(\mathbb{R}^n)$.
Conversely, assume that $\lim_{s\to 0^+} \frac{s}{\varphi(s)} > 0$, then $(M_\varphi)_{\mathrm{loc}}(\mathbb{R}^n) = L^1_{\mathrm{loc}}(\mathbb{R}^n)$, with equivalent norms on any given subset of $\mathbb{R}^n$ with finite measure (see e.g. [20, Theorem 5.3]). Hence, the norm $\|\cdot\|_{M_\varphi(\mathbb{R}^n)}$ has the Lebesgue point property, since $\|\cdot\|_{L^1(\mathbb{R}^n)}$ has it. □

References

[1] N. Aïssaoui, Maximal operators, Lebesgue points and quasicontinuity in strongly nonlinear potential theory, *Acta Math. Univ. Comenian.* (N.S.) **71** (2002), no. 1, 35–50.

[2] R. J. Bagby & J. D. Parsons, Orlicz spaces and rearranged maximal functions, *Math. Nachr.* 132 (1987), 15–27.

[3] J. Bastero, M. Milman & F. J. Ruiz, Rearrangement of Hardy-Littlewood maximal functions in Lorentz spaces, *Proc. Amer. Math. Soc* 128 (2000), 65 – 74.

[4] C. Bennett & R.Sharpley, "Interpolation of operators", Academic Press, Boston, 1988.

[5] M. S. Braverman & A. A. Sedaev, L. Kh. Poritskaya's characterization of symmetric spaces. (Russian) *Qualitative and approximate methods for investigating operator equations (Russian)*, 21–33, 167, Yaroslav. Gos. Univ., Yaroslavl', 1982.

[6] M. S. Braverman & A. A. Sedaev, On a characteristic of symmetric spaces. (Russian) *Funktsional. Anal. i Prilozhen*, 15 (1981), no. 2, 65–66.

[7] P. Cavaliere & A. Cianchi, Classical and approximate Taylor expansions of weakly differentiable functions, *Ann. Acad. Sci. Fenn. Math.* 39 (2014), no. 2, 527–544.

[8] M. Ciesielski & A. Kaminska, Lebesgue's differentiation theorems in R.I. quasi-Banach spaces and Lorentz spaces $\Gamma_{p,w}$, *J. Funct. Spaces Appl.* (2012), Art. ID 682960, 28 pp.

[9] L. Grafakos, "Classical Fourier Analysis, Second edition", Grad. Texts Math. Vol. 49 (Springer, Berlin, 2008).

[10] S. G. Krein, Yu. I. Petunin & E. M. Semënov, "Interpolation of Linear Operators", American Mathematical Society, Providence, R.I., 1982.

[11] M. A. Leckband, A note on maximal operators and reversible weak type inequalities, *Proc. Amer. Math. Soc* **92** (1984), no. 1, 19–26.

[12] A. K. Lerner, A new approach to rearrangements of maximal operators, *Bull. London Math. Soc.* **37** (2005), no. 5, 771–777.

[13] F.E. Levis, Weak inequalities for maximal functions in Orlicz-Lorentz spaces and applications, *J. Approx. Theory* **162** (2010), no. 2, 239–251.

[14] M. Mastyło & C. Pérez, The Hardy-Littlewood maximal type operators between Banach function spaces, *Indiana Univ. Math. J.* **61** (2012), no. 3, 883–900.

[15] C. Pérez, Two weighted inequalities for potential and fractional type maximal operators, *Indiana Univ. Math. J.* **43** (1994), no. 2, 663–683.

[16] L. Pick, A. Kufner, O. John & S. Fučík, "Function spaces. Vol. 1. Second revised and extended edition", de Gruyter & Co., Berlin, 2013.

[17] L. H. Poritskaya, Lebesgue - Banach points of functions in symmetric spaces, *Mat. Zametki* **23** (1978), no. 4, 581–592.

[18] M. M. Rao & Z. D. Ren "Theory of Orlicz spaces", Marcel Dekker, Inc., New York, 1991.

[19] D. V. Salehov, E. M. Semenov, On Lebesgue - Banach points (Russian), *Uspehi Mat. Nauk* **20** (1965), no. 6 (126), 151–156.

[20] L. Slavíková, Almost-compact embeddings, *Math. Nachr.* **285** (2012), no. 11–12, 1500–1516.

[21] E.M. Stein, "Singular integrals and differentiability properties of functions", Princeton University Press, Princeton, N.J. 1970.

[22] E.M. Stein, Editor's note: The differentiability of functions in $\mathbb{R}^n$, *Ann. of Math.* (2) **113** (1981), no. 2, 383–385.

Dipartimento di Matematica, Università di Salerno, Via Giovanni Paolo II, 84084 Fisciano (SA), Italy

E-mail address: pcavaliere@unisa.it

Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 57/A, 50134 Firenze, Italy

E-mail address: cianchi@unifi.it

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: pick@karlin.mff.cuni.cz

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: slavikova@karlin.mff.cuni.cz

Paper IV

L. Slavíková. On the necessity of bump conditions for the two-weighted maximal inequality. Preprint, 2015.

ON THE NECESSITY OF BUMP CONDITIONS FOR THE TWO-WEIGHTED MAXIMAL INEQUALITY

LENKA SLAVÍKOVÁ

Abstract. We study the necessity of bump conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p spaces with different weights. The conditions in question are obtained by replacing the $L^{p'}$-average of $\sigma^{\frac{1}{p'}}$ in the Muckenhoupt A_p-condition by an average with respect to a stronger Banach function norm, and are known to be sufficient for the two-weighted maximal inequality. We show that these conditions are in general not necessary for such an inequality to be true.

1. Introduction and statement of the result

The Hardy–Littlewood maximal operator M is defined for every measurable function f on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{Q: x\in Q} \frac{1}{|Q|}\int_Q |f|, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q containing x. By a "cube" we always mean a compact cube with sides parallel to coordinate axes.

Assume that $1 < p < \infty$. A longstanding open problem in harmonic analysis is to characterize those couples (w, σ) of nonnegative locally integrable functions, called weights in the sequel, which satisfy the inequality

$$\int_{\mathbb{R}^n} w(M(f\sigma))^p \le C \int_{\mathbb{R}^n} \sigma|f|^p \tag{1.1}$$

for all measurable functions f and some positive constant C.

In the special case when $\sigma = w^{1-p'}$, where $p' = \frac{p}{p-1}$, inequality (1.1) was characterized by Muckenhoupt [14]. He showed that the correct necessary and sufficient condition is the A_p-condition

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q \sigma\right)^{p-1} < \infty. \tag{1.2}$$

We note that throughout this paper, the notation $\sup_Q$ means that the supremum is taken over all cubes Q in $\mathbb{R}^n$.

The situation is much more complicated in the two-weighted case, when we do not assume any relationship between w and σ. It is well known that the A_p-condition (1.2) is still necessary for (1.1) in this setting, but it is not sufficient any more (see, e.g., [9, Chapter 4, Example 1.15]). A solution to the two-weighted problem was given by Sawyer [21], who showed that (1.1) holds

---

2010 *Mathematics Subject Classification.* 42B25, 42B35.

Key words and phrases. Bump condition, two-weighted inequality, Hardy-Littlewood maximal operator.

This research was partly supported by the grant P201-13-14743S of the Grant Agency of the Czech Republic.

if and only if there is a positive constant C such that

$$\int_Q w(M(\chi_Q\sigma))^p \leq C \int_Q \sigma \tag{1.3}$$

for every cube Q. This characterizing condition, however, still involves the operator M itself, and hence does not give a quite satisfactory answer to the above-mentioned problem.

Another approach to the two-weighted problem (1.1) consists in finding sufficient conditions for (1.1) that are close in form to the A_p-condition (1.2). These conditions are called "bump conditions" in the literature. They are more explicit than (1.3), and thus more appropriate for the use in applications. On the other hand, as we will show in the present paper, these conditions are not necessary for (1.1) - at least not in their currently available form.

To introduce the bump theory, let us first observe that the A_p-condition (1.2) can be written in the form

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{L^{p'},Q} < \infty, \tag{1.4}$$

where, for any $q \in (1,\infty)$ and any cube Q, $\|\cdot\|_{L^q,Q}$ denotes the L^q-norm on Q with respect to the normalized Lebesgue measure $dx/|Q|$.

Neugebauer [17] showed that if the norms in (1.4) are replaced by stronger Lebesgue norms, namely, if

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^{pr},Q}\|\sigma^{\frac{1}{p'}}\|_{L^{p'r},Q} < \infty \tag{1.5}$$

holds for some $r > 1$, then the two-weighted maximal inequality (1.1) is fulfilled.

Pérez [19] found a way how to weaken the sufficient condition (1.5). He noticed that in order to obtain (1.1) one just needs to "bump" in a suitable way the $L^{p'}$-norm of $\sigma^{\frac{1}{p'}}$ in (1.4). He also showed that more general norms than just those of Lebesgue can be used in this connection. For instance, if L^B denotes the Orlicz space induced by the Young function B and $\|\cdot\|_{L^B,Q}$ stands for the normalized Orlicz norm on a cube Q, then the bump condition

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{L^B,Q} < \infty \tag{1.6}$$

was proved in [19] to be sufficient for (1.1) provided that the complementary Young function $\overline{B} \in B_p$, that is, $\overline{B}$ satisfies the B_p-condition

$$\int_1^\infty \frac{\overline{B}(t)}{t^p}\frac{dt}{t} < \infty. \tag{1.7}$$

(See Section 2 for definitions regarding Orlicz spaces.) We point out that condition (1.7) is sharp in the sense that whenever L^B is an Orlicz space such that (1.6) implies (1.1) for every couple (w,σ), then (1.7) has to be fulfilled.

A basic example of an Orlicz space for which this result can be applied is the Lebesgue space L^q with $q > p'$. The strength of the result lies, however, in Orlicz spaces that are "closer to $L^{p'}$", such as, for instance, the space $L^{p'}(\log L)^\gamma$ with $\gamma > p'-1$.

The last result can be further improved if yet more general spaces of measurable functions, the so called "Banach function spaces" (see Section 2 for the definition), are brought into play. For any Banach function space X, we define the normalized X-norm on a cube Q by

$$\|f\|_{X,Q} = \|\tau_{\ell(Q)} f\chi_Q\|_X,$$

where τ_δ denotes, for $\delta > 0$, the dilation operator $\tau_\delta f(x) = f(\delta x)$, and $\ell(Q)$ stands for the sidelength of the cube Q. The maximal operator M_X is then given by

$$M_X f(x) = \sup_{Q: x\in Q} \|f\|_{X,Q}, \quad x \in \mathbb{R}^n.$$

Notice that if $X = L^1$ then M_X coincides with the classical Hardy-Littlewood maximal operator M.

A sufficient condition for (1.1), proved in [19] again, has the form

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q} \|\sigma^{\frac{1}{p'}}\|_{X,Q} < \infty, \tag{1.8}$$

where X is any Banach function space whose associate space X' fulfils

$$\int_{\mathbb{R}^n} (M_{X'} f)^p \le C \int_{\mathbb{R}^n} |f|^p \tag{1.9}$$

for all measurable functions f and some positive constant C. Condition (1.9) can be reduced to the B_p-condition if X is an Orlicz space.

Condition (1.9) can be weakened if we allow it to depend on σ. Namely, the following implication holds: if X is a Banach function space such that (1.8) is fulfilled and there is a positive constant C for which

$$\int_Q (M_{X'}(\sigma^{\frac{1}{p}} \chi_Q))^p \le C \int_Q \sigma \tag{1.10}$$

for every cube Q, then (1.1) holds. This was proved by Pérez and Rela [20] as a consequence of the Sawyer characterization of the two-weighted maximal inequality. We note that the result in [20] is restricted only to Orlicz spaces, however, it is easy to observe that the proof given there works equally well for an arbitrary Banach function space over $\mathbb{R}^n$. Moreover, the paper [20] gives even a quantitative version of this result which is shown to hold, at least for Orlicz spaces, not only in the Euclidean setting, but also in the more general context of spaces of homogeneous type.

It is worth noticing that (1.10) is in many situations considerably weaker than (1.9). For instance, one can easily observe that (1.9) is not valid when $X' = L^p$, while (1.10) holds with $X' = L^p$ if and only if

$$\int_Q (M_{L^p}(\sigma^{\frac{1}{p}} \chi_Q))^p = \int_Q M(\sigma \chi_Q) \le C \int_Q \sigma \tag{1.11}$$

for all cubes Q. It was shown by Fujii [8] and rediscovered later by Wilson [22] that the validity of condition (1.11) is equivalent to the fact that σ is an A_∞-weight, that is, a weight which satisfies the one-weighted A_p-condition for some $p > 1$.

Let us mention that the bump theory is an active area of research not only in connection with the two-weighted inequality for the Hardy-Littlewood maximal operator, but especially in connection with a similar inequality for singular integral operators. The situation is, however, significantly more complicated in that setting, and there are still important open problems that wait to be resolved. So far, it has been shown that the bump condition

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^A,Q} \|\sigma^{\frac{1}{p'}}\|_{L^B,Q} < \infty \tag{1.12}$$

is sufficient for the two-weighted inequality for singular integral operators provided that $\overline{A} \in B_{p'}$ and $\overline{B} \in B_p$. Several partial results regarding the sufficiency of (1.12) appeared in the literature [3, 4, 5, 6, 12, 18] until the proof in full generality was found by Lerner [13] and independently by Nazarov, Reznikov, Treil and Volberg [15] (for $p=2$). It was conjectured that the weaker condition

$$\sup_Q \|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{L^B,Q}<\infty \quad \& \quad \sup_Q \|w^{\frac{1}{p}}\|_{L^A,Q}\|\sigma^{\frac{1}{p'}}\|_{L^{p'},Q}<\infty$$

with $\overline{A}\in B_{p'}$ and $\overline{B}\in B_p$ might be sufficient as well, however, only partial results have been proved so far - see, e.g., [1, 7, 10, 11, 16].

The principal question we shall discuss in this paper is the necessity of bump conditions for the two-weighted maximal inequality. As we have seen, several versions of bump conditions are now available in the literature. We shall focus on the one due to Pérez and Rela [20], which has been the weakest so far.

Question 1.1. Given a couple (w,σ) of weights satisfying (1.1), is it true that there is a Banach function space X fulfilling (1.8) and (1.10)?

We notice that the answer to this question is positive whenever σ is an A_∞-weight. Indeed, in this situation it suffices to take $X=L^{p'}$. We already know that (1.10) is then fulfilled (see (1.11)). Further, condition (1.8) is in this case just the standard A_p-condition, which is well known to be necessary for (1.1). In fact, according to the reverse Hölder inequality (see, e.g., [9, Chapter 4, Lemma 2.5]), condition (1.1) implies even (1.8) with $X=L^{p'+\varepsilon}$ for some $\varepsilon>0$, depending on σ. Since the space $X=L^{p'+\varepsilon}$ satisfies not only (1.10), but also the stronger condition (1.9) (or, equivalently, condition (1.7) with $B(t)=t^{p'+\varepsilon}$), one can obtain even a better conclusion in this case.

The interesting problem is whether a similar result holds without the A_∞-assumption. We show that this is not the case in general.

Given $x\in\mathbb{R}^n$, we shall denote by $|x|_{\max}$ the maximum norm of x, that is, if $x=(x_1,\dots,x_n)$ then $|x|_{\max}=\max_{i=1,\dots,n}|x_i|$. We shall also use the notation $\log_+ x=\max\{\log x,0\}$, $x>0$.

Theorem 1.2. *Let $1<p<\infty$, and let*

$$\begin{aligned} w(x)&=\frac{|x|_{\max}^{n(p-1)}}{(1+\log_+|x|_{\max}^n)^p},\\ \sigma(x)&=\frac{1}{|x|_{\max}^n(1+\log_+\frac{1}{|x|_{\max}^n})^{p'}}, \quad x\in\mathbb{R}^n\setminus\{0\}. \end{aligned}$$

Then the couple (w,σ) fulfils (1.1), but there is no Banach function space X for which (1.8) and (1.10) hold simultaneously.

Remark 1.3. Assume that $\alpha\in(0,n)$ and $\beta\in\mathbb{R}$, and set

$$\sigma(x)=\frac{1}{|x|_{\max}^\alpha(1+\log_+\frac{1}{|x|_{\max}^n})^{\beta}}, \quad x\in\mathbb{R}^n\setminus\{0\}.$$

Then the answer to Question 1.1 is positive, regardless of what w is. This follows from the fact that σ is an A_∞-weight, combined with our previous observations.

2. Preliminaries

In this section we collect necessary prerequisities from the theory of Banach function spaces. An interested reader can find more details in [2].

Let $n\in\mathbb{N}$. We denote by $\mathcal{M}$ the set of all Lebesgue measurable functions on $\mathbb{R}^n$ having their values in $[-\infty,\infty]$. If F is a measurable subset of $\mathbb{R}^n$, then $|F|$ denotes the Lebesgue measure of F.

We say that a functional $\|\cdot\|_X : \mathcal{M} \to [0,\infty]$ is a *Banach function norm* if, for all functions f, $g \in \mathcal{M}$, for all sequences $(f_k)_{k=1}^{\infty}$ in $\mathcal{M}$ and for all constants $a \in \mathbb{R}$, the following properties hold:

(P1) $\|f\|_X = 0$ if and only if $f = 0$ a.e.; $\|af\|_X = |a|\|f\|_X$; $\|f+g\|_X \le \|f\|_X + \|g\|_X$;
(P2) $|f| \le |g|$ a.e. implies $\|f\|_X \le \|g\|_X$;
(P3) $|f_k| \nearrow |f|$ a.e. implies $\|f_k\|_X \nearrow \|f\|_X$;
(P4) if $F \subseteq \mathbb{R}^n$ with $|F| < \infty$ then $\|\chi_F\|_X < \infty$;
(P5) if $F \subseteq \mathbb{R}^n$ with $|F| < \infty$ then $\int_F |f(x)|\,dx \le C_F\|f\|_X$ for some constant C_F depending on F but independent of f.

The collection of all $f \in \mathcal{M}$ for which $\|f\|_X < \infty$ is denoted by X and is called a *Banach function space*.

To every Banach function norm $\|\cdot\|_X$ there corresponds another functional on $\mathcal{M}$, denoted by $\|\cdot\|_{X'}$ and defined, for $g \in \mathcal{M}$, by

$$\|g\|_{X'} = \sup_{\|f\|_X \le 1} \int_{\mathbb{R}^n} |f(x)g(x)|\,dx. \tag{2.1}$$

It turns out that $\|\cdot\|_{X'}$ is also a Banach function norm, we call it the *associate norm* of $\|\cdot\|_X$. The Banach function space X' built upon the Banach function norm $\|\cdot\|_{X'}$ is called the *associate space* of X. It is known (see, e.g., [2, Chapter 1, Theorem 2.7]) that $(X')' = X$.

Let us now mention particular examples of Banach function spaces. The basic examples are the *Lebesgue spaces* L^p, given by

$$\|f\|_{L^p} = \begin{cases} \left(\int_{\mathbb{R}^n} |f(y)|^p\,dy\right)^{\frac{1}{p}}, & 1 \le p < \infty; \\ \operatorname{esssup}_{y\in\mathbb{R}^n} |f(y)|, & p = \infty, \quad f \in \mathcal{M}. \end{cases}$$

A generalization of Lebesgue spaces is provided by the notion of *Orlicz spaces*. Given a *Young function* B, namely, a nonnegative continuous increasing convex function on $[0,\infty)$ such that $\lim_{t\to 0_+} \frac{B(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{B(t)}{t} = \infty$, the Orlicz norm $\|\cdot\|_{L^B}$ is given by

$$\|f\|_{L^B} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} B\left(\frac{|f(y)|}{\lambda}\right) dy \le 1\right\}, \quad f \in \mathcal{M}. \tag{2.2}$$

It can be shown that $\|\cdot\|_{L^B}$ is indeed a Banach function norm and, for any cube Q, the normalized Orlicz norm on Q can be expressed in the form

$$\|f\|_{L^B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right) dy \le 1\right\}, \quad f \in \mathcal{M}.$$

The associate norm to $\|\cdot\|_{L^B}$ is equivalent to another Orlicz norm induced by the *complementary Young function* $\overline{B}$ defined by

$$\overline{B(t)} = \sup_{s\ge 0}(st - B(s)), \quad t \in [0,\infty).$$

For any $p \in (1,\infty)$, the particular choice of $B(t) = t^p$ in (2.2) yields the Lebesgue space L^p. We note that $(L^p)' = L^{p'}$, where we employ the usual notation $p' = \frac{p}{p-1}$. The Orlicz space induced by the Young function $B(t) = t^p \log^{\gamma}(e+t)$ for $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$ is denoted by $L^p(\log L)^{\gamma}$ and one has $(L^p(\log L)^{\gamma})' = L^{p'}(\log L)^{-\gamma}$.

3. Proof of Theorem 1.2

We devote this section to the proof of Theorem 1.2. Throughout the proof, we shall denote

$$Q_r = \{x \in \mathbb{R}^n : |x|_{\max} \le r\}, \quad r > 0;$$

in other words, Q_r will stand for the cube centered at 0 and with sidelength $2r$. We shall write "$\approx$" in order to express that the two sides of an equation are equivalent up to multiplicative constants independent of appropriate quantities.

Proof of Theorem 1.2. We first prove that

$$M\sigma(x) \approx \frac{1 + \left|\log \frac{1}{|x|^n_{\max}}\right|}{|x|^n_{\max}(1+\log_+ \frac{1}{|x|^n_{\max}})^{p'}}, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{3.1}$$

up to multiplicative constants depending on p and n.

Consider the function

$$f(t) = \frac{1}{t(1+\log_+ \frac{1}{t})^{p'}}, \quad t > 0.$$

Since $\lim_{t\to 0_+} f(t) = \infty$ and f is nonincreasing on some neighbourhood of 0, we can find $a \in (0,1)$ such that $f(a) \ge 1$ and f is nonincreasing on $(0,a)$. Let us set

$$g(t) = \begin{cases} \frac{1}{t(1+\log \frac{1}{t})^{p'}}, & t \in (0,a), \\ 1, & t \in [a,1], \\ \frac{1}{t}, & t \in (1,\infty). \end{cases}$$

Then g is nonincreasing on $(0,\infty)$ and $f \approx g$ on $(0,\infty)$, since $f(t) = g(t)$ unless $t \in [a,1]$, and $c_1 \le f(t) \le c_2$ for some $c_1 > 0$, $c_2 > 0$ and every $t \in [a,1]$. Therefore,

$$\sigma(x) = f(|x|^n_{\max}) \approx g(|x|^n_{\max}) =: h(x), \quad x \in \mathbb{R}^n \setminus \{0\},$$

and, by the coarea formula,

(3.2)

$$\begin{aligned} M\sigma(x) \approx Mh(x) &= \frac{1}{Q_{|x|_{\max}}} \int_{Q_{|x|_{\max}}} h(y)\,dy = \frac{1}{2^n |x|^n_{\max}} \int_0^{|x|_{\max}} \int_{\{y\in\mathbb{R}^n : |y|_{\max} = r\}} h(y)\,d\mathcal{H}^{n-1}(y)\,dr \\ &\approx \frac{1}{|x|^n_{\max}} \int_0^{|x|_{\max}} g(r^n) r^{n-1}\,dr \approx \frac{1}{|x|^n_{\max}} \int_0^{|x|^n_{\max}} g(s)\,ds \approx \frac{1}{|x|^n_{\max}} \int_0^{|x|^n_{\max}} f(s)\,ds, \end{aligned}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

Given $t \in (0,1]$, we have

$$\int_0^t f(s)\,ds = \int_0^t \frac{ds}{s(1+\log\frac{1}{s})^{p'}} \approx \frac{1}{(1+\log\frac{1}{t})^{p'-1}} = \frac{1+|\log\frac{1}{t}|}{(1+\log_+\frac{1}{t})^{p'}}. \tag{3.3}$$

Also, if $t \in (1,\infty)$ then

$$\int_0^t f(s)\,ds = \int_0^1 f(s)\,ds + \int_1^t f(s)\,ds = C + \int_1^t \frac{ds}{s} = C + \log t \approx 1 + \log t = \frac{1+|\log\frac{1}{t}|}{(1+\log_+\frac{1}{t})^{p'}}. \tag{3.4}$$

A combination of (3.2), (3.3) and (3.4) yields (3.1).

Using (3.1), we obtain

$$\begin{aligned}
(M\sigma)^p(x)w(x) &\approx \frac{\left(1+\left|\log\frac{1}{|x|^n_{\max}}\right|\right)^p}{|x|^n_{\max}(1+\log_+\frac{1}{|x|^n_{\max}})^{p'p}(1+\log_+|x|^n_{\max})^p}\\
&= \begin{cases}\frac{1}{|x|^n_{\max}(1+\log\frac{1}{|x|^n_{\max}})^{p'}}, & |x|_{\max}\le 1;\\ \frac{1}{|x|^n_{\max}}, & |x|_{\max}>1\end{cases}\\
&= \frac{1}{|x|^n_{\max}(1+\log_+\frac{1}{|x|^n_{\max}})^{p'}} = \sigma(x), \quad x\in\mathbb{R}^n\setminus\{0\}.
\end{aligned}$$

Hence, for any cube Q,

$$\int_Q (M(\chi_Q\sigma))^p(x)w(x)\,dx \le \int_Q (M\sigma)^p(x)w(x)\,dx \approx \int_Q \sigma(x)\,dx,$$

and Sawyer's characterization (1.3) of the two-weighted maximal inequality yields that the couple (w,σ) satisfies (1.1).

Let X be any Banach function space. Given $b\in(0,1)$, we have

$$\begin{aligned}
\left\|\sigma^{\frac1p}\right\|_{X',Q_b} &= \left\|\left(\sigma^{\frac1p}\chi_{Q_b}\right)(2by)\right\|_{X'}\\
&= \frac{1}{(2b)^{\frac np}}\left\|\frac{\chi_{Q_{\frac12}}(y)}{|y|^{\frac np}_{\max}\left(1+\log_+\frac{1}{(2b|y|_{\max})^n}\right)^{\frac{p'}{p}}}\right\|_{X'}\\
&\ge \frac{1}{(2b)^{\frac np}}\left\|\frac{\chi_{Q_{\frac12}\setminus Q_{\frac b2}}(y)}{|y|^{\frac np}_{\max}\left(1+\log_+\frac{1}{b^{2n}}\right)^{\frac{p'}{p}}}\right\|_{X'}\\
&= \frac{1}{(2b)^{\frac np}\left(1+2\log_+\frac{1}{b^n}\right)^{\frac{p'}{p}}}\left\|\frac{\chi_{Q_{\frac12}\setminus Q_{\frac b2}}(y)}{|y|^{\frac np}_{\max}}\right\|_{X'}\\
&\ge \frac{1}{2^{\frac{n+p'}{p}}b^{\frac np}(1+\log_+\frac{1}{b^n})^{\frac{p'}{p}}}\left\|\frac{\chi_{Q_{\frac12}\setminus Q_{\frac b2}}(y)}{|y|^{\frac np}_{\max}}\right\|_{X'}.
\end{aligned}$$

Thus, for any $a\in(0,1)$,

$$\begin{aligned}
\int_{Q_a}\left(M_{X'}(\sigma^{\frac1p}\chi_{Q_a})\right)^p(x)\,dx &\ge \int_{Q_a}\left\|\sigma^{\frac1p}\right\|^p_{X',Q_{|x|_{\max}}}dx\\
&\ge \int_{Q_a}\frac{1}{2^{p'+n}|x|^n_{\max}(1+\log_+\frac{1}{|x|^n_{\max}})^{p'}}\left\|\frac{\chi_{Q_{\frac12}\setminus Q_{\frac{|x|_{\max}}{2}}}(y)}{|y|^{\frac np}_{\max}}\right\|^p_{X'}dx\\
&\ge \frac{1}{2^{p'+n}}\left\|\frac{\chi_{Q_{\frac12}\setminus Q_{\frac a2}}(y)}{|y|^{\frac np}_{\max}}\right\|^p_{X'}\int_{Q_a}\frac{dx}{|x|^n_{\max}(1+\log_+\frac{1}{|x|^n_{\max}})^{p'}}\\
&= \frac{1}{2^{p'+n}}\left\|\frac{\chi_{Q_{\frac12}\setminus Q_{\frac a2}}(y)}{|y|^{\frac np}_{\max}}\right\|^p_{X'}\int_{Q_a}\sigma(x)\,dx.
\end{aligned}\tag{3.5}$$

Assume that X fulfils (1.10). Then there is a constant $C>0$, independent of $a\in(0,1)$, such that

$$\int_{Q_a}\left(M_{X'}(\sigma^{\frac{1}{p}}\chi_{Q_a})\right)^p(x)\,dx\le C\int_{Q_a}\sigma(x)\,dx. \tag{3.6}$$

Since $\int_{Q_a}\sigma(x)\,dx$ is positive and finite, a combination of (3.5) and (3.6) yields that

$$\left\|\frac{\chi_{Q_{\frac{1}{2}}\setminus Q_{\frac{a}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|_{X'}\le 2^{\frac{p'+n}{p}}C^{\frac{1}{p}}=:D.$$

Passing to limit when a tends to 0 and using the property (P3) of $\|\cdot\|_X$, we obtain

$$\left\|\frac{\chi_{Q_{\frac{1}{2}}}(y)}{|y|_{\max}^{\frac{n}{p}}}\right\|_{X'}\le D. \tag{3.7}$$

To get a contradiction, assume that condition (1.8) is satisfied as well. Since

$$\left(\int_{Q_{\frac{1}{2}}}w(x)\,dx\right)^{\frac{1}{p}}\|\sigma^{\frac{1}{p'}}\|_{X,Q_{\frac{1}{2}}}=\|w^{\frac{1}{p}}\|_{L^p,Q_{\frac{1}{2}}}\|\sigma^{\frac{1}{p'}}\|_{X,Q_{\frac{1}{2}}}\le\sup_Q\|w^{\frac{1}{p}}\|_{L^p,Q}\|\sigma^{\frac{1}{p'}}\|_{X,Q}<\infty$$

and $\int_{Q_{\frac{1}{2}}}w(x)\,dx$ is clearly positive, we deduce that $\|\sigma^{\frac{1}{p'}}\|_{X,Q_{\frac{1}{2}}}<\infty$. However, by (3.7) and by the identity $X=(X')'$, we have

$$\begin{aligned}\|\sigma^{\frac{1}{p'}}\|_{X,Q_{\frac{1}{2}}}&=\sup_{\|u\|_{X'}\le1}\int_{Q_{\frac{1}{2}}}\sigma^{\frac{1}{p'}}(x)|u(x)|\,dx\\&\ge\frac{1}{D}\int_{Q_{\frac{1}{2}}}\frac{\sigma^{\frac{1}{p'}}(x)}{|x|_{\max}^{\frac{n}{p}}}\,dx\\&=\frac{1}{D}\int_{Q_{\frac{1}{2}}}\frac{dx}{|x|_{\max}^n(1+\log_+\frac{1}{|x|_{\max}^n})}=\infty,\end{aligned}$$

a contradiction. Thus, conditions (1.8) and (1.10) cannot be fulfilled simultaneously. The proof is complete. ☐

Acknowledgements

I would like to thank Carlos Pérez for fruitful discussions on two-weighted inequalities and, in particular, for suggesting to me the problem of the necessity of bump conditions. I am grateful to Luboš Pick for careful reading of this paper and helpful comments. I also thank the referee for her/his valuable suggestions.

References

[1] T. C. Anderson, D. Cruz-Uribe, and K. Moen. Logarithmic bump conditions for Calderón-Zygmund operators on spaces of homogeneous type. *Publ. Mat.*, 59(1):17–43, 2015.

[2] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[3] D. Cruz-Uribe, J. M. Martell, and C. Pérez. Sharp two-weight inequalities for singular integrals, with applications to the Hilbert transform and the Sarason conjecture. *Adv. Math.*, 216(2):647–676, 2007.

[4] D. Cruz-Uribe, J. M. Martell, and C. Pérez. Sharp weighted estimates for classical operators. *Adv. Math.*, 229(1):408–441, 2012.

[5] D. Cruz-Uribe and C. Pérez. Two-weight, weak-type norm inequalities for fractional integrals, Calderón-Zygmund operators and commutators. *Indiana Univ. Math. J.*, 49(2):697–721, 2000.
[6] D. Cruz-Uribe and C. Pérez. On the two-weight problem for singular integral operators. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 1(4):821–849, 2002.
[7] D. Cruz-Uribe, A. Reznikov, and A. Volberg. Logarithmic bump conditions and the two-weight boundedness of Calderón-Zygmund operators. *Adv. Math.*, 255:706–729, 2014.
[8] N. Fujii. Weighted bounded mean oscillation and singular integrals. *Math. Japon.*, 22(5):529–534, 1977/78.
[9] J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.
[10] T. Hytönen and C. Pérez. Sharp weighted bounds involving A_∞. *Anal. PDE*, 6(4):777–818, 2013.
[11] M. T. Lacey. On the separated bumps conjecture for Calderón-Zygmund operators. *Hokkaido Math. J.*, To appear.
[12] A. K. Lerner. A pointwise estimate for the local sharp maximal function with applications to singular integrals. *Bull. Lond. Math. Soc.*, 42(5):843–856, 2010.
[13] A. K. Lerner. On an estimate of Calderón-Zygmund operators by dyadic positive operators. *J. Anal. Math.*, 121:141–161, 2013.
[14] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.
[15] F. Nazarov, A. Reznikov, S. Treil, and A. Volberg. A Bellman function proof of the L^2 bump conjecture. *J. Anal. Math.*, 121:255–277, 2013.
[16] F. Nazarov, A. Reznikov, and A. Volberg. Bellman approach to the one-sided bumping for weighted estimates of Calderón–Zygmund operators. *Preprint*, arXiv:1306.2653.
[17] C. J. Neugebauer. Inserting A_p-weights. *Proc. Amer. Math. Soc.*, 87(4):644–648, 1983.
[18] C. Pérez. Weighted norm inequalities for singular integral operators. *J. London Math. Soc. (2)*, 49(2):296–308, 1994.
[19] C. Pérez. On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights. *Proc. London Math. Soc. (3)*, 71(1):135–157, 1995.
[20] C. Pérez and E. Rela. A new quantitative two weight theorem for the Hardy-Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 143(2):641–655, 2015.
[21] E. T. Sawyer. A characterization of a two-weight norm inequality for maximal operators. *Studia Math.*, 75(1):1–11, 1982.
[22] J. M. Wilson. Weighted inequalities for the dyadic square function without dyadic A_∞. *Duke Math. J.*, 55(1):19–50, 1987.

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: `slavikova@karlin.mff.cuni.cz`