

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

**DOCTORAL THESIS**

Michal Zamboj

# Synthetic projective geometry

Mathematical Institute

Supervisor of the doctoral thesis: Mgr. Lukáš Krump, Ph.D.
Study programme: Mathematics (P1101)
Study branch: 4M8 (1101V026)

Prague 2018

I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

I understand that my work relates to the rights and obligations under the Act No. 121/2000 Sb., the Copyright Act, as amended, in particular the fact that the Charles University has the right to conclude a license agreement on the use of this work as a school work pursuant to Section 60 subsection 1 of the Copyright Act.

In ..... date ..... signature of the author

I am most grateful to my supervisor, Mgr. Lukáš Krump, Ph.D., for his human approach, unceasing enthusiasm in my research during the years, and many great remarks that helped to refine my papers and the proposed thesis.

I also want to express my sincerest gratitude to Professor Hellmuth Stachel, with whom I consulted and revised the thesis during my stay in Vienna. Thanks to his significant effort and invitation, I was awarded an Aktion scholarship at TU Wien, where I wrote the last parts. Particularly valuable was the library of the Department of Discrete Mathematics and Geometry with original historical sources.

Finally, I wish to thank my beloved wife Katarína for her permanent support and taking care of our son Richard who was born while I was writing this thesis and has brought me the greatest joy. This work is dedicated to them.

*Michal Zamboj*

Title: Synthetic projective geometry

Author: Michal Zamboj

Institute: Mathematical Institute

Supervisor: Mgr. Lukáš Krump, Ph.D., Mathematical Institute

Abstract: A synthetic approach to the construction of projective geometry, its methods and selected results are given in the proposed thesis. The main historical drawbacks of the original proof of Chasles's theorem for non-developable ruled surfaces and von Staudt's formalization of projective geometry are commented. The corresponding theoretical background is elaborated on visual demonstrations with the accent to interrelations of classical synthetic, axiomatic and analytic points of view. Synthetic methods of projective geometry and their mixture with analytic methods are described on examples including numerous alternative proofs and generalizations of some theorems. A method of four-dimensional visualization is introduced in details. Elementary constructions of images of points, lines, planes and 3-spaces are followed by models of polychora, their sections and shadows. Chasles's theorem is proven for non-developable ruled quadrics on synthetic visualizations, then generalized and proven within the pure projective framework for algebraic surfaces. The synthetic classification of regular quadrics is derived from descriptive geometry constructions of sections of four-dimensional cones and analytically verified in the projective extension of the real space. An integral part of the thesis is a supplementary online book with interactive constructions.

Keywords: synthetic geometry, projective geometry, multidimensional geometry, descriptive geometry, algebraic geometry

# Contents

**Introduction** **6**

**1 Development and structure of projective geometry** **7**
1.1 Foundation of projective geometry . . . . . 7
1.2 Non-projective problems of Chasles's theorem. . . . . 9
1.2.1 Chasles's original proof . . . . . 11
1.2.2 Polar properties of quadrics . . . . . 14
1.2.3 Conclusion . . . . . 15
1.3 The formalization of synthetic projective geometry . . . . . 15
1.3.1 Geometrie der Lage . . . . . 16
1.3.2 Beiträge . . . . . 17
1.3.3 Conclusion . . . . . 23
1.4 Constructions of the projective space . . . . . 24
1.4.1 The axiomatic construction of the projective plane . . . . 24
1.4.2 Extending the real plane . . . . . 29
1.4.3 The analytic construction of the projective plane . . . . . 32

**2 Methods of synthetic projective geometry** **37**
2.1 Projective correspondences . . . . . 37
2.1.1 Projective generation of a conic . . . . . 42
2.2 Synthetic methods . . . . . 47
2.2.1 The method of projection . . . . . 47
2.2.2 The method of selection of an improper hyperplane . . . . 49
2.3 Mixing analytic and synthetic methods . . . . . 61
2.3.1 Analytic methods . . . . . 61
2.3.2 Mixed methods . . . . . 62

**3 A method of visualization of the four-dimensional space** **68**
3.1 Introduction . . . . . 68
3.2 Projecting points, lines, planes, 3-spaces . . . . . 71
3.3 Measuring lengths and angles . . . . . 80
3.4 Projecting solids . . . . . 83
3.4.1 A regular pentachoron . . . . . 83
3.4.2 A construction of a tesseract . . . . . 85
3.5 Constructing sections . . . . . 87
3.5.1 A construction of a section cut of a cubic pyramid . . . . . 87
3.6 Constructing shadows . . . . . 89
3.6.1 Parallel and central lighting . . . . . 90
3.6.2 A construction of a shadow of a cubic pyramid . . . . . 91
3.7 A pole of a plane with respect to a tetrahedron . . . . . 93
3.8 Conclusion . . . . . 94

**4 Selected results** **96**
4.1 A projective proof of Chasles's theorem for non-developable ruled quadrics . . . 96
4.2 A projective proof and generalization of Chasles's theorem along non-torsal lines . . . 100
4.2.1 Chasles's theorem along non-torsal lines . . . 100
4.2.2 Conclusion . . . 106
4.3 The synthetic classification of regular quadrics . . . 107
4.3.1 Introduction . . . 107
4.3.2 The analytic approach . . . 107
4.3.3 Synthetic approach . . . 110
4.3.4 Conclusion . . . 119

**Conclusion** **121**

**Bibliography** **122**

**List of publications** **129**

**A Attachments** **130**
A.1 Interactive constructions . . . 130
A.2 The original proof of Chasles's theorem . . . 131
A.3 Chasles's text on polarity . . . 134
A.4 Von Staudt's algebra of throws . . . 138
A.5 The real field with 3D models as its elements . . . 148

# Introduction

Projective geometry has its stable place in the list of mathematical subjects. Not only is it a theory on its own, but also a basic concept to grasp many other subjects in mathematics, physics or computer science. Well written introductory textbooks from the synthetic point of view are easily accessible on internet archives and many analytic/algebraic texts are used in the present time, and so, the purpose of this work is not to create a complete textbook on projective geometry. Instead, in the selected topics we focus on the synthetic approach, which, as it seems, disappeared from (not only) projective geometry in the second half of the 20th century, and has become again suitable with computer graphics. This opens the fundamental questions: What do we mean by "Synthetic Geometry"? What are its methods? Answers to both of these questions have been changing in time depending on authors and the current state of mathematical knowledge. The whole discourse about the analytic and synthetic method accompanies mathematics and philosophy of mathematics from antiquity. Kvasz in his *Patterns of change* [Kvasz, 2008, pp. 8–9], where he writes about relativization of the language, determines the relevance to synthetic geometry by its fundamental elements. *"The second kind of linguistic change leaves the ways of generation of descriptions untouched but it changes the relation between the linguistic expressions and the objects that they stand for. On this level the linguistic innovations unite into a developmental line Euclid's Elements, projective geometry, Lobachevski's non-Euclidean geometry, Klein's Erlangen Program, and algebraic topology. Along this line the rules for the construction of linguistic descriptions did not change. All these theories use the same ingredients – points, straight lines, and circles – in constructing their objects, and so they all belong to synthetic geometry (in contrast to analytic or fractal geometry, where the ways of construction of objects (or rather of their linguistic descriptions) are radically different)."*

With the synthetic language, one builds geometry in consecutive steps by induction. While, for example, von Staudt constructed projective geometry synthetically without the use of figures, other educators often incorporate methods of pictorial demonstrations into synthetic geometry. An exemplary presentation with a deeper commentary may be seen in Byrne's reedition of his famous book with its full title *The first six books of the elements of Euclid in which coloured diagrams and symbols are used instead of letters for the greater ease of learners*, [Byrne and , ed.]. Byrne starts with the following lines: *"This Work has a greater aim than mere illustration; we do not introduce colours for the purpose of entertainment, or to amuse by certain combinations of tint and form, but to assist the mind in its researches after truth, to increase the facilities of instruction, and to diffuse permanent knowledge."* In his recent book, Akopyan [2017] gives a geometric statement and its proof in one figure. This method is usually called proof without words (see also [Bogomolny]). Throughout the thesis we strongly accent the visual element in projective geometry. However, this should not be taken as a dogma, and instead of pushing the scales to one side, we try to show interrelations between different approaches.

Another aspect of synthetic geometry is that it is coordinate-free. Oppositely, analytic geometry uses some a priori given reference space, which determines its

language. Another light into the distinction of the synthetic and analytic method in projective geometry is shed by [Hlavatý, 1944, p. 8]: *"The synthetic method of projective geometry is very suitable for studying special problems, because it does not use mathematical formulae, but it requires better skill in the direct judgement. The process in this method proceeds from special to general. On the contrary, the analytic method is a suitable tool for the study of general problems, because it uses mathematical formulae, which may contain not only special, but also general cases, at once. In the development of projective geometry, the analytic method ruled the thought content, the synthetic method have allowed the study of such detailed relations, which are less easily examined by the analytic method. Nowadays, a good expert must master both methods equally well."* But, reaching the symbiosis of the analytic and synthetic method takes a long journey. It seems that the analytic method is more direct, especially if the projective geometry is applied as a practical tool to reach another discipline. After description of Cayley's impact on analytic projective geometry in [Kolmogorov and Yushkevich, 1996, pp. 43–44], Laptev and Rozenfel'd review: *"The subsequent development of mathematics showed the superiority of analytic methods in geometry. Despite the beauty of the proofs of certain theorems by synthetic methods; despite the intuitiveness and great pleasure that this approach offers in the contemplation of geometric figures; finally, despite the great assistance of spatial intuition in the solution of particular problems, on the whole these methods are cumbersome and inaccessible and frequently preclude the formulation and solution of more general problems."* As an example, the contributors add Klein's Erlangen Program. Yet, was it not Klein in the just mentioned work [Klein, 1893], who also accented the equality of methods in the note entitled *On the Antithesis between the Synthetic and the Analytic Method in Modern Geometry*? *"The distinction between modern synthesis and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject-matter and methods of reasoning have gradually taken a similar form in both. We choose therefore in the text as common designation of them both the term projective geometry. Although the synthetic method has more to do with space-perception and thereby imparts a rare charm to its first simple developments, the realm of space-perception is nevertheless not closed to the analytic method, and the formulae of analytic geometry can be looked upon as a precise and perspicuous statement of geometrical relations. On the other hand, the advantage to original research of a well formulated analysis should not be underestimated, - an advantage due to its moving, so to speak, in advance of the thought. But it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident, and the progress made by the aid of analysis is only a first, though a very important, step."*

At last, we should stop by another important term mentioned above — intuition. When geometry itself is the object of examination, it is synthetic reasoning that leads to intuitive conclusions – properties, objects, differentiations, etc. Of course, they may or may not be true. That is the time when the analytic method is used to prove or verify them. And that is the spirit of this thesis, which could be not better expressed than with the words of Poincaré in his Science and Method [Poincaré, 2003, pp. 129–130]: *"For the pure geometrician himself this faculty is necessary: it is by logic that we prove, but by intuition*

*that we discover. To know how to criticize is good, but to know how to create is better. You know how to recognize whether a combination is correct, but much use this will be if you do not possess the art of selecting among all the possible combinations. Logic teaches us that on such and such a road we are sure of not meeting an obstacle; it does not tell us which is the road that leads to the desired end. For this it is necessary to see the end from afar, and the faculty which teaches us to see is intuition. Without it, the geometrician would be like a writer well up in grammar but destitute of ideas. Now how is this faculty to develop, if, as soon as it shows itself, it is hounded out and proscribed, if we learn to distrust it before we know what good can be got from it."*

The majority of the thesis consists of published articles: Section 1.2 is the edited conference paper [Zamboj, 2016b]. The text of the papers [Zamboj, 2017, 2018d] is distributed in Chapter 2. Chapter 3, with an edited introduction, conclusion and added Section 3.7, consists of two consecutive articles [Zamboj, 2018a,b]. In the last chapter, Section 4.2 is, with small introductory changes, the publication [Zamboj, 2016a], and Section 4.3 is the paper [Zamboj, 2018c].

In Chapter 1 we approach projective geometry from its historical development to see its "childhood illnesses", drops and descents. We start with a brief chronological summary of foundations. The historical facts are taken from the books by [Kolmogorov and Yushkevich, 1996, Scriba and Schreiber, 2015, Ostermann and Wanner, 2012, Cajori, 1919], where the detailed history and interrelations are to be found, and also from the primary and other secondary sources referenced in the text. To understand the underlaying processes of projective geometry and its differences with other geometries, we discuss main problems, which occurred before its proper definition. Our commentary is aimed particularly to Chasles's theorem for non-developable ruled surfaces and its proof, which is chosen due to its importance and complexity, and furthermore, the original proof involves many formal missteps. Further on, we proceed to commentary on von Staudt's formal synthetic construction, which grounded main pillars of projective geometry. The last sections are dedicated to review the interrelations and to give demonstrative definitions of the projective space: synthetic-axiomatic, constructive as the projective extension of the affine space, and existential as a vector space. Utmost care is taken to use as simple symbolic language as possible.

Synthetic and analytic definitions of projective transformations and projective generation of a conic are parts of the necessary theoretical part in Chapter 2. Later on, synthetic methods — the method of projection and the method of selection of an improper hyperplane — are described and demonstrated on many examples, including their applications in Euclidean and affine geometry. In some cases it is convenient to mix synthetic and analytic methods. As examples, Menelaus's and Ceva's theorems are generalized and proven. At last, synthetic reasoning with analytic representations in computer graphics is shown on von Staudt's construction of the real field.

A visualization method of the four-dimensional space is explained in terms of descriptive geometry in Chapter 3. The method is an analogue of Monge's projection, in which we project the four-dimensional space onto two orthogonal three-dimensional spaces. The chapter is introduced with the history and description of the current state of art in four-dimensional visualizations. Then, we

methodically describe projections of points, lines, planes, 3-spaces, and their incidence and metric properties. Afterwards, models of polytopes and projections of sections and shadows are visualized. A construction of the pole of a plane with respect to a tetrahedron is given with the use of four-dimensional methods. For better understanding, the whole part is strongly supported by step-by-step constructions in interactive 3D graphics and also by analogies in two and three-dimensional spaces.

The last Chapter 4 includes selected synthetically obtained results in projective geometry. It begins with a constructive proof of Chasles's theorem for non-developable ruled quadrics. In Section 4.2 we give a generalization and purely projective proof of the Chasles's theorem along non-torsal lines of algebraic surfaces. Quadrics as sections of four-dimensional cones with three-dimensional spaces are constructed with the use of the four-dimensional visualizing method. This connection and application of four-dimensional descriptive geometry and synthetic projective geometry concludes the thesis.

The first two chapters of the thesis are convenient as supplementary explanatory and motivational material for undergraduate students who participated in projective geometry courses, or courses which involved projective geometry. Chapter 3 about the four-dimensional visualization is not dependent on the previous text and is accessible for students and teachers of descriptive geometry. It contains a novel extension of the theory. Basic knowledge of projective, algebraic, and descriptive geometry is assumed to read the last chapter with selected results of research. Throughout the text we use a large amount of figures of constructions, which are also supported with corresponding interactive models in supplementary files attached to USB flash drive and also online book `https://www.geogebra.org/m/k7hpk2zn` (see Attachment A.1). The figures and models are exported from the 3D modeling software *GeoGebra 5* [ggb], and the parts, which involve computer programming are created in *Wolfram Mathematica 11* [wol].

# 1. Development and structure of projective geometry

## 1.1 Foundation of projective geometry

Throughout the time, the name of projective geometry varied due to the authors' points of view — geometry of position, incidence geometry, elementary geometry, pure geometry, geometry of the cross-ratio, and so on. Each of these names emphasize some elementary feature. To clearly understand the blood circulation of projective geometry we examine the main problems, which accompanied it from the beginning until the proper definitions. From our distant perspective, we can track down isolated problems of projective geometry back to the ancient synthetic geometry of Apollonius of Pergé (3rd–2nd century BC), Menelaus of Alexandria (1st–2nd century AD), and Pappus of Alexandria (4th century AD). A demand for realistic visualizations in Renaissance implied the foundation of linear perspective in painting and sculpture in the 14th–15th century. The mathematical theory of linear perspective, and so the visualization of improper points, were described by Jean Pélerin (Viator) [Pélerin, 1505], more precisely by Dürer [1525] and their followers in the 16th century. It seems that the first ground-breaking text on projective geometry was the unfortunate *Brouillon Project d'une atteinte aux événemens des recontres du Cône avec un plan* by Desargues [1639]. He used the improper points to examine properties of conic sections, and moreover involutions to find the intersecting points of lines and conic sections — Desargues's involution. The treatise was rejected by mathematicians and disappeared, until the transcript by La Hire was found two hundred years later by Chasles. The one who saw and understood Desargues's text was Pascal, who wrote his famous generalization of Pappus's theorem — Pascal's theorem in *Essay pour les coniques* [Pascal, 1640]. An important contribution to conic sections and their polar properties and harmonic ratio was also made by La Hire himself at the end of the 17th and beginning of the 18th century (see [Šír, 2002]). With *Brouillon project...* lost, projective geometry stayed still until the beginning of the 19th century and the school of Monge. To reflect the time, it is important to remind Gauss's work on the theory of geometric constructions and Monge's *Géometrie Descriptive* [Monge, 1799]. Carnot published the following works *De la corrélation des figures en géometrie* [Carnot, 1801], *Géométrie de position* [Carnot, 1803], and *Essai sur les transversales* [Carnot, 1806], in which he used the polarity, cross-ratio, and principle of continuity. His approach is built on continuous transformations between figures, which is also the main difference to the point of view of Poncelet — another Monge's student. Before Poncelet's pioneer work, we should also mention Brianchon, who described the so-called Brianchon's theorem dual to Pascal's theorem in [Brianchon, 1806], duality in [Brianchon, 1817], and attempted to build projective geometry systematically in [Brianchon, 1818]. The principle of polar correspondence was in focus of Servois and Gergonne, who named the pole and its polar with respect to a conic, respectively, in 1810. A complex and commonly accepted foundation of projective geometry was given in *Traité des propriétés projectives des figures* by Poncelet in 1813/14, published in 1822, with

the extended reedition [Poncelet, 1865]. The construction of projective geometry is based on compositions of central projections. Poncelet described elementary constructions and methods of projective geometry, such as the principle of duality, principle of continuity, dual construction of conics, polar correspondence and went even further to describe the construction of the foci of a conic with the use of the isotropic points, and to briefly examine spatial properties. An important role in the projective geometry of French authors during 1810–1832 played the journal *Annales de mathématiques pures et appliquées*, also abbreviated to *Annales de Gergonne*, in which Gergonne published his augmented version of the principle of duality [Gergonne, 1826, 1827]. The barycentric coordinates and the property of four points named *cross-ratio* by W. K. Clifford in 1878, were defined by Möbius [1827] in his famous *Der barycentrische Calcul*. Furthermore, Möbius systematically categorized geometric transformations to collineations, affinities, similarities and congruencies. In [Möbius, 1837], he continued to properly describe the principle of signs and to show that the cross-ratio is invariant with respect to projective transformations. An analytic framework in the language of homogenous coordinates was given by Plücker, another German author, in *Analytisch-geometrische Entwicklungen* [Plücker, 1828] and *System der analytischen Geometrie* [Plücker, 1835]. Later on, Plücker added important base ideas to the point of view of algebraic projective geometry in [Plücker, 1839, 1846], and finally [Plücker, 1868], where he introduced *Plücker's coordinates of a line*, linear complexes, and congruencies. Chasles in 1830 finished and in 1837 published his historical work about methods in modern geometry [Chasles, 1837], in which he studied the method of reciprocal polars related to the principle of duality and also defined the cross-ratio and shown its invariance. In [Chasles, 1880] and [Chasles, 1865], he built extensive background for the projective geometry of linear figures, conics and quadric surfaces based on projective transformations and cross-ratio. In contrast to Plücker's analytic approach, Steiner, a synthetic geometer and strong adherent of the progressive education, in [Steiner and Oettingen , ed.] (with first version in 1832) created the system of projective geometry on synthetic constructions, analogically to the French school. He switched the order and defined the projectivity as the transformation which preserves the cross-ratio. Steiner is also the one, who started to type dual formulations of theorems into two columns (Figure 1.1). Until now projective geometry had become an independent theory full of advanced theorems and applications, but it still stumbled on some basic assumptions. In the following section, we discuss their nature on the example of Chasles's theorem — one of the summits of this era. If we accept Hilbert's partition of a mathematical theory into a naive, formal and critical stage, Freudenthal in his essay *The Impact of von Staudt's Foundations of Geometry* in [Plaumann and Strambach, 1980] considers von Staudt with his *Geometrie der Lage* [von Staudt, 1847] and *Beiträge zur Geometrie der Lage* [von Staudt, 1856] the one, who started the formal era of projective geometry. A detailed description of von Staudt's contribution to the formalization of projective geometry is in Section 1.3. After the results of French school, Plücker's algebraic support, and von Staudt's pure non-metric definitions, projective geometry became a complete theory. The homogenous coordinates highly influenced development of algebraic (e.g. by Cayley, Salmon, Sylvester, Hesse, Clifford), non-Euclidean (e.g. by Laguerre, Klein, Minkowski, Poincaré), and differential geometry (e.g. by Fubini, Čech, Hlavatý).

In von Staudt's texts, even before Hilbert's axiomatization, we can see a transparent axiomatic character mastered for projective geometry by Veblen and Young [1910] and Coxeter [2003]. The axiomatic construction will be shortly discussed in Section 1.4.1. In the 20th century, the axiomatic and analytic approaches to projective geometry have became integral parts of many mathematical theories such as geometric structures, linear algebra, computational geometry, algebraic geometry, non-Euclidean geometry . . . , while the synthetic method somehow dissolved (with some bright exceptions) into books on descriptive geometry and computer graphics in the last decades. On the other hand, it seems that computer geometry and graphics brought visualizations in projective geometry back to life, and a great recent example of an interpretation of projective geometry in both — synthetic and analytic way, with relation to non-Euclidean geometries is in [Richter-Gebert, 2011].

*»Bei allen Strahlbüscheln, von welchen vier Strahlen durch die nämlichen vier bestimmten Punkte* (a, b, b, c) *einer Geraden A gehen, haben die drei Doppelverhältnisse, die sich aus den Sinussen der von den jedesmaligen vier Strahlen eingeschlossenen Winkel zusammensetzen lassen, einerlei Werthe;«* nämlich diese Werthe sind jedesmal den Werthen der drei Doppelverhältnisse gleich, welche aus den Abständen der vier festen Punkte von einander zusammengesetzt sind.

*Bei allen Geraden, welche die nämlichen vier bestimmten Strahlen* (a, d, b, c) *eines Strahlbüschels B schneiden, haben die drei Doppelverhältnisse, die sich aus den Abständen der jedesmaligen vier Durchschnittspunkte* (a, b, b, c) *von einander zusammensetzen lassen, einerlei Werthe;«* nämlich diese Werthe sind jedesmal den Werthen der drei Doppelverhältnisse gleich, welche aus den Sinussen der von den vier festen Strahlen eingeschlossenen Winkel zusammengesetzt sind.

Figure 1.1: The first use of the two-column typeset of dual theorems from [Steiner and Oettingen , ed., p. 18]. The theorem describes the invariance of the cross-ratio with the use of metrics.

## 1.2 Non-projective problems of Chasles's theorem.

Chasles's theorem is an effective tool in descriptive geometry for constructing tangent planes and tangent surfaces to a non-developable ruled surface. The theorem was formulated by Michel Floréal Chasles as a corollary of previous thoughts about the cross-ratio in the first section of the article *Sur les surfaces engendrées par une ligne droite; particulièrement sur l'hyperboloïde, le paraboloïde et le cône du second degré* [Chasles, 1839] (see also the excerpt in Attachment A.2). The theorem says (Figure 1.2):

**Theorem 1.2.1** (Chasles's theorem)**.** *Four tangent planes of a non-developable ruled surface through a ruling have the same cross-ratio as their contact points.*

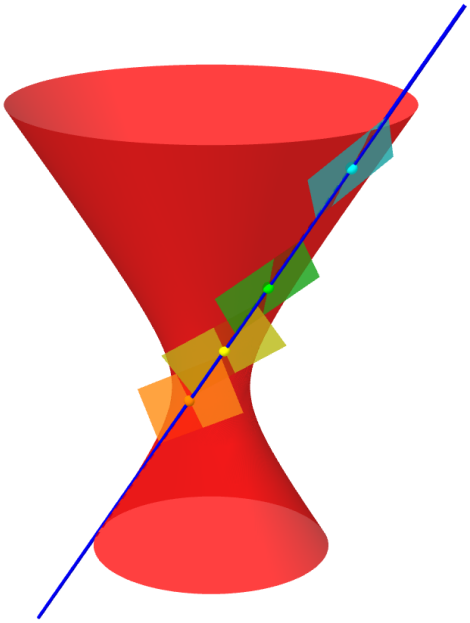


Figure 1.2: Tangent planes of a one-sheet hyperboloid.
Interactive model: `https://www.geogebra.org/m/zvwbfjqa`

The theorem belongs to present projective geometry axiomatically built on the incidence property. However, the axiomatization of geometry is an invention of the late 19th century. We inspect main problems, which Chasles and other geometers were, or better said were not, dealing with. An application of Chasles's theorem in descriptive geometry is in Figure 1.3 from [Kadeřávek et al., 1932].

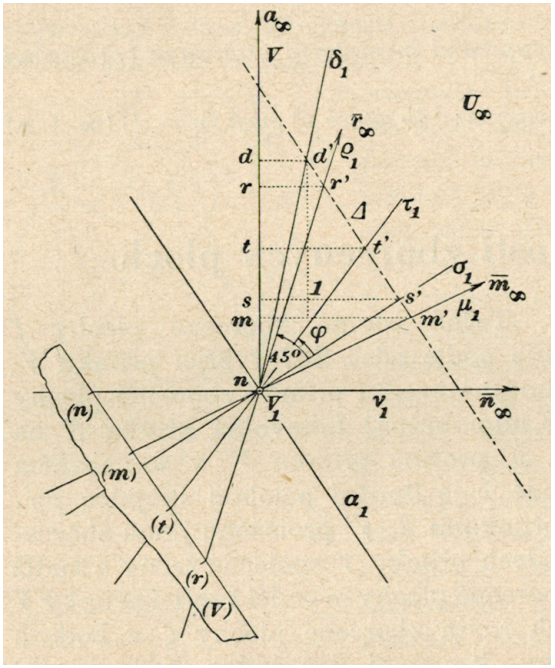



Figure 1.3: A construction of a tangent plane in a point on a ruling of a ruled surface showing that one can use a strip of paper to construct projective pencils of lines.

### 1.2.1 Chasles's original proof

#### The cross-ratio

In 1837, Chasles published his historical work about methods in modern geometry [Chasles, 1837]. He especially studied the method of reciprocal polars related to the principle of duality, which had recently been discovered by Poncelet and enriched by Gergonne. He drew attention to Proposition 129, from Pappus' Book 7 of *Synagógé*, which is the first known reference of the invarious property of four points on a line. Pappus proved this proposition with the use of the Euclidean parallelism. Chasles in [Chasles, 1837, Note IX, p. 304], utilizes and defines this property as the *rapport anharmonique*, presently known as the *cross-ratio*.[1]. In [Chasles, 1839, pp. 51–52], he denoted: *"I had made great use of ratio* $\frac{ca}{cb} : \frac{da}{db}$ *in my Traité historique des méthodes géométriques, and especially in the article which follows, and I considered this ratio for the unique type of all length relationships transformable by the principles of duality and homography. For this reason, I had to give it a special name, and I called it the anharmonic ratio, because in the case of equality to one, which is often present in geometric research, the four points* $a, b, c, d$ *form the harmonic proportion."* Note that, according to Chasles, four points make the harmonic ratio if the value of the cross ratio is 1, not $-1$. This problem is treated later in [Chasles, 1880, Chapters 1 and 2], in which the principle of signs is explained, and the harmonic ratio has the definitive value $-1$. Chasles generalized the cross-ratio from a range of points to a pencil of lines[2] or planes in [Chasles, 1837, p. 302]. In fact, he naturally used the cartesian coordinate system with Euclidean distances and angles to measure the coefficients of the cross-ratio. For example, in Poncelet's *Traité des propriétés projectives des figures* [Poncelet, 1865, pp. 6–7], the method of interchanging lengths of segments and sizes of angles in different formulas using elementary (Euclidean) trigonometry can be found. We show a construction of the cross-ratio led in that spirit (Figure 1.4).

**Theorem 1.2.2.** *Let us have four points* $A, B, C, D$ *on a line and a point* $O$ *not incident with the line. Join* $O$ *and the points* $A, B, C, D$ *with the lines* $a, b, c, d$*. Then*

$$\frac{|AC|}{|BC|} : \frac{|AD|}{|BD|} = \frac{\sin(a,c)}{\sin(b,c)} : \frac{\sin(a,d)}{\sin(b,d)}.$$

*Where* $|AC|$ *is the oriented length of the segment* $\overline{AC}$*, and* $\sin(a,c)$ *is the sinus of the oriented angle* $\sphericalangle(a,c)$*. The orientation of the angle is given by the orientation of the segment such that* $\operatorname{sgn}(\sin(ac)) = \operatorname{sgn}(|AC|)$.

*Proof.* Let $p$ be the altitude to $\overline{AC}$ and $m$ the altitude to $\overline{OA}$ in $\triangle ACO$. The area of $\triangle ACO$ is

$$\tfrac{1}{2}|AC| \cdot p = \tfrac{1}{2}|OA| \cdot m = \tfrac{1}{2}|OA| \cdot |OC| \cdot \sin(a,c).$$

Therefore

[1] Möbius in 1827 in *Der barycentrische Calcul* Möbius [1827, p. 220] used the name *Doppelschnittsverhältnisse*.

[2] Also in [Möbius, 1827, p. 388].

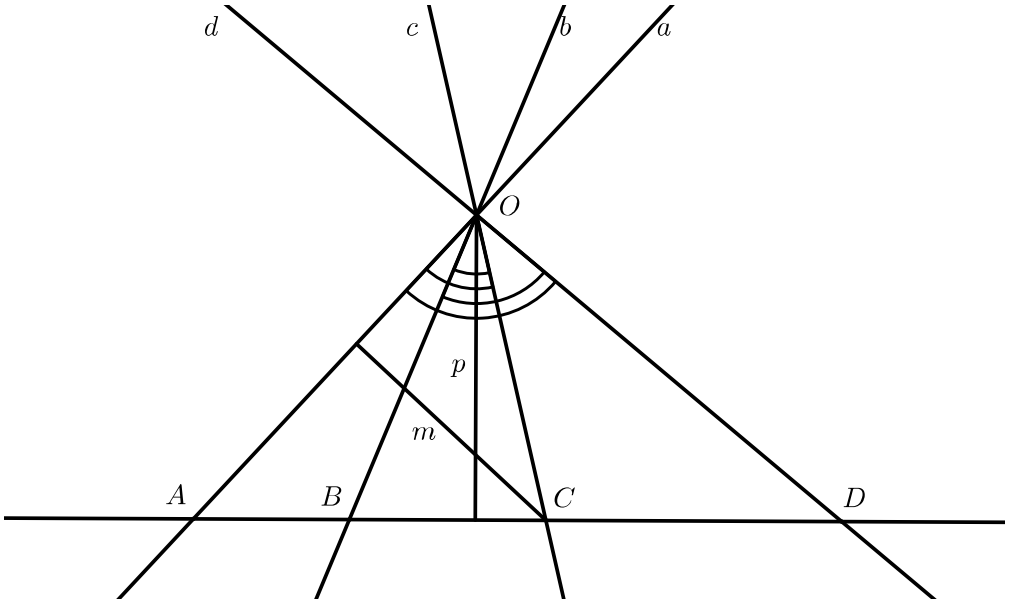



Figure 1.4: Theorem 2.2.1.

$$\sin(a,c) = \frac{|AC| \cdot p}{|OA| \cdot |OC|}$$

and similarly for $\sphericalangle(b,c), \sphericalangle(a,d), \sphericalangle(b,d)$. As a result, we have

$$\frac{\sin(a,c)}{\sin(b,c)} : \frac{\sin(a,d)}{\sin b,d} = \frac{\frac{|AC|\cdot p}{|OA|\cdot|OC|}}{\frac{|BC|\cdot p}{|OB|\cdot|OC|}} : \frac{\frac{|AD|\cdot p}{|OA|\cdot|OD|}}{\frac{|BD|\cdot p}{|OB|\cdot|OD|}} = \frac{|AC|}{|BC|} : \frac{|AD|}{|BD|}.$$

□

The use of Euclidean measures leads to a conflict with (more general) projective geometry or geometry of position, built only on the relation of incidence. Coolidge in *The Rise and Fall of Projective Geometry* [Coolidge, 1934, pp. 222–223] identifies one of the weak spots in synthetic geometry of this early period of projective geometry: *"The basis of projective geometry was the cross ratio. This is projectively invariant but, as previously given, was based on distances and angles which are not in themselves unalterable."* The incorrect use of the cross-ratio is a removable metrical problem. The (projectively) correct way of defining the cross-ratio was made by von Staudt in [von Staudt, 1847] with his idea of the *Wurf*. The concept is based on unifying objects — points of range, lines or planes of a pencil — which are elements of some system and making projective transformations in-between these systems (Figure 1.5). More details are described in Section 1.3.2.

### Infinite closeness

After clarifying the use of the cross-ratio, Chasles proceeded to the following lemma:

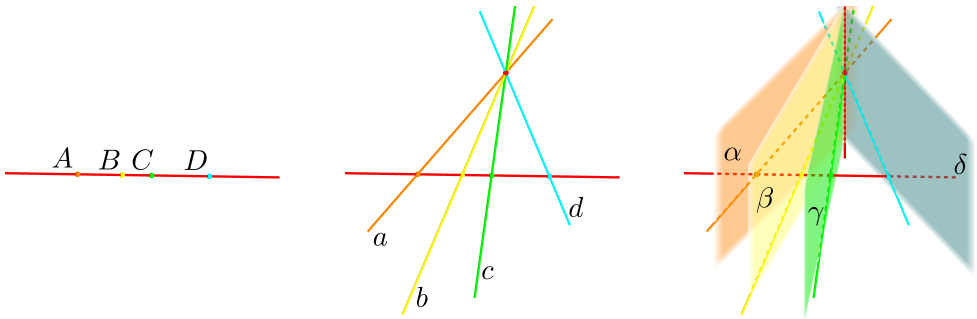



Figure 1.5: Projective transformations between heterogenous objects — a range of points, pencil of lines and pencil of planes.

**Lemma 1.2.1.** *Let us have three fixed lines located in any positions in space and another four lines, which intersect the three given ones. Then the cross-ratios of four points of intersection on each of the fixed lines are equal.*

*Proof.* Let $l, l'$ and $l''$ be fixed lines, $A, B, C, D; A', B', C', D'$ and $A'', B'', C'', D''$ be the points of intersection of the lines $l, l'$ and $l''$ with the considered four lines $AA'A''$, $BB'B''$, $CC'C''$, and $DD'D''$, respectively (Figure 1.6). We prove that the following holds

$$\frac{CA}{CB} : \frac{DA}{DB} = \frac{C'A'}{C'B'} : \frac{D'A'}{D'B'} = \frac{C''A''}{C''B''} : \frac{D''A''}{D''B''}.$$

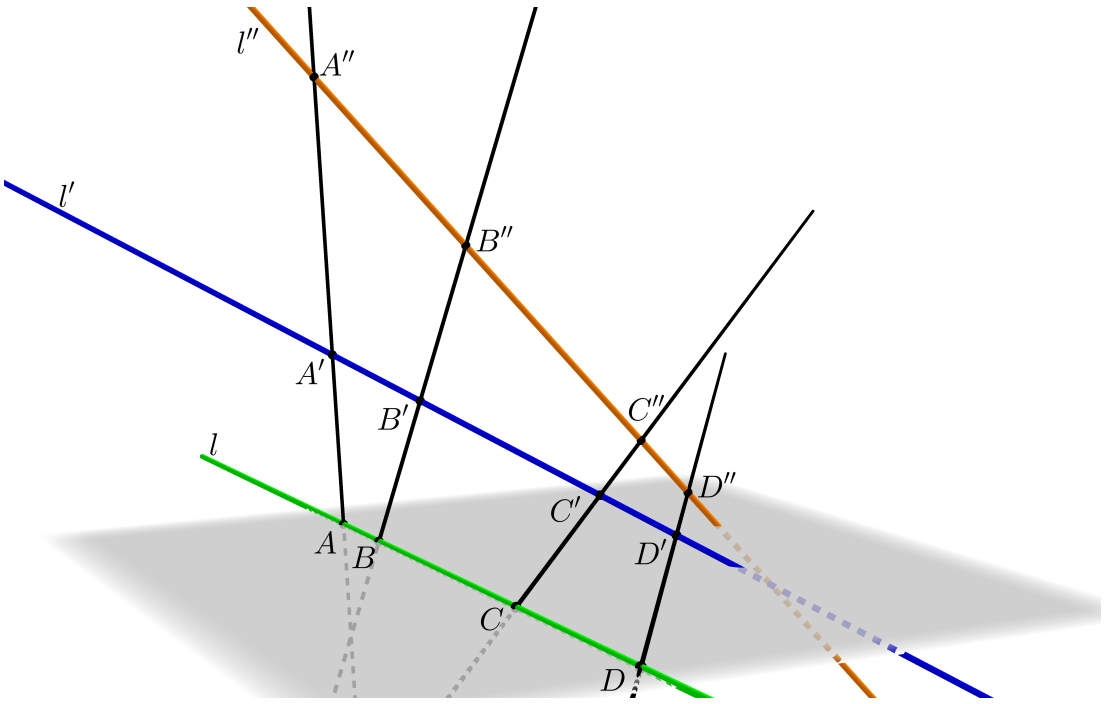



Figure 1.6: A projectivity between skew lines.

The four lines $AA', BB', CC', DD'$ generate four planes passing through $l$. The four points $A', B', C', D'$ are intersections of the planes with the line $l'$, so the cross-ratio of these points is equal to the cross-ratio of the pencil of planes with the axis $l$. The same holds for the cross-ratio of the points $A'', B'', C'', D''$. Hence

the cross-ratios of the points $A', B', C', D'$ and $A'', B'', C'', D''$ are equal. We prove that the points $A, B, C, D$ have the same cross-ratio, if we consider the four planes of the pencil with the axis $l''$ through the four lines $AA'', BB'', CC'', DD''$, respectively. □

Obviously the theorem above does not work for all positions of lines in space; lines must be skew to each other. However, Chasles did not state this explicitly.

Chasles's theorem is a corollary with an elegant proof:

**Corollary 1.2.1** (Chasles's theorem)**.** *Four tangent planes of a non-developable ruled surface through a ruling have the same cross-ratio as their contact points.*

*Proof.* Assume that three lines $l, l', l''$ are three consecutive rulings of the non-developable ruled surface infinitely close to each other. Let $l$ be the axis of a pencil of planes. Let four planes $\alpha, \beta, \gamma, \delta$ of this pencil intersecting $l'$ in points $A', B', C', D'$, respectively, be the tangent planes of the surface at $A, B, C, D$. The cross-ratios of the tangent planes and the points $A', B', C', D'$ are equal, because $l'$ intersects the pencil of planes in these points. The same assumption can be made for $l''$. As a consequence of the previous lemma, the cross-ratio of the planes is equal to the cross-ratio of their tangent points $A, B, C, D$. □

The proof is straightforward, with one disadvantage — it is not projective. The crucial problem is the definition of the non-developability. From the point of view of differential geometry, a ruled surface is defined as the surface developed by moving a *ruling (generatrix)*. If two infinitely close ("neighboring") positions of each ruling do not intersect (except for a finite number of such lines), the surface is called a *non-developable ruled surface*. Chasles intuitively used this differential definition of the non-developability. The main drawback is the infinite closeness of two positions. With only the relation of incidence, we cannot declare how close the two lines are. Chasles did not explicitly define a non-developable surface in his works. However, he clearly used the definition of a non-developable surface based on the infinitesimal calculus of that era. Such definitions can be found for example in Dupin's *Développements de Géométrie* [Dupin, 1813, p. 241], and Cauchy's *Du Calcul Infinitésimal* [Cauchy, 1826, p. 210]. We give proper projective definitions and proof of the theorem in Section 4.2. The infinite closeness is essential in the proof, so we cannot simply exchange the definitions. We have to find a new way to prove the theorem. Therefore the problem is unremovable. In [Chasles, 1837], Chasles was very close to an algebraic proof based on polar properties of surfaces of the 2nd degree.

### 1.2.2 Polar properties of quadrics

At the beginning of the section *Mémoire de géometrie sur duex principes généraux de la science: la dualité et l'homographie* in [Chasles, 1837] (see also Attachment A.3), Chasles formulated a definition of a pole in the space. He never stated what induces the polarity but from the theorems and examples we can conclude that the polarity is induced by a quadric. He shown that if poles lie on a line, their polar planes also have a common line. The final property of projectivity is expressed in the following theorem:

**Theorem 1.2.3.** *If a pole takes four positions $A, B, C, D$ on a line, the polar plane takes four corresponding positions $\alpha, \beta, \gamma, \delta$. These four planes pass through one line, and they have the same cross-ratio as the points $A, B, C, D$.*

The main idea of the proof is to show that the intersection points of the planes $\alpha, \beta, \gamma, \delta$ and the given points $A, B, C, D$ have the same cross-ratio. If Chasles had used this theorem on non-developable ruled quadrics — a hyperbolic paraboloid and one-sheet hyperboloid, he would have obtained Chasles's theorem for surfaces of the 2nd degree with a "correct" proof. We discuss the synthetic projective proof of Chasles's theorem for non-developable ruled quadrics and a generalization of the theorem along non-torsal lines of algebraic surfaces in Chapter 4.

### 1.2.3 Conclusion

Chasles's theorem is by its nature a great result in projective geometry. However, proofs of the theorem have always slipped into the use of more powerful methods of projective differential and differential geometry. We have identified two basic problems of non-projective character in Chasles's original proof of the theorem for non-developable ruled surfaces. The first one is the metric definition of the cross-ratio and the second one is the use of infinite closeness. While the definition of the cross-ratio was resolved by von Staudt, the infinite closeness cannot be removed so simply, and we must avoid it by using different algebraic methods. It is also important to say that Chasles developed his results a long time before the axiomatization of geometry, and it is not of our interest to blame the use of non-projective methods on him.

## 1.3 The formalization of synthetic projective geometry

Karl Georg Christian von Staudt (1798–1867) was a German, or rather Bavarian, geometer, a student of Gauss and solver of Feuerbach's problem. As a professor at the University of Erlangen he published *Geometrie der Lage* [von Staudt, 1847] and during 1856–60 three issues of *Beiträge zur Geometrie der Lage* [von Staudt, 1856]. Von Staudt was certainly familiar[3] with findings of Steiner, Plücker, Möbius, Chasles, Gergonne, Poncelet, and their predecessors, but he was the first, who successfully liberated projective geometry from the fetters of the metric. Historical facts and description of von Staudt's influence are well noted in [Cajori, 1919, p. 294], [Coolidge, 1934, Žitný and Zolotarev, 1996]. To highlight Von Staudt's work, he is glorified in Halsted's epic preface in [Halsted, 1906]: *"... Yet man is no mole. Infinite feelers radiate from the windows of his soul, whose wings touch the fixed stars. The angel of light in him created for the guidance of eye-life an independent system, a radiant geometry, a visual space, codified in 1847 by a new Euclid, by the Erlangen professor, Georg von Staudt, in his immortal Geometrie der Lage published in the quaint and ancient Nürnberg of Albrecht Dürer. ..."* However, there is no doubt about the modern point of view on geometry, the form of von Staudt's interpretation through the

[3]Even though he did not refer to other authors or publications.

language of synthetic geometry is cumbersome and sometimes hard to understand. Furthermore, any reader of von Staudt would be disappointed, if he or she expects pictures of objects or constructions. The interpretation by Reye [1899] has become more popular and better understandable. On the other hand, the synthetic language is used in an intuitive way — we can see how constructions of general objects progress, while imagining specific ones in our mind. Hence we can track back any construction from words instead of trying to reconstruct the picture. At start, this seems to be a disadvantage inherited from Greek geometry, but, on the contrary, von Staudt set himself free from the coordinate system of the affine space (though he used parallelism) and, even more, from objects of the real space.

### 1.3.1 Geometrie der Lage

In *Geometrie der Lage* von Staudt clearly states, that the aim of the book is a description of geometry without measuring — the geometry of position, projective geometry based on the relation of incidence. Firstly, the author accompanies us into the variety of static geometrical objects, well-known from the real affine space, created from points, lines, planes, and their planar or spatial bundles. Afterwards, he systematically builds incidence properties, perspective and projective mapping, and involution. Freudenthal describes (see [Plaumann and Strambach, 1980, p. 411–412]): *"What we would call axioms is in von Staudt's approach copied from "real" space though it is not yet concentrated in a few "Grundsätze", as it would be in Pasch's work. It is circumstantially formulated in the first paragraph, where it can easily be identified by the characteristic fact that it is neither proved nor introduced as a definition. Anyhow though not as axioms, incidence axioms are explicitly formulated and — a rather important fact — duality is already axiomatically rooted. Even the order axioms are somehow satisfactory."* Since the duality implies equivalent properties of reciprocal objects, it was probably the most important factor, which led von Staudt to unify heterogenous objects to be the elements of some system — "Gründgebilde". This major step in the language of synthetic geometry opened gates to mapping between heterogenous systems with somehow similar inner structure, such as a range of points, pencil of lines, planes, conics, etc. and influenced the development of structural algebra. Von Staudt naturally accepted the invarious property of the harmonic ratio with respect to projectivities, or more precisely — harmonicity, since the harmonicity is a purely synthetically defined property of elements, not as a ratio of lengths of segments.

**Construction 1** (Harmonic conjugate, Figure 1.7)**.** *The construction of the harmonic conjugate $D$ of a point $C \in \overline{AB}$ with respect to $A, B$ goes as follows:*

1. *Choose an arbitrary distinct point $E$ not on the line $\overline{AB}$ and an arbitrary line $c$ through $C$ not incident with $E$ and $A, B$.*
2. *Find the intersections $G$ of the lines $\overline{AE}$ and $c$, and $H$ of the lines $\overline{BE}$ and $c$.*
3. *Join the intersection $I$ of the lines $\overline{AH}$ and $\overline{BG}$ with $E$.*

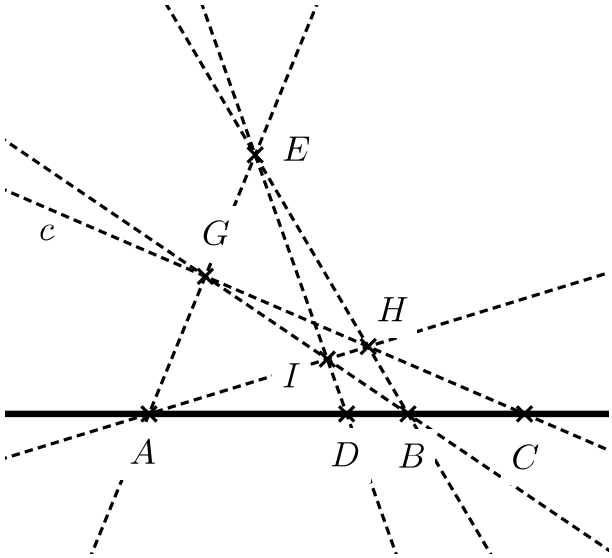



Figure 1.7: The pairs of points $A, B$ and $C, D$ are harmonically separated, or the point $D$ is the harmonic conjugate of $C$ with respect to $A, B$.
Step-by-Step construction: `https://www.geogebra.org/m/aa8z4dvb`

*4. The point $D$ is the intersection of the lines $\overline{AB}$ and $\overline{EI}$.*

A *perspectivity* $\overline{\overline{\wedge}}$ is in a wider sense a central projection of ranges of points with von Staudt's generalization into heterogenous objects (Figure 1.5).[4] A *projectivity* $\overline{\wedge}$ is then defined as the product of a finite number of perspectivities (Figure 1.8). We return to proper definitions of projective transformations in Section 2.1.

### 1.3.2 Beiträge

Von Staudt developed pure methods of projective geometry in *Geometrie der Lage*. In *Beiträge zur Geometrie der Lage*, he tried to integrate them with the algebraic interpretation of complex numbers.

#### *Wurf*

Möbius and Chasles defined the *cross-ratio* as a ratio of lengths of four segments. For von Staudt the invarious property is a position of objects in a system instead of preserving lengths of segments. Usage of the term *Wurf (throw)*, implies that what is really preserved by a projectivity is a visual property differentiable only by the order of elements. Even after describing algebraic operations with throws, von Staudt was very precise when he talked about a throw or its value — the cross ratio. Hlavatý discussed the new term in [Hlavatý, 1944, p. 6]: *"Existence of projective geometry is not a priori dependent on the term of metric geometry, oppositely, it is contained in projective geometry as its special case. Von Staudt's*

[4] We follow the notation of Veblen and Young [1910] with distinction between homogenous and heterogenous forms. Coxeter [2003] uses the term *elementary correspondence* for heterogenous transformation, and leaves *perspectivity* for the homogenous product of two elementary correspondances.

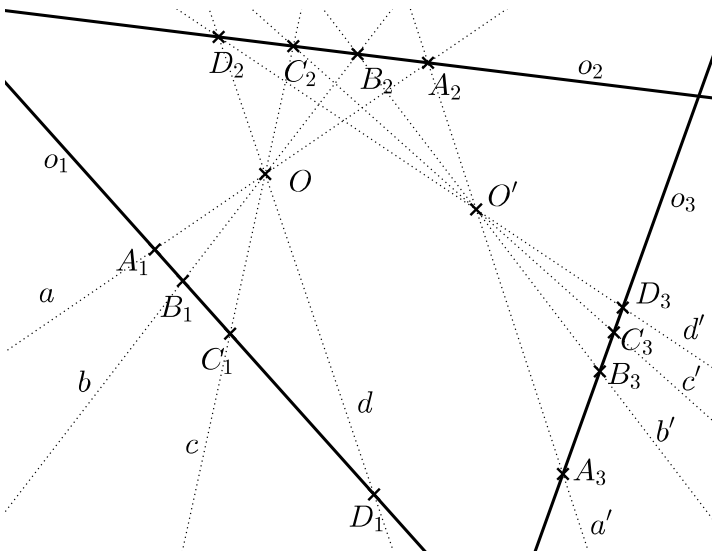



Figure 1.8: The ranges of points $A_1, B_1, C_1, D_1, \ldots$ with the axis $o_1$ and $A_2, B_2, C_2, D_2, \ldots$ with the axis $o_2$ are in the perspectivity with the center $O$: $A_1B_1C_1D_1 \cdots \overline{\overline{\wedge}}\, A_2B_2C_2D_2 \ldots$.
It also holds that $A_2B_2C_2D_2 \cdots \overline{\overline{\wedge}}\, A_3B_3C_3D_3 \ldots$. Therefore the ranges of points $A_1, B_1, C_1, D_1, \ldots$ with the axis $o_1$ and $A_3, B_3, C_3, D_3, \ldots$ with the axis $o_3$ are in the projectivity $A_1, B_1C_1D_1 \cdots \overline{\wedge}\, A_3B_3C_3D_3 \ldots$.
The pencils of lines $a, b, c, d, \ldots$ with the center $O$ and $a', b', c', d', \ldots$ with the center $O'$ are in the perspectivity with the axis $o_2$, $abcd \cdots \overline{\overline{\wedge}}\, a'b'c'd' \ldots$.
The range of points $A_1, B_1, C_1, D_1, \ldots$ with the axis $o_1$ and the pencil of lines $a, b, c, d, \ldots$ with the center $O$ are also in the (heterogenous) perspectivity $A_1B_1C_1D_1 \cdots \overline{\overline{\wedge}}\, abcd \ldots$.
Interactive model: `https://www.geogebra.org/m/mbuercm2`

*invention went even further: it was shown, that also a line or plane can be determined by numerical values without any measuring, only with the use of linear constructions (and irrational cuts). Some numerical values could therefore in geometry of position mean a point, line, or plane, and in their analytical representation was no difference."*

### Imaginary elements

One can watch attempts of geometers of the 19th century to visualize and display complex numbers. Since Cardano, mathematicians had to (and they sometimes obviously did not) accept that there are numbers, which are not real, but useful for solving algebraic equations. They existed, but there was no idea of how to construct them in the real plane. They also existed in Gauss's geometric model of complex numbers. Differently, Poncelet in his fundamental work *Traité des propriétés projectives des figures* [Poncelet, 1865] described their properties constructively by the so-called "principle of continuity". The principle can be understood as a geometric rhapsody to the fundamental theorem of algebra. Let us have a circle and a line in a position, in which the line intersects the circle in two points. Moving the line to the tangent position, and even further, changes the

number of real intersections. The questions are: Where have those intersecting points disappeared? How can points disappear just by continuous motion? So the answer is: there are some other "imaginary" intersecting points, which cannot be seen. Von Staudt found the way to describe them by a synthetic construction. The key was the *involution* — a special case of a projectivity used preferentially by Desargues, and defined by von Staudt in [von Staudt, 1847, pp. 118–120] as a projectivity, which interchanges pairs of points.

Let us step a little further and use a conic section $\kappa$ with a system of two pairs of conjugated points $A, A'$ and $B, B'$ (the explanation why a conic is an object of projective geometry is in Section 2.1.1). The pairs lie on lines through a point $S$, which is therefore the intersection of $\overline{AA'}$ and $\overline{BB'}$. If the point $S$ lies outside the conic $\kappa$, it is easy to observe (see Figure 1.9), that we have two points which are self-conjugated with respect to $\kappa$: $X = X'$ and $Y = Y'$. These self-conjugated points are the tangent points of the tangents to $\kappa$ through $S$ and also the intersections of the polar $s$ of the pole $S$ with respect to $\kappa$. In Figure 1.9 the construction fits with the picture and is called a *hyperbolic involution*, but if we change the configuration of points $A, A', B, B'$ such that the point $S$ lies inside the conic $\kappa$, we obtain an *elliptic involution* (Figure 1.10), and the polar $s$ will not intersect the conic section $\kappa$ in real points. Instead, we call the points *imaginary*. The same process can be applied on a range of points with pairs of corresponding points $A_1, A_1'$ and $B_1, B_1'$ and on a pencil of lines with pairs of corresponding lines $a, a'$ and $b, b'$ as shown in the figures. If the point $S$ lies on the conic $\kappa$, it becomes conjugated with all points of the conic $\kappa$. Hence the involution degenerates, and we do not call it parabolic. This way von Staudt constructed imaginary elements (complex conjugates) synthetically only with incidence properties and without motion. When imaginary elements had become objects of the synthetic language, von Staudt could have used them in his constructions in the same way as any

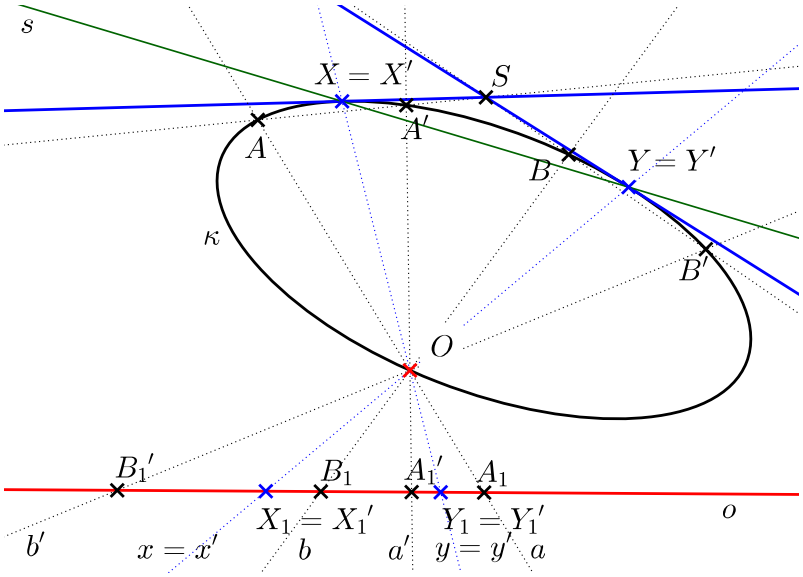



Figure 1.9: Hyperbolic involutions of a pencil of lines, range of points on a line and range of points on a conic.
Step-by-Step construction: `https://www.geogebra.org/m/bzahsgbt`

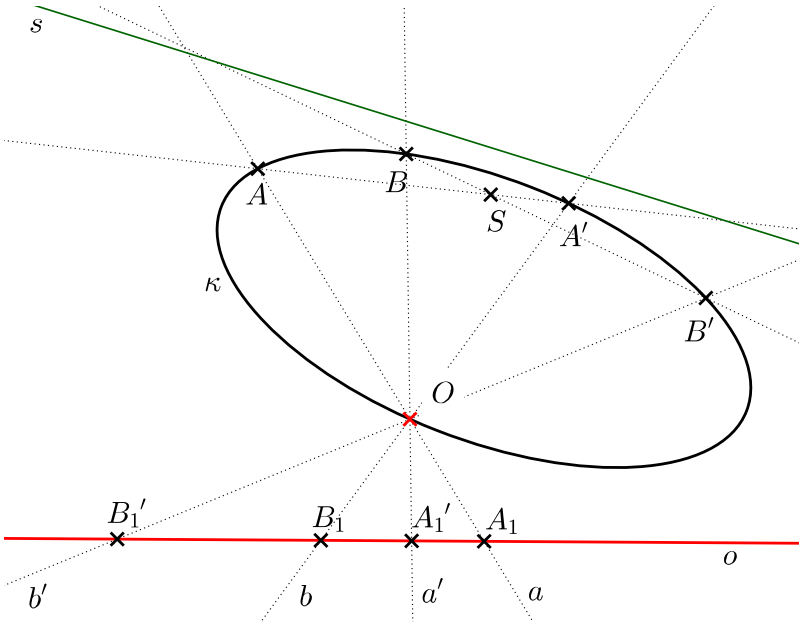



Figure 1.10: Elliptic involutions of a pencil of lines, range of points on a line and range of points on a conic.
Step-by-Step construction: `https://www.geogebra.org/m/khwddnhk`

real elements. At this point, we can see that the synthetic language has become more powerful than its graphical representation. Von Staudt gave examples of objects such as a triangle with two imaginary points and two real sides. Just to mention, he did not describe only intersecting imaginary lines in the real plane, but also skew lines on a given ruled surface. The further development of synthetic geometry with imaginary elements can be found in [Hatton, 1920].

### The algebra of throws

After introducing the throw and imaginary elements, von Staudt solved another important problem. The calculus of complex numbers was already known from their algebraic representation, again, existentially. Using the elliptic involution, von Staudt built algebraic operations with real and complex numbers[5] constructively, using the synthetic language. The original text with constructions of operations with real elements is in Attachment A.4. These constructions of a projective scale are often depicted on a range of points on a line [Richter-Gebert, 2011, pp. 89–91] or pencil of lines [Hlavatý, 1944, pp. 20–27]. Even though, they become apparent from affine geometry (we return to them in Section 2.2), it is not how they were originally presented. It is evident that von Staudt created his system on a range of points on a conic (he explicitly used a curve of the 2nd order on few places). We interpret these constructions and moreover project them onto a range of points and pencil of lines in our figures. Thus, we obtain three kinds of a projective scale at once.

[5]The question of continuity was not resolved in the time of von Staudt's publication. The continuity is a priori involved, otherwise, the constructions define rational and Gaussian rational fields. Hlavatý [1944] carefully supplemented the construction with Dedekind's cuts.

**Addition** We assign the coordinates $0, 1, \infty$ to three elements of some system — the range of points on a conic.[6] The join of the intersection point of the line $\overline{AB}$ and the tangent of the conic through $\infty$ with 0 intersects the conic in the second point $A + B$ which is the wanted sum ([von Staudt, 1856, p. 167], Figure 1.11). After the description of the sum of two elements, the commutativity

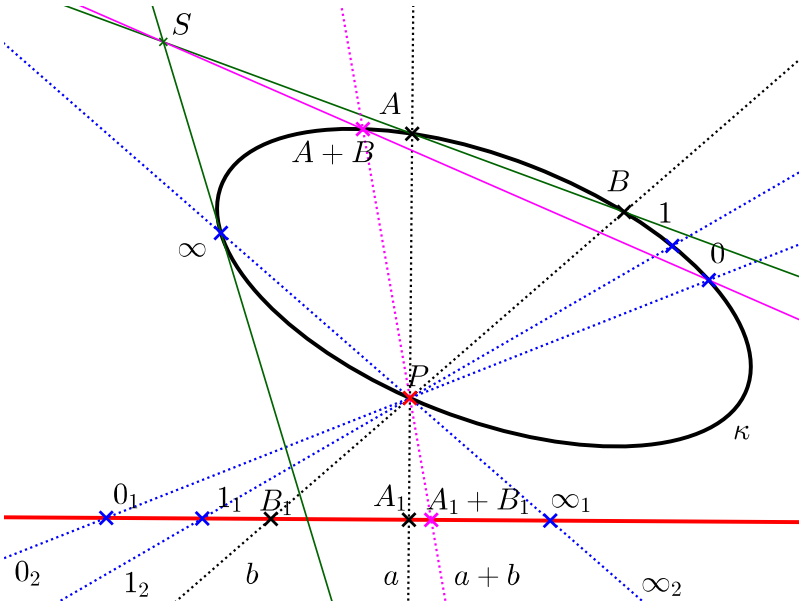



Figure 1.11: The sum $A + B$ on a conic $\kappa$. The construction is projected from the point $P$ on $\kappa$ onto the pencil of lines $0_1, 1_1\infty_1 \ldots$ with the center $P$, and then projected onto the range of points $0_2, 1_2, \infty_2 \ldots$.
Step-by-Step construction: `https://www.geogebra.org/m/enyragrv`

and additive identity is presented. Then the sum of a finite number of elements, associativity of addition, and additive inverse are shown.

**Multiplication** A product of two elements is constructed in Figure 1.12 [von Staudt, 1856, pp. 173–174]. If the intersection point $S$ of $\overline{0\infty}$ and $\overline{AB}$ is joined with 1, the second intersection point of this line with the conic $\kappa$ is the wanted product $A \cdot B$. The commutativity of multiplication, finite series, and multiplicative identity, and then multiplicative inverse or division, and distributivity over addition are consecutively described. Altogether von Staudt finished the algebraic construction of the field structure.[7]

**Complex numbers** Afterwards he discussed the exponentiation and incorporated the imaginary elements, to which he returned after the introduction of the value of a throw [von Staudt, 1856, pp. 256–261]. To construct the imaginary unit $i$ (and $-i$), we construct elements $I$ (and $J$), such that $I \cdot I = -1$. The value $-1$ is nothing else then the harmonic conjugate of 1 in the system $0, 1, \infty$. As a practical example, we construct these imaginary units (Figure 1.13). Let

[6] Later on we show that the three elements: $0, 1, \infty$ of a given throw create a projective basis and determine any other element of the system uniquely.

[7] Only the associativity of multiplication seems to be missing.

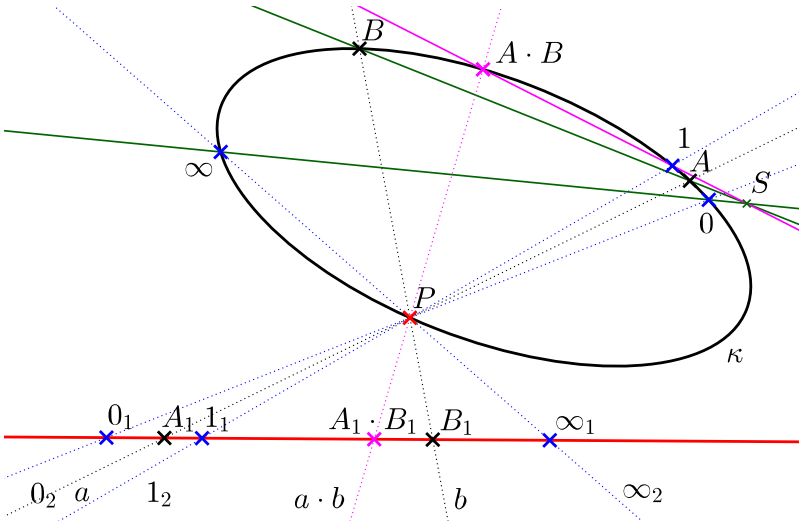



Figure 1.12: The product $A \cdot B$. Again, the construction on conic $\kappa$ is projected onto a pencil of lines and on a range of points on a line.
Step-by-Step construction: `https://www.geogebra.org/m/q8mpxxwe`

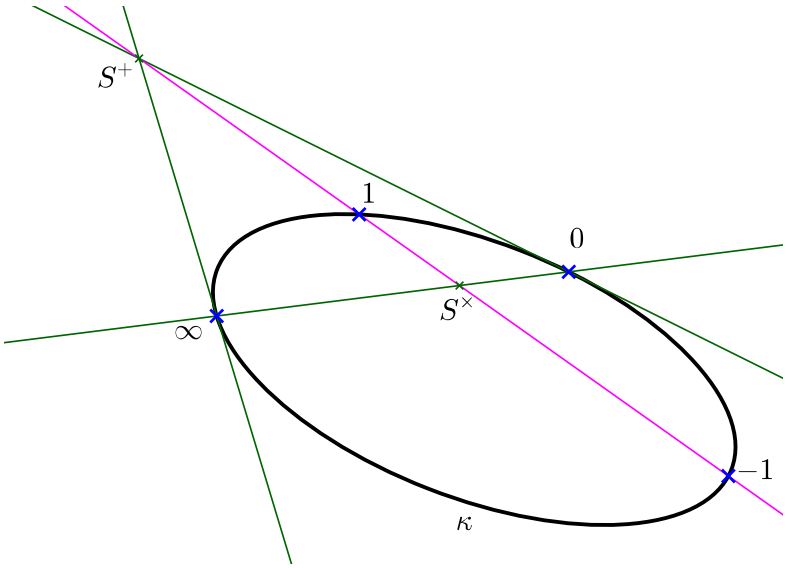



Figure 1.13: The construction of the imaginary numbers $\imath, -\imath$. The corresponding imaginary points are the tangent points of the tangent lines to $\kappa$ through the point $S^{\times}$.

us have elements $0, 1, \infty$ on a conic $\kappa$. Firstly, we find the harmonic conjugate $-1$ of 1. Such construction was demonstrated in Figure 1.7 on a range of points on a line. We could use this construction and project it on the conic, but, this time, for illustrative reasons, we express $0 = 1 + (-1)$. Reverting the construction of addition (Figure 1.11), we construct the tangents to $\infty$ and $0(= 0 + 0)$, which intersect in the point $S^{+}$. The intersection of the join of $S$ and 1 with the conic section $\kappa$ is the harmonic conjugate $-1$. Now, reverting the construction of multiplication, we join the elements $-1$ and 1 and the elements 0 and $\infty$. The intersection of the joins is the point $S^{\times}$ inside the conic $\kappa$ (this holds generally

due to properties of the order, not only for the ellipse in the figures). Thus, we have the case of elliptic involution, in which there are no real tangents from $S^{\times}$ to $\kappa$. These imaginary tangents touch the ellipse $\kappa$ in the imaginary points $I$ and $J$. Further on we can use the imaginary points for other constructions. Numerous synthetic constructions with imaginary elements is in [Hlavatý, 1944], pp. 179–207. Worth mentioning is that von Staudt derived even trigonometric representation of the complex numbers (with the use of a circle).

**The fundamental theorem of projective geometry**

Until now, we have been describing von Staudt's changes in the synthetic language, ordering of the starting points of projective geometry and formalization of unseizable results of algebra. The whole process has led to a purely projective proof of an important result, which is called the fundamental theorem of projective geometry. To be precise, von Staudt was working in the real projective space in [von Staudt, 1847], few decades before Klein and the research of topological properties. Von Staudt, as noted before, intuitively used the continuity in his proof. He proved that a projectivity, in the sense of mapping which preserves the harmonicity, of a line onto itself with three fixed points is an identity. Furthermore, he derived that a projectivity is determined by three pairs of conjugated elements, and also that a projectivity of a line onto another line with a fixed point is a perspectivity. The equivalence of these statements forms the fundamental theorem of projective geometry in the real plane. A further insight is given in Section 2.1. But when von Staudt let himself into the complex plane with the use of the elliptic involution, complications appeared. Freudenthal in [Plaumann and Strambach, 1980, pp. 416–417] explains the mistake: *"But on the ground of these definitions one had also to prove anew all elementary incidence properties. Collineations and correlations must be investigated and conics had to be redefined. And even this was not yet enough. In the second issue of "Beiträge" von Staudt had to suffer for the wrong he had done in "Geometrie der Lage": the definition of projectivities by the invariance of the harmonic separation. This definition is obviously wrong in the complex domain if the fundamental theorem of projective geometry is to be saved."* However, von Staudt enlarged the language to keep his own way and defined chains on the oriented complex line.

### 1.3.3 Conclusion

Let us slightly consult von Staudt's work in projective geometry in the sense of Kvasz's book *Patterns of Change* [Kvasz, 2008] and [Kvasz, 2013]. The historical placement of von Staudt's geometry is after the foundation of the fundamental theorem of algebra, non-Euclidean geometry (we should not forget that von Staudt was a student of Gauss), and projective geometry in its naive interpretation. On the other side, the axiomatic method and topology were almost knocking at the gates. First of all, von Staudt reshuffled the importance of geometric properties with incidence being the main relation and postulating the invariance of a harmonic quadruple. The logical power of synthetic language has obtained the ability of proving the fundamental theorem of projective geometry. By removing a picture, von Staudt increased the expressive power of the synthetic language to be able to construct imaginary elements and the algebraic field structure. At

that point, if the language of algebra was the language of what existed, the language of synthetic geometry became the language of how to construct it. Kvasz says [Kvasz, 2008, p. 44]: *"Thus the casus irreducibilis is not a failure of algebra or of the algebraist — on the contrary. Since algebraic expressions determine analytical curves, the formulas giving the solutions of algebraic equations must fail, in some cases, to give curves the freedom not to intersect."* The projective form of the language of synthetic geometry seized the infinity. And if Euclid gave lines freedom to be parallel and not to intersect, Desargues forbid such behavior. An analogue holds for the imaginary — Poncelet with the principle of continuity forbid curves not to intersect, and von Staudt formalized the process and constructed the intersecting points. Assigning the coordinates $0, 1, \infty$ to some elements of a system, von Staudt also showed how to explicitly interpret a model of such system (even though without figures). Von Staudt's systematic approach was precise and modern, and we can observe similar methods in the future works by Pasch, Hilbert and other axiomatists. He had swept floor for topologists such as Klein and modern algebraists such as Grassmann. All of this he did with the use of only the synthetic language.

## 1.4 Constructions of the projective space

After the century of revolutionary ideas in geometry finished with Hilbert's axiomatization, it was the time for the big clean up. While the jewels of projective geometry started their own lives in other theories, the elements were being reformulated. The call for the abstraction and generality was answered, and most authors built projective geometry over an arbitrary field. In the book by Veblen and Young [1910], projective geometry is formulated axiomatically, and it still incorporates extensive passages of the analytical (coordinate) method. On the other hand, Coxeter's pure axiomatic treatise [Coxeter, 2003] separates (up to the short last chapter) incidence projective geometry with no use of the cross-ratio. A good example of a modern analytic textbook on projective geometry is [Casas-Alvero, 2014]. A standard synthetic construction is well presented by Hlavatý [1944]. The point of this enumeration is that if a reader is introduced into projective geometry by any of Coxeter's, Casas-Alvero's or Hlavatý's textbooks, he or she will most probably not understand the same theorem in the other. We sum up the main features, advantages, and disadvantages of the axiomatic and analytic approaches. The inspiration for this section were mostly the books [Richter-Gebert, 2011] and [Glaeser et al., 2016, Chapters 5 and 6], in which connections between different approaches are extensively described and include stunning visual presentations.

### 1.4.1 The axiomatic construction of the projective plane

An axiomatic construction of some theory is based on undefined primitive concepts — *elementary objects* and *relations*. From the primitive concepts we state some simple unproved primitive propositions — *axioms*, which are consistent and independent. Then, we build further objects and theorems from the previous. The elementary objects of planar projective geometry are *points* and *lines* — as a finite or infinite range or set of points, and the elementary relation is *incidence*.

Instead of the statements "a point $P$ lies on a line $l$" and "a line $l$ passes through a point $P$" we use this unifying term of incidence to say "a point $P$ and a line $l$ are incident". This rhetorical exercise not only simplifies the language, but also shows the natural symmetry of the incidence relation. Hence, symmetrically, even points may be considered as pencils of lines (finite or infinite). This is the idea of the principle of duality, emphasized in the following set of axioms:

**The axioms of the projective plane** ([Coxeter, 2003, pp. 15, 24], [Richter-Gebert, 2011, p. 41])

(i) Any two distinct points are incident with just one line.

(ii) Any two distinct lines are incident with just one point.

(iii) There are four distinct points such that no line is incident with more than two of them.

For any consequence of these three axioms it holds that if we exchange words "point" $\longleftrightarrow$ "line" and their related terms such as "range of points" $\longleftrightarrow$ "pencil of lines", "two lines meet in a point" $\longleftrightarrow$ "two points join in a line", etc., we obtain a true dual statement. Axioms (i) and (ii) are evidently dual, and we assign the equivalent symbol[8] $\times$ to meets and joins in the following way:

$$l = P \times Q \text{ means "a line } l \text{ is the join of points } P \text{ and } Q\text{"},$$
$$L = p \times q \text{ means "a point } L \text{ is the meet of lines } p \text{ and } q\text{"}.$$

While the first axiom also holds in Euclid's system in his Elements, with the second axiom we trespass the border of Euclidean geometry. As simple as possible — in the projective plane there are no parallels. For future reference (see Section 2.2.2), this also means that we can state and prove general theorems of projective geometry without examination of special affine cases.

Without axiom (iii) (and even if we assume the existence of two distinct points and two distinct lines!) a simple triangle with three distinct points and sides would be a model of such a system. The minimal system of axioms (i)–(iii) is called the Fano plane (Figure 1.14). Since the dual statement to axiom (iii) is an easy consequence of the axioms, we do not consider it as an axiom.

A *projective plane* is a set of points and lines which satisfies axioms (i)–(iii). One can proceed to deduce new theorems and define new objects. For example, a consequence of the axioms is that every line is incident with at least three distinct points. If we include other non-contradicting undefined propositions, we obtain different projective geometries. In finite geometries, if $q$ is the minimal number of points on every line, then $q-1$ is called the *order* of the projective geometry. The second characteristic number is the *dimension* $n$ of the defined space. Therefore the Fano plane is denoted as $\mathrm{PG}(n, q-1) = \mathrm{PG}(2, 2)$.

The great feature of the axiomatic approach is that we do not have to fix our system only to objects from the geometric world. We give formulations of the following problems, which can be reformulated into the language of the finite geometries $\mathrm{PG}(2, 2)$ and $\mathrm{PG}(3, 2)$.

[8]The interpretation of the symbol $\times$ for the vector product will become apparent in the analytic approach.

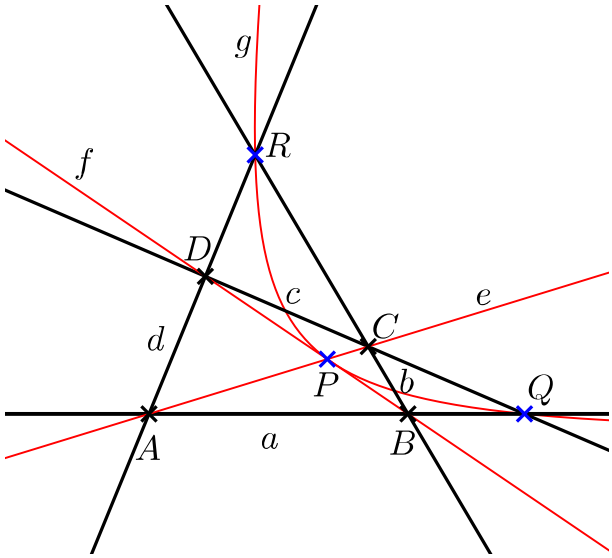



Figure 1.14: Axioms (i)–(iii) are satisfied in the Fano plane. See the abstraction of the straight line $g$. It is a mere triple of points $PQR$ and we must leave or deform our empirical intuition.

*Example.* [Havlíček, 1976, pp. 194–201]

Let us organize a toothpaste tasting. There are 7 brands of toothpastes and 7 degustators. The tasting is objective and so none of the degustators knows which brand he is currently tasting. Each degustator should taste the same number of brands of toothpastes and also each brand must be tasted by the same number of degustators. Furthermore each pair of toothpastes is also tasted by the same number of degustators, and, again, each pair of degustators tastes the same number of brands.

A possible scheme of the tasting is as follows:

| | Color | | |
|---|---|---|---|
| Degustator | Red | Green | Blue |
| A | a | d | e |
| D | d | c | f |
| C | c | e | b |
| P | e | f | g |
| B | f | b | a |
| R | b | g | d |
| Q | g | a | c |

If we swap the words degustator ≡ point, brand ≡ line, the configuration in the table is the same as the Fano plane in Figure 1.14, where three lines pass through each point. One can also notice, that only three colors are necessary to separate the toothpastes.

While the first example became obvious in the reformulation, just a step into the three-dimensional space (generalized Fano space) is far beyond the space of this thesis. At least, we state and geometrize the problem. A visual geometric solution is for example in [Falcone and Pavone, 2011].

*Example* (Kirkman's schoolgirl problem). [Woolhouse, 1850, Query VI, p. 48]

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

| | Rows | | | | |
|---|---|---|---|---|---|
| Day | 1 | 2 | 3 | 4 | 5 |
| Monday | Alice<br>Barbara<br>Catherine | Daphne<br>Elizabeth<br>Felicity | Gabriela<br>Helen<br>Irene | Julia<br>Kimberley<br>Linda | Mary<br>Nicole<br>Olivia |
| Tuesday | Alice<br>Daphne<br>Helen | Barbara<br>Elizabeth<br>Kimberley | Catherine<br>Irene<br>Olivia | Felicity<br>Linda<br>Nicole | Gabriela<br>Julia<br>Mary |
| Wednesday | Alice<br>Elizabeth<br>Mary | Barbara<br>Helen<br>Nicole | Catherine<br>Gabriela<br>Kimberley | Daphne<br>Irene<br>Linda | Felicity<br>Julia<br>Olivia |
| Thursday | Alice<br>Felicity<br>Irene | Barbara<br>Linda<br>Olivia | Catherine<br>Helen<br>Julia | Daphne<br>Kimberley<br>Mary | Elizabeth<br>Gabriela<br>Nicole |
| Friday | Alice<br>Gabriela<br>Linda | Barbara<br>Daphne<br>Julia | Catherine<br>Felicity<br>Mary | Elizabeth<br>Helen<br>Olivia | Irene<br>Kimberley<br>Nicole |
| Saturday | Alice<br>Julia<br>Nicole | Barbara<br>Irene<br>Mary | Catherine<br>Elizabeth<br>Linda | Daphne<br>Gabriela<br>Olivia | Felicity<br>Helen<br>Kimberley |
| Sunday | Alice<br>Kimberley<br>Olivia | Barbara<br>Felicity<br>Gabriela | Catherine<br>Daphne<br>Nicole | Elizabeth<br>Irene<br>Julia | Helen<br>Linda<br>Mary |

Each of the young ladies is one of the 15 points in the projective space $\mathrm{PG}(3, 2)$, there are 3 points "abreast" on a line. The system consists of 15 planes containing 7 points and 7 lines each. In total there are 35 lines. The question is to find 5 mutually skew lines of three points in 7 classes.

We have seen that finite projective geometry, or finite geometry in general, can be used to solve problems from other disciplines such as combinatorics above. The infiniteness of the projective plane is implied by the construction of a harmonic conjugate (Construction 1). Comparing the construction with the configuration of the Fano plane, we see that some other axiom is necessary, explicitly (see [Coxeter, 2003, p. 15]):

(iv) The three diagonal points of a complete quadrangle are never collinear.

By the *complete quadrangle* in (iv) we mean 4 points — *vertices* joined by 6 distinct lines — *sides*. The common point of the opposite sides (without a common vertex) is a *diagonal point*. The dual is called *complete quadrilateral* and the join of the opposite vertices (without a common side) is a *diagonal line* (Figure 1.15).

Repeatedly performing the construction of a harmonic conjugate in our system, it can be easily seen that we obtain infinitely many points on a rational line

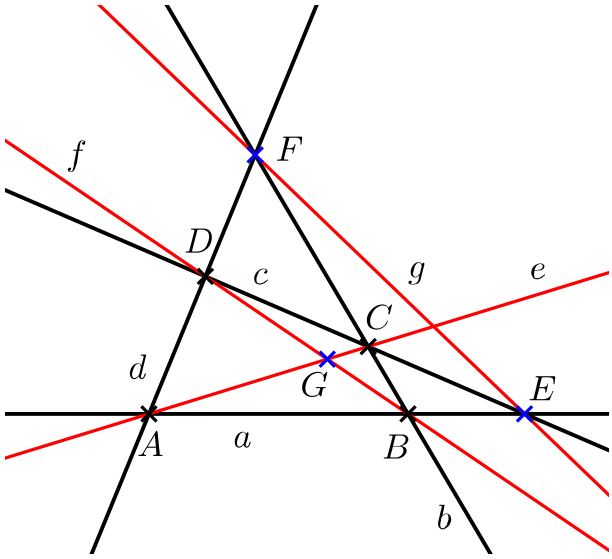



Figure 1.15: The complete quadrangle that satisfies (iv). Vertices $A, B, C, D$ are joined by sides $a, b, c, d, e, f$, the diagonal points are $E, F, G$. Dually, the complete quadrilateral with the sides $a, b, c, d$ has vertices $A, B, C, D, E, F$ and diagonal lines $e, f, g$.

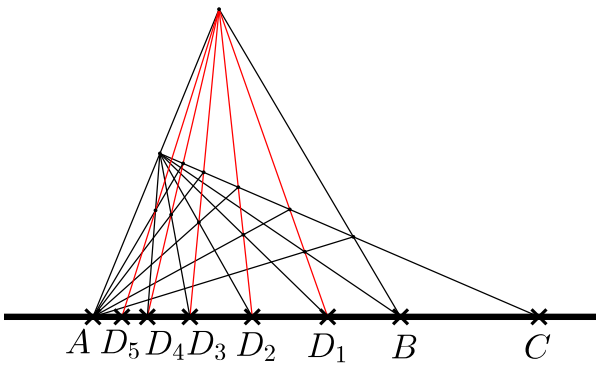



Figure 1.16: Consecutive constructions of the harmonic conjugates $D_1$ to $C$, with respect to $AB$; then $D_2$ to $C$ with respect to $AD_1$, ad infinitum.
Step-by-Step construction `https://www.geogebra.org/m/y5eje8py`

(Figure 1.16). Including an axiom on continuity (e.g. Dedekind's cuts, invariance with respect to a projectivity, etc.) we obtain a continuous real line. In the following chapters we use the three-dimensional space to describe a projective plane, and hence we should add one more remark in advance [Casse, 2006, p. 41]: A projective plane of an order greater than 2, immersed in a projective space of a dimension $n \geq 3$, is Desarguesian. Desargues's theorem 2.1.3 and its three-dimensional nature is discussed in Section 2.1.

For the rest of the thesis we continue to work with the infinite $n$-dimensional real space $\mathbb{RP}^n$ or, when explicitly stated, in the complex projective space $\mathbb{CP}^n$ (both of which are Desarguesian).

### 1.4.2 Extending the real plane

#### Central projection

Chronologically the notion of the real projective plane became apparent from perspective paintings. The famous perspective woodcut of a man drawing a lute in a central projection by Dürer (Figure 1.17) is self-explanatory. A central projection in the space is the key to visualize the infinite (*improper*) elements of the real (affine or Euclidean) plane.

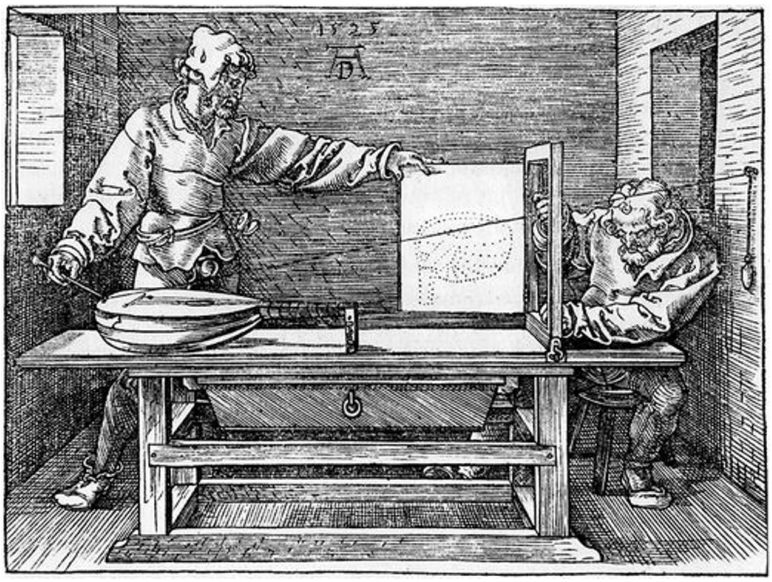


Figure 1.17: The lute lies on the horizontal plane and the man depicts it point-by-point into the vertical plane with the use of a rope — projecting line, leading through a fixed point on the wall — the centre of the projection. Source: `https://upload.wikimedia.org/wikipedia/commons/6/69/358durer.jpg`

A central projection between two planes $\pi$ and $\rho$ with a center $C$ is shown in Figure 1.18. Using elementary stereometric constructions we describe the connection between the real projective plane and the Euclidean plane.[9]

Let us have a plane $\pi$, three arbitrary noncollinear points $E_0, E_1, E_2$ and an arbitrary point $U$ inside the triangle $E_0E_1E_2$ in the plane $\pi$. We find the central projection, which projects the lines $\overline{E_0E_1}$ and $\overline{E_0E_2}$ onto perpendicular axes intersecting in the image $E_0'$ of the point $E_0$. Moreover, the image $U'$ of the point $U$ becomes the opposite vertex to $E_0'$ of the square with its two sides on the perpendicular axes. This slightly edited formulation is called Poncelet's principal lemma in [Ostermann and Wanner, 2012, p. 324].

Let $\rho'$ be the plane perpendicular to $\pi$ through the line $l_\infty = \overline{E_1E_2}$ and let a plane $\rho$ be parallel to $\rho'$. The join of the points $E_0$ and $U$ intersects the line $l_\infty$ in

[9] At this point, the reader may get confused, since in the figure, the planes are in some sense equivalent as the objects of the three-dimensional affine space in which we work. The difference is in the inner structure of the chosen reference system.

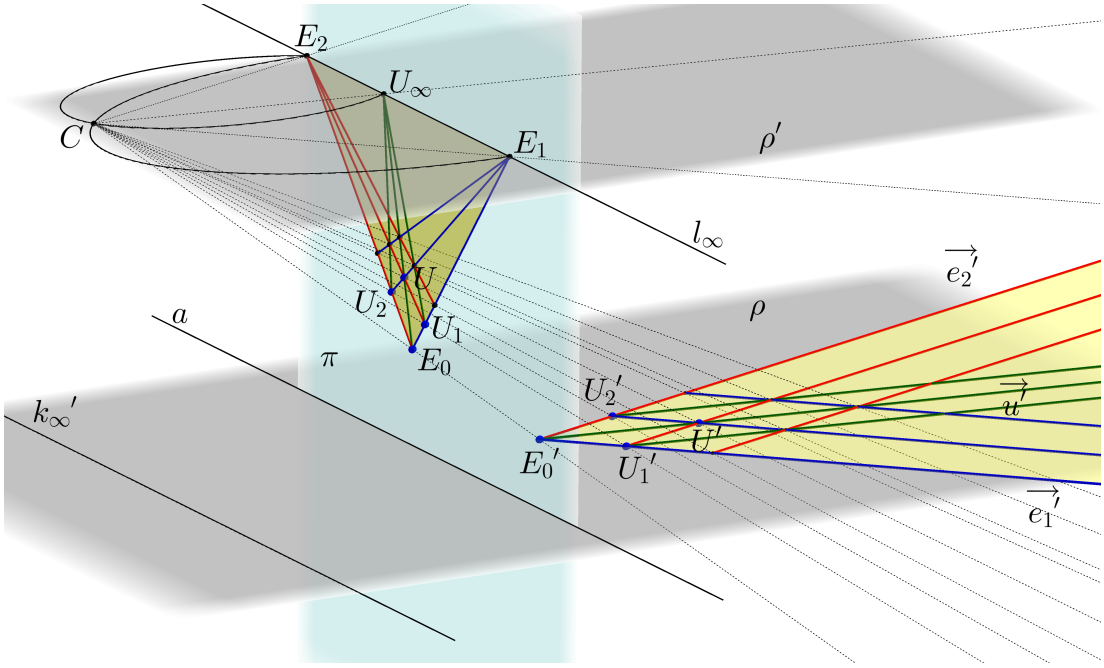



Figure 1.18: The relationship between the Euclidean plane $\rho$ and the projective plane $\pi$ in the central projection from the center $C$.
Step-by-Step construction: `https://www.geogebra.org/m/nmthuq87`

the point $U_\infty$. We find the center $C$ of the projection in the plane $\rho'$ such that the angle $\angle E_1CE_2$ is right, and the size of the angle $\angle E_1CU_\infty$ is 45°. Now we only need to project all points of the plane $\pi$ from the center $C$ into $\rho$. We obtain the images $E_0'$ and $U'$ of the points $E_0$ and $U$, respectively. The points $E_1, E_2, U_\infty$, and the whole line $l_\infty$ vanish and become directions in the plane $\rho$. One explanation is evident from the line $\overline{E_0U}$ on which $U_\infty$ lies. Therefore the point $\overrightarrow{u}' = U_\infty'$ lies on the image $\overline{E_0'U'}$. The images $\overrightarrow{e_1}'$ and $\overrightarrow{e_2}'$ of $E_1$ and $E_2$ are the directions of the mutually perpendicular lines $\overline{CE_1}$ and $\overline{CE_2}$ in the plane $\rho'$. See that in the plane $\rho$ we constructed a square net, and so, we created a Euclidean orthogonal reference system with orthogonal axes $e_1', e_2'$ and origin $E_0'$, in which $U'$ is the unit point with the coordinates $[1, 1]$. The points $U_1'$ and $U_2'$ are the unit points of the reference axes. In the original plane $\pi$ this also holds with the von Staudt's constructions on the projective lines $\overline{E_0E_1}$ and $\overline{E_0E_2}$ with the reference systems $E_0, U_1, E_1$ and $E_0, U_2, E_2$, respectively, and we can think of the (affine) plane $\pi$ as the model of the real projective plane in the one-to-one correspondence to the Euclidean plane $\rho$ unified with its directions (i.e. classes of parallel lines), which are images of the points on the horizon $l_\infty$. An important remark is that, at start, we needed at least (and only) four points $E_0, E_1, E_2, U$ (compare with Axiom (iii)). According to the formulation of the problem, we thought of a three-dimensional situation, but one should notice the whole construction is projected in a parallel projection onto our two dimensional projecting plane (a paper, computer screen). And so, there also exists a planar central collineation in a plane with the center $C$ and axis $a$, which maps the projective coordinate system $E_0, E_1, E_2, U$ onto the affine coordinate system $E_0', \overrightarrow{e_1}', \overrightarrow{e_2}', U'$. One disadvantage of the visualization of a projective plane onto an affine plane is that we do not see all projective points. The original points of the line $k_\infty'$ in the plane $\rho$ vanish from our picture.

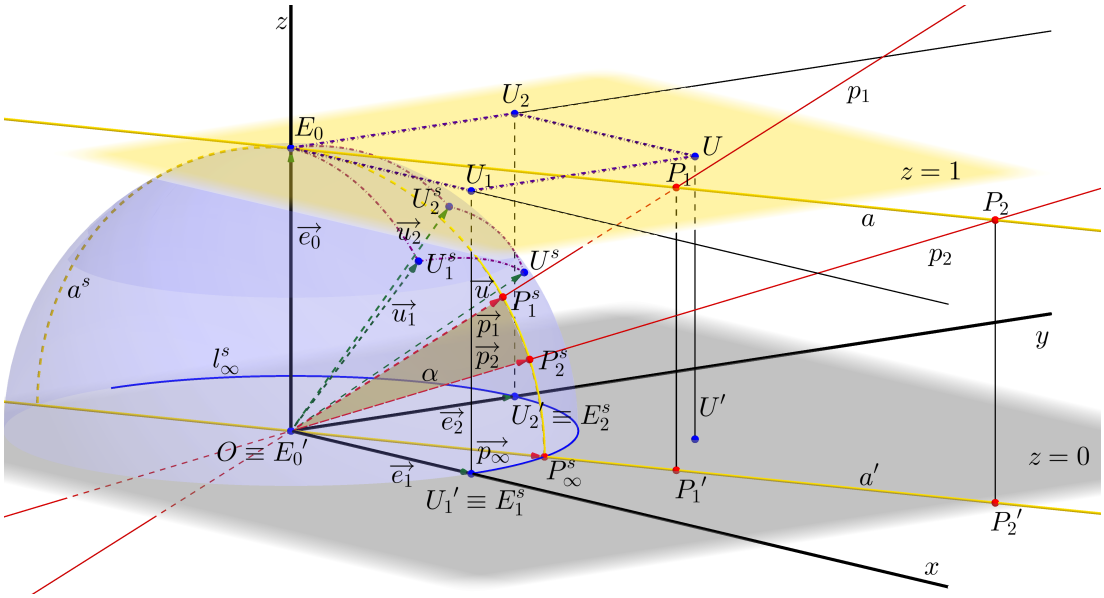



Figure 1.19: Models of the projective extension of the real plane. The point $P_1'$ in the plane $z = 0$ is orthogonally projected onto the point $P_1$ in the plane $z = 1$. The line $p_1 = \overline{P_1 O}$ represents the point $P_1'$ in the bundle of lines through $O$. The unit vector $\overrightarrow{p_1}$ is the directing vector of the line $p_1$ and represents the point $P_1'$ in the vector model, as would any of its nonzero scalar multiply. The point $P_1^s$ is the intersection of the line $p_1$ and the unit sphere in the origin with the positive $z$ coordinate, and it represents the point $P_1'$ in the half-sphere model. The line $a'$ is represented by the plane $\alpha$ in the bundle model (only a section is highlighted) and by the half of the great circle $a^s$ with one endpoint $P_\infty^s$ in the half-sphere model. The point $P_\infty^s$ is the image of the improper point of the line $a'$. The coordinate system $E_0, E_1, E_2$ and the unit square $E_0U_1UU_2$ in the models correspond also to Figure 1.18.
Interactive model: `https://www.geogebra.org/m/jstrkjyw`

Observe that in the last construction we used a three-dimensional space to describe a two-dimensional projective plane. But this is not the first time we used such a trick, the same happened with all the constructions on a line or conic in the previous sections. In synthetic constructions we always stepped out from the line or conic to its plane, performed some joins and meets, and returned back. The idea of the three-dimensional surrounding Euclidean space is also used for other models of the projective extension of the real plane. We describe some of them to make a bridge into the analytic approach.

Let us have a three-dimensional Euclidean space with the system of orthogonal axes $x, y, z$ (Figure 1.19). Let the plane $z = 0$ be a domain with a point $P'$. We translate all its points into the plane $z = 1$ in the direction of the $z$-axis and join the image $P$ of the point $P'$ with the origin $O$. If the point $P'$ and so $P$ is moved further from the origin, the line $\overline{PO}$ approaches to the parallel position with the plane $z = 0$, but it will never lie there. Apart from the lines in the plane $z = 0$, all the lines through $O$ have their corresponding points in $z = 1$. By adding the pencil of lines with the center $O$ in the plane $z = 0$, we extend the plane

$z = 1$ (represented by the three-dimensional bundle of lines) with the directions of the plane $z = 0$. Using the motion imagery, the directions correspond to the points at infinity. Furthermore, each line of the bundle may be represented by its directing vector, and, oppositely, since we covered all the lines through $O$, each three-dimensional nonzero vector describes a point of the extended plane. Any nonzero multiply of a directing vector represents the same line. Hence, finally, points of the projective plane are also represented by the set of the equivalence classes of the three-dimensional vector space without the zero vector. If we use the unit vectors placed in the originas the representatives, we obtain a half sphere with a half meridian circle with one ending point with the center in the origin. Similarly we could find the intersections of the bundle of lines and the unit sphere and identify the antipodal points.

### 1.4.3 The analytic construction of the projective plane

The following section is inspired by [Richter-Gebert, 2011, Casas-Alvero, 2014, Lávička, 2006]. The models of the projective extension of the real plane are direct motivation for the use of homogenous coordinates in the algebraic calculations. In the vector space model of the real plane, points are treated as vectors. The existential analytic definition of the projective space is:

**Definition 1.4.1** (Projective space)**.** *Let $\mathbf{V}^{n+1}$ be an $(n+1)$-dimensional, $n \geq 0$, vector space over a field $\mathbb{T}$.*
*Let $\sim$ be an equivalence relation on $\mathbf{V}^{n+1} \setminus \{\overrightarrow{o}\}$ defined as follows*

$$\text{for } \overrightarrow{u}, \overrightarrow{v} \in \mathbb{P}^n,\ \overrightarrow{u} \sim \overrightarrow{v} \Leftrightarrow \overrightarrow{u} = \lambda \overrightarrow{v}, 0 \neq \lambda \in \mathbb{T}.$$

*An $n$-dimensional projective space $\mathbb{P}^n$ is the set of the equivalence classes $(\mathbf{V}^{n+1} \setminus \{\overrightarrow{o}\})/\sim$.*
*If $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{C}$, we denote the real or complex projective space $\mathbb{RP}^n$ or $\mathbb{CP}^n$, respectively.*

Hence in the real projective plane, a point $P \in \mathbb{RP}^2$ is represented by a vector $(p_1, p_2, p_0) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ and its equivalent non-zero scalar multiplies $(\lambda p_1, \lambda p_2, \lambda p_0), \lambda \in \mathbb{R} \setminus \{0\}$. Later on, we denote this relation $P(p_1, p_2, p_0)$, and for the simplicity of the text we unify the terms point and its vector representative. If $p_0 = 0$, the point $P$ is an infinite point — direction, in the embedded affine space, and we call it an *improper point*. Otherwise, for $p_0 \neq 0$, the point is called *proper*. Observe that this classification is only a property of our affine point of view. In the projective space, points are indistinguishable in this sense.

Now when we treat points as vectors, we can conveniently use the properties of the underlying vector space. For example, *points $P(p_1, p_2, p_0)$ and $Q(q_1, q_2, q_0)$ are linearly (in)dependent* when their vector representatives $(p_1, p_2, p_0)$ and $(q_1, q_2, q_0)$ are linearly (in)dependent. Let a point $X(x_1, x_2, x_0)$ be a *linear combination of distinct points $P(p_1, p_2, p_0)$ and $Q(q_1, q_2, q_0)$*, we denote:

$$X = \alpha Q + \beta P, (\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

For different choices of the parameters $\alpha$ and $\beta$, we obtain points on the line through $P$ and $Q$ (or corresponding 2-dimensional linear subspace spanned by their vectors in the vector space). The choice of $(\alpha, \beta) = (0, 1)$ corresponds to

the point $X = P$, for $(\alpha, \beta) = (1, 0)$ we have $X = Q$. To be precise, the defined relation of equivalence $\sim$ holds for any multiply $(\lambda\alpha, \lambda\beta), \lambda \in \mathbb{R} \setminus \{0\}$ of $(\alpha, \beta)$. The pair $(\alpha, \beta)$, up to the multiplication, is called the *projective (homogenous) coordinates* of the point $X$ in the range of points $\overline{PQ}$. Actually, if we look back, we have just defined the real projective line $\mathbb{RP}^1$ as a vector subspace.

The projective plane that satisfies Axioms (i)–(iii) contains at least three points on each line. We discussed the importance of the elements $0, 1, \infty$ for the construction of the real field in Section 1.3. Analytically, a triple of elements $0(0, 1), 1(1, 1)$, and $\infty(1, 0)$ in a range of points on a line is called its *projective coordinate system.* According to Axiom (iii), and also in Poncelet's principal lemma (Figure 1.18), the real projective plane contains four points $E_0, E_1, E_2, U$. In the embedding into the Euclidean space, the natural choice of the corresponding vectors for the *fundamental triangle* is $E_0(0, 0, 1), E_1(1, 0, 0), E_2(0, 1, 0)$, and the *unit point* $U(1, 1, 1) = E_1 + E_2 + E_0$. Generally we can state the following formal definition:

**Definition 1.4.2** (Projective coordinate system and homogenous coordinates)**.** *A projective coordinate system (projective reference or coordinate frame) of the projective space $\mathbb{P}^n$ is an ordered set of $(n+2)$ points $(E_1, \ldots, E_n, E_0; U)$ from $\mathbb{P}^n$, such that any $(n + 1)$ of them are linearly independent.*

*An ordered set of $(n + 1)$ vectors $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}, \overrightarrow{e_0})$ is called a basis adapted to the projective coordinate system $(E_1, \ldots E_n, E_0; U)$ if and only if the vectors $\overrightarrow{e_i}$ represent the points $E_i$, and the unit point $U$ is represented by their sum $\sum_{i=0}^{n} \overrightarrow{e_i}$.*

*The $(n + 1)$-tuple $(x_1, \ldots, x_n, x_0) \in \mathbf{V}^{n+1}$ assigned to each point $X \in \mathbb{P}^n$ is called the projective coordinates of the point $X$ relative to $(E_1, \ldots E_n, E_0; U)$ if and only if there is a basis $(\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}, \overrightarrow{e_0})$ adapted to $(E_1, \ldots E_n, E_0; U)$, for which $X = x_1\overrightarrow{e_1} + \cdots + x_n\overrightarrow{e_n} + x_0\overrightarrow{e_0}$.*

In the thesis we choose the unit vectors $\overrightarrow{e_i} = (\underbrace{0, \ldots, 0}_{\times(i-1)}, 1, \underbrace{0, \ldots, 0}_{\times(n-i+1)})$ for $i = 1, \ldots, n$ and $e_0 = (0, \ldots, 0, 1)$, and we choose them also as the representatives of the points $E_i$. Therefore we can still use the simplification and unify points with their vector representatives.

In the axiomatic construction, the duality of points and lines in $\mathbb{RP}^2$ has been apparent from the beginning. Yet analytically, we have seen a line $a$ through points $P, Q$ as the set of points $X$ hidden in the parametric form

$$a : X = \alpha Q + \beta P, (\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

In the algebraic sense, we can obtain the implicit equation of the line $a$, when we think of it again as a 2-dimensional subspace $\alpha \in \mathbb{R}^3$. The subspace orthogonal, or in this context better said incident, to $\alpha$ is a 1-dimensional subspace $\alpha^\perp$ of the dual space of $\mathbb{R}^3$. The linear forms $\mathbf{F}$ in $\alpha^\perp$ vanish on all the points $X(x_1, x_2, x_0) \in \alpha$. Hence we have the implicit equation of the line

$$a : \mathbf{F}(X) = a_1x_1 + a_2x_2 + a_0x_0 = 0, (a_1, a_2, a_0) \in \alpha^\perp.$$

If we assign coordinates to a line $a(a_1, a_2, a_0)$, we can rewrite the equation with the use of the dot product $a \cdot X = 0 = X \cdot a$. The commutativity of the dot product enlightens the duality of lines and points. For a simple demonstration, let us find

the join $a(a_1, a_2, a_0)$ of two distinct points $P(p_1, p_2, p_0)$ and $Q(q_1, q_2, q_0)$. It holds that both $P$ and $Q$ lie on the line $a$, and hence $a \cdot P = 0$ and also $a \cdot Q = 0$. In other words, the vectors $(p_1, p_2, p_0)$ and $(q_1, q_2, q_0)$ are both perpendicular to the vector $(a_1, a_2, a_0)$. Points $P$ and $Q$ are distinct and so linearly independent as well as their vectors. Therefore the line $a$ is the cross product $a = P \times Q$ (due to the equivalence relation, it does not matter if we change the order of $P$ and $Q$). Oppositely, the intersection point $A(a_1, a_2, a_0)$ of two distinct lines $p(p_1, p_2, p_0)$ and $q(q_1, q_2, q_0)$ is incident with both of them, and we can write $A = p \times q$. As a consequence, three points $P, Q$, and $R$ are collinear if and only if the determinant $(P \times Q) \cdot R = [PQR] = 0$. Dually, lines $p, q$, and $r$ are concurrent if and only if it holds that $[pqr] = 0$. This way computations in the real projective plane become the algebra of dot and cross products and determinants.

The implicit equation of a line $a(a_1, a_2, a_0)$ holds for its proper points $X(x_1, x_2, x_0), x_0 \neq 0$. After the substitution $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$, we obtain the affine equation $a : a_1 x + a_2 y + a_0 = 0$. The improper point of a line is the one with the last coordinate $x_0 = 0$, and hence the first two coordinates of the point $X(x_1, x_2, x_0)$ satisfy $a_1 x_1 + a_2 x_2 = 0$. Since $(x_1, x_2) \neq (0, 0)$ it is the directing vector $(-a_2, a_1)$, or its multiplies, of the line $a$. All improper points of the projective plane lie on the improper line $l_\infty(0, 0, a_0)$.

### The cross-ratio

With the analytic framework, we can finally properly define the cross ratio of four points on a real projective line without Euclidean measurement.

**Definition 1.4.3** (Cross-ratio)**.** *Let us have a line $l$ through distinct points $P$ and $Q$, with projective coordinates $P(p_1, p_0), Q(q_1, q_0)$, and two other distinct points $A(a_1, a_0)$ and $B(b_1, b_0)$ on $l$. The points $A$ and $B$ are linear combinations of $P$ and $Q$:*

$$\begin{gathered} A = \alpha_1 P + \beta_1 Q \\ a_1 = \alpha_1 p_1 + \beta_1 q_1 \\ a_0 = \alpha_1 p_0 + \beta_1 q_0 \\ \text{for } (\alpha_1, \beta_1) \in \mathbb{R}^2 \setminus \{(0,0)\} \end{gathered}$$

$$\begin{gathered} B = \alpha_2 P + \beta_2 Q \\ b_1 = \alpha_2 p_1 + \beta_2 q_1 \\ b_0 = \alpha_2 p_0 + \beta_2 q_0 \\ \text{for } (\alpha_2, \beta_2) \in \mathbb{R}^2 \setminus \{(0,0)\}. \end{gathered}$$

*The number $(A, B; P, Q) = \dfrac{\beta_1 \alpha_2}{\alpha_1 \beta_2}$ is called the cross-ratio of the ordered quadruple $A, B, P, Q$ for $A \neq Q$ and $B \neq P$. In case $A = Q$ or $B = P$ we assign $(A, B; P, Q) = \infty$.*

Using Cramer's rule, it is easy to show the formulation in the form of determinants

$$(A, B; P, Q) = \frac{\begin{vmatrix} a_1 & p_1 \\ a_0 & p_0 \end{vmatrix} \begin{vmatrix} b_1 & q_1 \\ b_0 & q_0 \end{vmatrix}}{\begin{vmatrix} b_1 & p_1 \\ b_0 & p_0 \end{vmatrix} \begin{vmatrix} a_1 & q_1 \\ a_0 & q_0 \end{vmatrix}} = \frac{[AP][BQ]}{[BP][AQ]}.$$

Observe how we naturally generalized Chasles's distances used in Section 1.2.1 with determinants. Let us briefly describe what is the relation between the affine ratio with oriented lengths and the cross-ratio (see also [Ostermann and Wanner, 2012, pp. 201–204]).

On a projective line with the projective coordinate system $0(0,1)$, $1(1,1)$, $\infty(1,0)$, and a proper point $A(a,1)$ (the only improper point of the line is $\infty(1,0)$), the cross ratio $(A,1;0,\infty)$ gives the coordinate of the point $A$:

$$(A,1;0,\infty) = \frac{\begin{vmatrix} a & 0 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} a & 1 \\ 1 & 0 \end{vmatrix}} = \frac{a \cdot (-1)}{1 \cdot (-1)} = a.$$

Now, when we work with proper points $A(a,1)$ and $B(b,1)$ on the given projective line, they are uniquely defined by their coordinates $a$ and $b$. Thus, the determinant

$$\begin{vmatrix} a & b \\ 1 & 1 \end{vmatrix} = a - b$$

well characterizes the oriented length of the segment $AB$ (see that this fails if we choose one improper point, and the value of the determinant will be arbitrary).

Let us have four proper distinct points $A(a,1), B(b,1), C(c,1)$, and $X(x,1)$ on a projective line. If we send the point $X$ to infinity, the cross-ratio changes into the so-called *affine ratio*

$$\lim_{x\to\infty}(A,B;C,X) = \lim_{x\to\infty}\frac{[AC][BX]}{[BC][AX]} = \lim_{x\to\infty}\frac{[AC](b-x)}{[BC](a-x)} = \frac{[AC]}{[BC]} = (A,B;C).$$

Oppositely, the cross-ratio is often defined as the ratio of affine ratios in the affine setting. Certainly, for four proper distinct points $A(a,1), B(b,1), C(c,1)$, and $D(d,1)$ we have

$$\frac{(A,B;C)}{(A,B;D)} = \frac{\dfrac{[AC]}{[BC]}}{\dfrac{[AD]}{[BD]}} = \frac{[AC][BD]}{[BC][AD]} = (A,B;C,D).$$

We already mentioned the special place of the harmonic quadruple and defined it by the synthetic construction in Figure 1.7. With the use of the cross-ratio, four collinear points $C, D, A$, and $B$ form a harmonic quadruple if $(C,D;A,B) = -1$. Let us explain the connection with the above-mentioned construction of the harmonic conjugate.

Let $C$ be a point on a line through two given distinct points $A$ and $B$, then

$$C = c_1A + c_2B \text{ for } (c_1,c_2) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

We choose a point $E$ non-collinear with $A, B, C$ and join it with $A$ and $B$, and then we construct a line through $C$, which intersects the lines $\overline{AE}$ and $\overline{BE}$ in points $G$ and $H$, respectively. We have

$$\begin{aligned} E &= a_1A + a_2G \text{ for } (a_1,a_2) \in \mathbb{R}^2 \setminus \{(0,0)\} \\ E &= b_1B + b_2H \text{ for } (b_1,b_2) \in \mathbb{R}^2 \setminus \{(0,0)\}. \end{aligned} \tag{1.1}$$

Hence we derive

$$\begin{aligned} a_1A + a_2G &= b_1B + b_2H \\ a_1A - b_1B &= -a_2G + b_2H. \end{aligned} \tag{1.2}$$

The last expression holds for the intersection point $C$ of the lines $\overline{AB}$ and $\overline{GH}$ and so $c_1 = a_1, c_2 = -b_1$. The intersection of the lines $\overline{AH}$ and $\overline{BG}$ is the point $I$. From 1.3, we obtain

$$I = a_1A - b_2H = -a_2G + b_1B. \tag{1.3}$$

Now we find the intersection $D$ of the lines $\overline{EI}$ and $\overline{AB}$. After the substitution $b_2H = a_1A - I$ from Equation 1.3 into Equation 1.1, we have

$$\begin{aligned} E &= b_1B + a_1A - I \\ a_1A + b_1B &= E + I = D. \end{aligned}$$

In result we express the points $C$ and $D$ on the line $\overline{AB}$ as

$$\begin{aligned} C &= a_1A - b_1B \\ D &= a_1A + b_1B. \end{aligned}$$

The cross-ratio of the quadruple $C, D, A, B$ is

$$(C, D; A, B) = \frac{-b_1 \cdot a_1}{a_1 \cdot b_1} = -1.$$

# 2. Methods of synthetic projective geometry

## 2.1 Projective correspondences

In Section 1.4 we built a rigid puppet — $\mathbb{RP}^n$. In this section, we move its strings and blow some life into it by introducing mappings performed on fundamental elements of the projective real plane. Let us define these correspondences (transformations) in a synthetic-axiomatic and afterwards analytic formulation (see [Veblen and Young, 1910, pp. 55–78], [Coxeter, 2003, pp. 8–13, 49–50, 57], [Hlavatý, 1944, pp. 31, 40–41, 55], [Glaeser et al., 2016, pp. 188–189], [Richter-Gebert, 2011, p. 86], [Casas-Alvero, 2014, p. 16]). The following definitions could be generalized for other objects which have the same properties as ranges of points (one-dimensional forms). For the sake of brevity and demonstration, we do not define them in general but only on ranges of points on a line and pencils of lines in the real projective plane. On the other hand, for example, in Section 1.3 we constructed a projective coordinate system on a conic section (one-dimensional form of the second degree) and projected it on a range of points and pencil of lines. Even the elements of a system can be generalized and projected — constructions on a set of points of a virtual model of a cow are shown in Section 2.3. Whole books on projective geometry are devoted to the classification and properties of projective transformations and their connections with other geometries. Mostly synthetic and exhaustive classification of projective transformations is given in major parts of [Hlavatý, 1944, 1945].

### Perspectivity

(Figure 1.8) A *perspectivity between ranges of points* with distinct axes $o_1$ and $o_2$ with a *center of perspectivity* $O$, such that $O$ is not incindent with $o_1$ and $o_2$, maps $X_1$ on $o_1$ to $X_2$ on $o_2$ whenever $X_1, X_2, O$ are collinear. The perspectivity between the ranges of points is denoted

$$o_1 \overset{(O)}{\barwedge} o_2 \text{ or } A_1B_1C_1\ldots \overset{(O)}{\barwedge} A_2B_2C_2\ldots.$$

A *perspectivity between pencils of lines* with distinct centers $O$ and $O'$ with an *axis of perspectivity* $o$, such that $o$ is not incindent with $O$ and $O'$, maps $x$ through $O$ to $x'$ through $O'$ whenever $x, x', o$ are concurrent. The perspectivity between the pencils of lines is denoted

$$O \overset{(o)}{\barwedge} O' \text{ or } a_1b_1c_1\ldots \overset{(o)}{\barwedge} a_2b_2c_2\ldots.$$

*A range of points* $A, B, C, \ldots$ with an axis $o$ *and a pencil of lines* $a, b, c, \ldots$ with a center $O$, such that $O$ is not incident with $o$, *are in a perspectivity* if for each point $X$ on $o$ the corresponding line $x$ passes through $O$. The perspectivity between the range of points and the pencil of lines is denoted

$$o \barwedge O \text{ or } ABC\cdots \barwedge abc\ldots.$$

**Projectivity**

A composition (or product) of finitely many perspectivities between ranges of points and pencils of lines is called a *projectivity, projective transformation, projective correspondence, homography* (between two one-dimensional forms). If a projectivity is between inhomogeneous systems (e.g. pencils and ranges), it is sometimes called a *correlation*. Two ranges of points with a common axis or two pencils of lines with a common center may also be in a projectivity. An involutive projectivity is called an *involution*, i.e. a projectivity of period 2.

We denote two systems in a projectivity with the symbol $\overline{\wedge}$, e.g.:
a projectivity between ranges of points

$$o_1 \,\overline{\wedge}\, o_2 \text{ or } A_1B_1C_1 \cdots \overline{\wedge}\, A_2B_2C_2 \ldots,$$

a correlation between a range of points and pencil of lines

$$o_1 \,\overline{\wedge}\, O_2 \text{ or } A_1B_1C_1 \cdots \overline{\wedge}\, a_1b_1c_1 \ldots,$$

an involution on a range of points (see Figures 1.9 and 1.10)

$$A_1A_1' \,\overline{\wedge}\, A_1'A_1.$$

A projectivity with two real fixed points is called *hyperbolic*, with one real fixed point *parabolic*, and without any real fixed points *elliptic*. An involution with two real fixed points is called *hyperbolic*, without any fixed points *elliptic*.

Synthetically we can define projectivities on more-dimensional forms (e.g. points in a plane, planes through a point) by keeping the projectivity of one-dimensional forms: A *projectivity* between the elements of two $n$-dimensional forms is any one-to-one reciprocal correspondence between the elements of the two forms, such that to every one-dimensional form of one there corresponds a projective one-dimensional form of the other.

A *collineation* is any one-to-one correspondence in the real projective space that maps collinear points to collinear points. If the correspondence is projective, we call it *projective collineation*.[10]

The modern abstract analytic version of the definition of a projectivity (on points) deals with the underlying vector space model.

Let $\mathbb{RP}^n(\mathbf{V}^{n+1})$ and $\mathbb{RP}'^m(\mathbf{V}'^{m+1})$ be real projective spaces. Assume $\varphi$ is an isomorphism between $\mathbf{V}^{n+1}$ and $\mathbf{V}'^{m+1}$. A point $P \in \mathbb{RP}^n$ is represented by the vector $\overrightarrow{v} \in \mathbf{V}^{n+1}$, denoted $P = \langle \overrightarrow{v} \rangle$. Then the map $f : \mathbb{RP}^n \to \mathbb{RP}'^m$ defined by the rule $\langle \overrightarrow{v} \rangle \to \langle \varphi(\overrightarrow{v}) \rangle$ is called a projectivity $f$ induced by $\varphi$. Because of the associated isomorphism, the images of projective points are points, the dimensions of the projective spaces are equal, so $n = m$, and the projectivity is also a collineation. If we assumed $\varphi$ to only be a homomorphism, we could obtain a projection of collinear points into one point between spaces of different dimensions.

A projectivity $f$ is described by a square matrix of the linear transformation (isomorphism) $\varphi$ on its representatives. In homogenous coordinates we map $P(p_1, \ldots, p_n, p_0) \in \mathbb{RP}^n$ onto $P'(p_1', \ldots, p_n', p_0') \in \mathbb{RP}'^n$:

[10] Collineations are always projective transformations in $\mathbb{RP}^2$ and $\mathbb{CP}^2$. However, the distinction is necessary in other projective spaces.

$$P' = f(P) = \mathbf{T}P$$
$$\begin{pmatrix} p'_1 \\ \vdots \\ p'_n \\ p'_0 \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} & a_{10} \\ \vdots & & & \vdots \\ a_{n1} & \dots & a_{nn} & a_{n0} \\ a_{01} & \dots & a_{0n} & a_{00} \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_n \\ p_0 \end{pmatrix}.$$

In this simplified form, we incorporate the equivalence relation in the analytic definition of a projective space. The matrix $\mathbf{T}$ is also unique only up to a non-zero real factor.

With the following theorem we can create an important link between von Staudt's "Wurf" as a visual property and the cross-ratio as a numerical value of four points on a line (see also [Lávička, 2006, p. 45]).

**Theorem 2.1.1.** *Projectivities in $\mathbb{RP}^2$ preserve the cross-ratio.*

*Proof.* Let us have four distinct collinear points $A, B, P, Q \in \mathbb{RP}^n$ such that

$$A = \alpha_1 P + \beta_1 Q, \text{ for } (\alpha_1, \beta_1) \in \mathbb{R}^2 \setminus \{(0,0)\}$$
$$B = \alpha_2 P + \beta_2 Q, \text{ for } (\alpha_2, \beta_2) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Let $f$ be a projectivity $f : \mathbb{RP}^n \to \mathbb{RP}'^n$ induced by an isomorphism $\varphi : \mathbf{V}^{n+1} \to \mathbf{V}'^{n+1}$. The isomorphism $\varphi$ on the vector representatives $\langle \overrightarrow{a} \rangle, \langle \overrightarrow{b} \rangle, \langle \overrightarrow{p} \rangle, \langle \overrightarrow{q} \rangle$ of points $A, B, P, Q$ implies:

$$f(A) = \varphi(\langle \overrightarrow{a} \rangle) = \varphi(\alpha_1 \langle \overrightarrow{p} \rangle + \beta_1 \langle \overrightarrow{q} \rangle) = \alpha_1 \langle \varphi(\overrightarrow{p}) \rangle + \beta_1 \langle \varphi(\overrightarrow{q}) \rangle = \alpha_1 f(P) + \beta_1 f(Q)$$

and similarly

$$f(B) = \alpha_2 f(P) + \beta_2 f(Q)$$

and the cross ratio $(f(A), f(B); f(P), f(Q)) = \dfrac{\beta_1 \alpha_2}{\alpha_1 \beta_2} = (A, B; P, Q)$.

In case $A = Q$ or $B = P$ also $f(A) = f(Q)$ or $f(B) = f(P)$ and the cross-ratio equals $\infty$. □

A straightforward consequence is that projectivities in $\mathbb{RP}^2$ preserve harmonicity.

### The fundamental theorem

Let us sum up how to determine projectivities in the so-called fundamental theorem of projective geometry, nowadays formulated as a list of equivalent statements. Since it is the object of a majority of textbooks on projective geometry in any (synthetic, axiomatic, analytic) exposition, we do not aim to prove these theorems in this work. We include them to show important properties of projective correspondences and fulfill our commitment from Section 1.3.

From the modern axiomatic point of view, Pickert gives (for details, history and proofs see [Plaumann and Strambach, 1980, pp. 1–12]) the following formulation in a projective plane based on Axioms (i)–(iii):

**Theorem 2.1.2** (The fundamental theorem of projective geometry, axiomatically)**.** *For a projective plane the following conditions are equivalent:*

1. *A projectivity of a range of points with three fixed points is an identity.*

2. *A projectivity is already determined by the images of 3 points.*

3. *A projectivity is a product of at most 3 perspectivities.*

4. *A projectivity of a range of points to itself with a fixed point is a product of two perspectivities.*

5. *A projectivity of a range of points onto another range of points with a fixed point is a perspectivity.*

6. *(Pappus's theorem, Figure 2.2) If the three points $A, B, C$ are on a line and the three points $A', B', C'$ are on another line but not on the first, then the intersecting points of the lines $\overline{AB'}, \overline{CA'}, \overline{BC'}$ with $\overline{BA'}, \overline{AC'}, \overline{CB'}$, respectively, are collinear.*

Historically, von Staudt proved the first statement. If we look at it from the analytic derivation in $\mathbb{RP}^2$, and we fix the points $0(0,1), 1(1,1), \infty(1,0)$ of the projective coordinate system on a projective line, then each point $P(p,1)$ is uniquely determined by its cross ratio $(P, 1; 0, \infty) = p$.

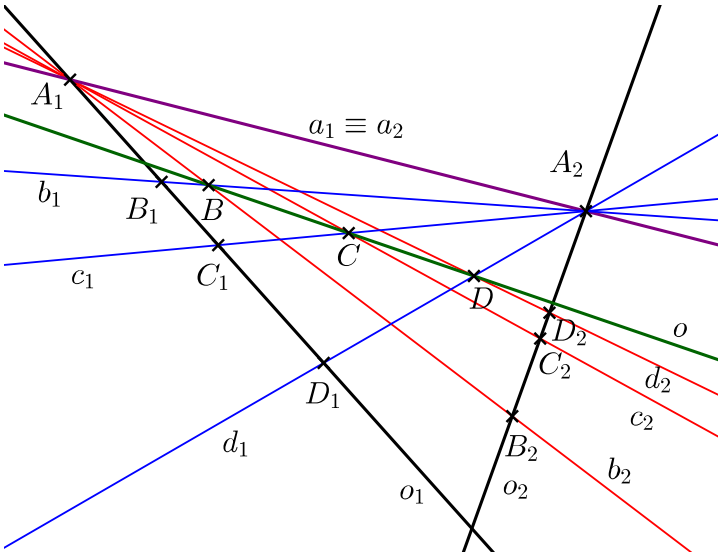



Figure 2.1: A demonstration of the second, third and fifth condition of Theorem 2.1.2.
Step-by-Step construction: `https://www.geogebra.org/m/ntrnzagf`

The meaning of the second, third and fifth statement is depicted in Figure 2.1. The third condition on two distinct ranges in $\mathbb{RP}^2$ can be even more simplified, and any projectivity can be reduced into the product of two perspectivities (see [Glaeser et al., 2016, p. 190], [Hlavatý, 1944, pp. 44–46]). For example, let us have ranges of points $A_1, B_1, C_1, D_1, \ldots$ and $A_2, B_2, C_2, D_2, \ldots$ in a projectivity. A pencil of lines $a_1, b_1, c_1, d_1, \ldots$ with the center $A_2$ is in a perspectivity with the range $A_1, B_1, C_1, D_1, \ldots$, and similarly a pencil of lines $a_2, b_2, c_2, d_2, \ldots$ with the center $A_1$ is in a perspectivity with the range $A_2, B_2, C_2, D_2, \ldots$. The pencils $a_1, b_1, c_1, d_1$ and $a_2, b_2, c_2, d_2$ have a fixed line $a_1 = a_2$ and with respect to the fifth condition they are also in a perspectivity. The common points $B, C, D, \ldots$ of

the pencils lie on the line $o$, called the *axis of projectivity*. See that the second statement holds, since the point $D_1$ is not necessary to find the projective axis, and its image $D_2$ can be constructed afterwards with the use of the point $D$.

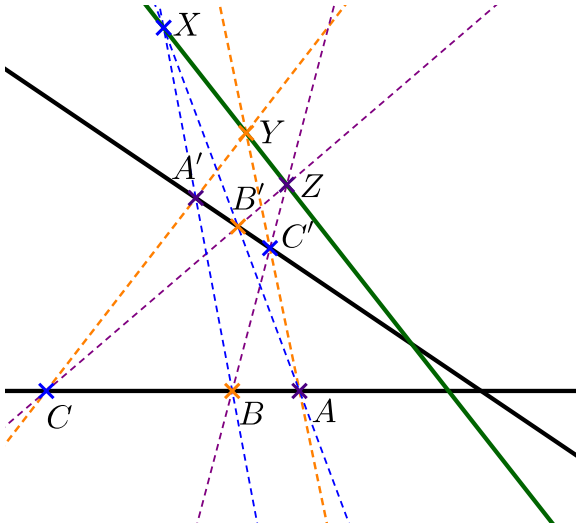



Figure 2.2: A configuration in Pappus's theorem.
Step-by-Step construction: `https://www.geogebra.org/m/pehvjqpf`

Pappus's theorem in general projective planes is often stated as an axiom, and the projective plane which satisfies it, is called *Pappian*. In the real projective plane the "axiom" is implied by the commutativity of the inner field. We examine the proof in Section 2.3.

One more axiom in general and theorem in $\mathbb{RP}^2$ should not be omitted. Desargues's theorem gives a better look into perspectivities of planar figures [Glaeser et al., 2016, pp. 184–185], [Coxeter, 2003, p. 19], [Plaumann and Strambach, 1980, pp. 6–7]:

**Theorem 2.1.3** (Desargues's theorem, Figure 2.3)**.** *If two triangles are perspective to a point, then their sides are perspective to a line.*

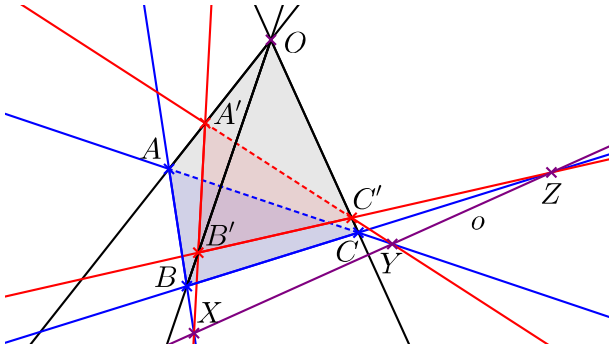



Figure 2.3: A suggestive visualization of the configuration in Desargues's theorem. The triangles $ABC$ and $A'B'C'$ are perspective to the *perspector* — point $O$ and to the *perspectrix* — line $o$. The pyramid $ABCO$ is highlighted.
Step-by-Step construction: `https://www.geogebra.org/m/hdgeuw4c`

The theorem is due to Hessenberg's theorem a consequence of Pappus's axiom. The projective planes in which Desargues's theorem holds are called *Desarguesian*. Instead of the formulation of a proper axiomatic proof, this theorem is the ideal representative of intuitive thinking. Rather then having two triangles in one plane, one can use the third dimension to obtain a triangular pyramid and its sections, the perspector is the apex of the pyramid, the perspectrix is the common line of the intersecting planes.

Richter-Gebert [2011, p. 62] refers to the name fundamental theorem with the following analytic version in $\mathbb{RP}^2$

**Theorem 2.1.4** (The fundamental theorem of projective geometry, analytically)**.** *If $f : \mathbb{RP} \to \mathbb{RP}$ is any bijective map that preserves the collinearity of points, then $f$ can be expressed as multiplication by a $3 \times 3$ matrix.*

To find such a matrix we need four points in $\mathbb{RP}^2$, from which no three are collinear or analogically $n + 2$ points in $\mathbb{RP}^n$. The generalization of the fundamental theorem over another fields is given further in [Richter-Gebert, 2011, p. 88]

### 2.1.1 Projective generation of a conic

Until now, we have worked only with straight lines, but projective geometry is a very convenient theory to examine conics. Standard definitions of a conic again differ in the synthetic and analytic approach. Synthetically a (point) conic can be constructed by joining conjugated elements of two projective pencils of lines (the dual formulation defines a conic as the envelope of its tangents).

**Definition 2.1.1.** *Let us have two pencils of lines $a, b, c, \ldots$ with a center $O$ and $a', b', c', \ldots$ with a center $O' \neq O$ in a projectivity. The set of all intersecting points of conjugated elements in the projectivity is called a conic.*

*If the projectivity is not a perspectivity, the conic is called regular. Otherwise, the conic is singular.*

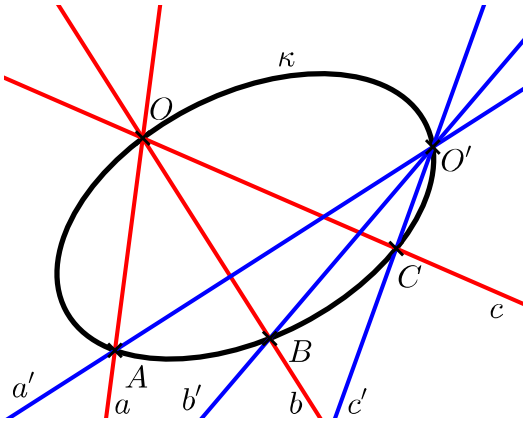



Figure 2.4: Projective generation of a conic.
Step-by-Step construction: `https://www.geogebra.org/m/ahvpvx6s`

Algebraically speaking, a conic $\kappa$ in $\mathbb{RP}^2$ is the zero set of a quadratic form over $\mathbb{R}$

$$\boldsymbol{F}_2(X) = 0$$

with the associated matrix $\mathbf{A}$

$$\mathbf{A} = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}, (a, b, c, d, e, f) \in \mathbb{R}^6 \setminus \{(0, 0, 0, 0, 0, 0)\}.$$

Hence a conic $\kappa$ is given in the form of a homogeneous equation:

$$X^{\mathrm{T}}\mathbf{A}X = ax_1^2 + 2bx_1x_2 + cx_2^2 + 2dx_1x_0 + 2ex_2x_0 + fx_0^2 = 0. \tag{2.1}$$

If $\det \mathbf{A} \neq 0$ the conic is *regular*, otherwise, it is *singular*. From Equation 2.1, one can observe that if we fix a point $X(x_1, x_2, x_0)$, we obtain a linear equation on the six parameters $a, b, c, d, e, f$; from which one can be reduced by factorization. Therefore five distinct points give five such linear equations, which determine the conic. See that the same property is hidden in the synthetic definiton. Two of the five given points are centers of the pencils of lines and three other points are the intersections of the conjugated lines in the projectivity. By Theorem 2.1.2, for any other line in one pencil, we obtain its image in the second pencil and also their intersection. Let us see the equivalence of both definitions (the algebraic construction is given in [Richter-Gebert, 2011, pp. 167–172]).

**Theorem 2.1.5** (Figure 2.4)**.** *Let $A, B, C, P, Q$ be 5 points on a conic. Let $P, Q$ be the centers of pencils of lines through $A, B, C$ such that $\overline{PA}, \overline{PB}, \overline{PC}$ are conjugated with $\overline{QA}, \overline{QB}, \overline{QC}$, respectively. Points on the conic are intersecting points of conjugated pairs of these pencils.*

*Proof.* Let us have 4 points $A, B, C, D$, each 3 of them are non-collinear. A choice of the 5th point $P$ determines one of the conics of the pencil of conics through $A, B, C, D$. Two intersecting lines make a singular case of a conic, and there are 3 such pairs on points $A, B, C, D$ : $(\overline{AB}, \overline{CD}); (\overline{AC}, \overline{BD}); (\overline{AD}, \overline{BC})$ (Figure 2.5). Let $\kappa$ be the singular conic given by pair the $(\overline{AB}, \overline{CD})$. If we

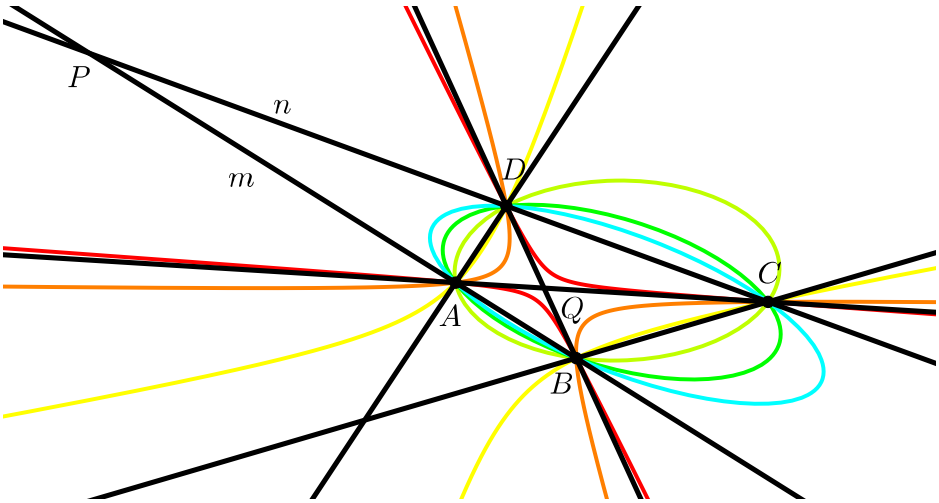



Figure 2.5: A pencil of conics through four points $A, B, C$, and $D$.

choose the point $P(p_1, p_2, p_0)$ to be the intersecting point of $m(m_1, m_2, m_0) = \overline{AB}$ and $n(n_1, n_2, n_0) = \overline{CD}$, then

$$p_1 m_1 + p_2 m_2 + p_0 m_0 = 0$$
$$p_1 n_1 + p_2 n_2 + p_0 n_0 = 0,$$

and also

$$(p_1 m_1 + p_2 m_2 + p_0 m_0)(p_1 n_1 + p_2 n_2 + p_0 n_0) = P^{\mathrm{T}} m n^{\mathrm{T}} P = P^{\mathrm{T}} G P = 0$$
$$(P^{\mathrm{T}} G P = 0 \Leftrightarrow P^{\mathrm{T}} G^{\mathrm{T}} P = 0) \Rightarrow P^{\mathrm{T}} \frac{(G + G^{\mathrm{T}})}{2} P = 0.$$

The matrix $\frac{G+G^{\mathrm{T}}}{2}$ is the matrix associated with $\kappa$. The coordinates of $m, n$ are

$$(m_1, m_2, m_0) = (a_1, a_2, a_0) \times (b_1, b_2, b_0)$$
$$(n_1, n_2, n_0) = (c_1, c_2, c_0) \times (d_1, d_2, d_0),$$

where $(a_1, a_2, a_0)$ are the coordinates of the point $A$, and so on. Then

$$P^{\mathrm{T}} m n^{\mathrm{T}} P = [P, A, B][P, C, D] = 0.$$

If we assume $P$ to be a variable point, then the last equation is quadratic. A similar equation holds for a different pair of intersecting lines, for example $\overline{AC}$ and $\overline{BD}$. These two singular conics give us the equation of the pencil of conics

$$k_1[P, A, C][P, B, D] + k_2[P, A, B][P, C, D] = 0, \text{ for } (k_1, k_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}. \quad (2.2)$$

Let $Q$ be another fixed point of the conic, then we use the Plücker's trick (described in [Richter-Gebert, 2011, p. 96–98]). We choose

$$k_1 = [Q, A, B][Q, C, D], k_2 = -[Q, A, C][Q, B, D].$$

Equation 2.2 changes to

$$[Q, A, B][Q, C, D][P, A, C][P, B, D] - [Q, A, C][Q, B, D][P, A, B][P, C, D] = 0,$$

which is the quadratic equation for a variable point $P$. Performing the last modification, we have

$$\frac{[Q, A, B][Q, C, D]}{[Q, A, C][Q, B, D]} = \frac{[P, A, B][P, C, D]}{[P, A, C][P, B, D]}. \quad (2.3)$$

Without loss of generality, we choose the respective projective coordinate systems of pencils of lines through $A, B, C$, and $D$ with the centers $P$ and $Q$:

$$\overline{PB} = (0, 1)_P; \ \overline{PC} = (1, 0)_P; \ \overline{PA} = (1, 1)_P; \ \overline{PD} = (\alpha_1, \alpha_2)_P$$
$$\overline{QB} = (0, 1)_Q; \ \overline{QC} = (1, 0)_Q; \ \overline{QA} = (1, 1)_Q; \ \overline{QD} = (\beta_1, \beta_2)_Q$$
$$(\alpha_1, \alpha_2) \text{ and } (\beta_1, \beta_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

The cross ratios are

$$(\overline{PB}, \overline{PC}; \overline{PA}, \overline{PD}) = \frac{\alpha_2}{\alpha_1}, \alpha_1 \neq 0$$
$$(\overline{QB}, \overline{QC}; \overline{QA}, \overline{QD}) = \frac{\beta_2}{\beta_1}, \beta_1 \neq 0.$$

The homogenous coordinates of the lines $\overline{PA}, \overline{PD}$ are

$$(P \times A) = \overline{PA} = \overline{PC} + \overline{PB} = (P \times C) + (P \times B)$$
$$(P \times D) = \overline{PD} = \alpha_1 \overline{PC} + \alpha_2 \overline{PB} = \alpha_1 (P \times C) + \alpha_2 (P \times B).$$

Therefore

$$\frac{[P,A,B][P,C,D]}{[P,A,C][P,B,D]} = \frac{B(P\times A)(-C(P\times D))}{C(P\times A)(-B(P\times D))} =$$

$$\frac{B((P\times C)+(P\times B))(C(\alpha_1(P\times C)+\alpha_2(P\times B)))}{C((P\times C)+(P\times B))(B(\alpha_1(P\times C)+\alpha_2(P\times B)))} =$$

$$\frac{[P,C,B]\alpha_2[P,B,C]}{[P,B,C]\alpha_1[P,C,B]} = \frac{\alpha_2}{\alpha_1}.$$

In like manner for the pencil of lines through $Q$. From Equation 2.3 we have

$$\frac{\beta_2}{\beta_1} = \frac{\alpha_2}{\alpha_1},$$

and the cross-ratios of the corresponding pencils are equal. □

### Polarity

The next essence we should add into the theoretical background is the underlying harmonicity of conics. This time we start analytically and give the synthetic construction as a consequence.

Let us have a conic $\kappa$ in $\mathbb{RP}^2$ given by its quadratic form $\boldsymbol{F}_2$ with the matrix $\mathbf{A}$, and let $\boldsymbol{F}$ be its associated symmetric bilinear (polar) form with the same matrix. Then two points $P(p_1,p_2,p_0), Q(q_1,q_2,q_0)$ are called the *polar conjugates* with respect to the conic $\kappa$ if

$$\boldsymbol{F}(P,Q) = (p_1,p_2,p_0)\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_0 \end{pmatrix} = P^{\mathrm{T}}\mathbf{A}Q = 0.$$

A point $P$ is called a *singular point* of the conic $\kappa$ if it is the polar conjugate of every point in $\mathbb{RP}^2$. A point of the conic $\kappa$ which is not singular is called a *regular* point of the conic $\kappa$. Straightforward from the definition of a conic, a point on a conic is its own polar conjugate. If a point $P$ is not a singular point of a conic $\kappa$, its polar conjugates form the *polar line* (or simply *polar*) $p'$ of the pole $P$ with respect to $\kappa$.

$$p' : P^{\mathrm{T}}\mathbf{A}X = (ap_1+cp_2+dp_0)x_1 + (cp_1+bp_2+ep_0)x_2 + (dp_1+ep_2+fp_0)x_0 = 0.$$

From the linear equation, it is obvious that the polar is a line. See that the homogenous coordinates of the line $p'$ are extracted from $(p'_1,p'_2,p'_0) = P^{\mathrm{T}}\mathbf{A}$. The polar of a point on the conic is a tangent of the conic with the pole being its contact point. The duality of a pole and its polar is also apparent from ${p'}^{\mathrm{T}}\mathbf{A}^{-1} = P^{\mathrm{T}}$, for a regular matrix $\mathbf{A}$. The described map from points to lines and vice-versa with respect to a conic is called a *polarity*.

Let us proceed to the following property in $\mathbb{CP}^2$. The join of the pole $P$ and an arbitrary point $Q$ on its polar intersects the conic in points $U,V$, which are separated harmonically by $P,Q$.

Let $\kappa$ be a conic, with an associated matrix $\mathbf{A}$, a point $P(p_1,p_2,p_0)$ be a non-singular point of a conic, and a line $p'$ be the polar of the pole $P$ with respect to $\kappa$.

Let $Q(q_1,q_2,q_0) \neq P$ be a point on the line $p'$ such that

$$\boldsymbol{F}(P,Q) = 0. \tag{2.4}$$

The projective homogenous coordinates of a point $W(w_1, w_2, w_0)$ on the line $\overline{PQ}$ in the projective coordinate system of $\mathbb{RP}^2$ are $W = k_1Q + k_2P$, for $(k_1, k_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and the projective homogenous coordinates of $W$ in the coordinate system of the projective line $\overline{PQ}$ are $(k_1, k_2)$ . By the choice $(k_1, k_2) = (0, 1)$, we obtain the projective coordinates of $P$, and $(k_1, k_2) = (1, 0)$ are the projective coordinates of $Q$. If $W$ lies on $\kappa$, then

$$\boldsymbol{F_2}(k_1Q + k_2P) = (k_1Q + k_2P)^{\mathrm{T}}\mathbf{A}(k_1Q + k_2P) = 0.$$

After the expansion

$$\begin{aligned} &k_1^2\boldsymbol{F_2}(Q) + 2k_1k_2\boldsymbol{F}(P,Q) + k_2^2\boldsymbol{F_2}(P) = \\ &k_1^2Q^{\mathrm{T}}\mathbf{A}Q + 2k_1k_2P^{\mathrm{T}}\mathbf{A}Q + k_2^2P^{\mathrm{T}}\mathbf{A}P = 0. \end{aligned} \tag{2.5}$$

From 2.4, $\boldsymbol{F}(P,Q) = P^{\mathrm{T}}\mathbf{A}Q = 0$. We substitute $l = \frac{k_1}{k_2}, k_2 \neq 0$ into 2.5. If $k_2 = 0$, we obtain the projective coordinates of $Q$. Otherwise, we have a quadratic equation for $l$

$$\boldsymbol{F_2}(P) + l^2\boldsymbol{F_2}(Q) = 0.$$

Using Vieta's formulas for the roots $l$ and $l'$, we have $l + l' = 0$, and so $l = -l'$. Without loss of generality, we choose $k_2 = k_2' = 1$, then $k_1 = l, k_1' = l'$ and assign these coordinates to the points $U(l, 1), V(l', 1)$ on the conic $\kappa$. Hence the cross-ratio

$$(P, Q; U, V) = \frac{\begin{vmatrix} 0 & 1 \\ l & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ l' & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ l & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ l' & 1 \end{vmatrix}} = \frac{-l}{-l'} = -1.$$

In result, the points $P, Q, U$, and $V$ form a harmonic quadruple. On the contrary, we could have stated the above property as the synthetic definition of a pole and its polar.

As a result, we have the following synthetic construction.

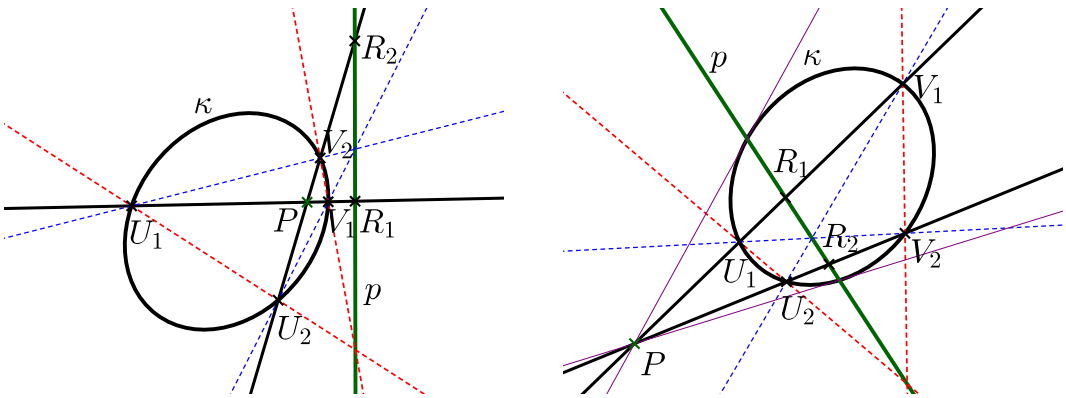



Figure 2.6: Synthetic construction of a polar. The right figure also emphasize the relation between a polar and tangents.
Step-by-Step construction: `https://www.geogebra.org/m/zd3xhkmw`

**Construction 2** (Figure 2.6)**.** *Let $\kappa$ be a conic and $P$ its non-singular point, find the polar line $p$ of the pole $P$ with respect to $\kappa$.*

1. *We choose two lines through $P$, which intersect $\kappa$ in points $U_1, V_1; U_2, V_2$.*

2. *Using Construction 1 of a harmonic conjugate, we find the points $R_1, R_2$ such that $(P, R_1; U_1, V_1) = (P, R_2; U_2, V_2) = -1$ (in Figure 2.6 we conveniently used the complete quadrangle with the vertices $U_1, V_1, U_2, V_2$).*

3. *The line $p$ through the points $R_1$ and $R_2$ is the desired polar.*

The construction also gives another point of view on imaginary elements. If the pole $P$ is an outer point of the conic $\kappa$, we have two real tangents, if the pole $P$ is an inner point of the conic $\kappa$, the tangents are imaginary (compare with Figures 1.9 and 1.10).

**Projectivity of conics**

The definition of a conic with pencils of lines implies an important property of projectivities (for more details see [Veblen and Young, 1910]). Let us have in a real projective plane $\pi$ two projective pencils of lines: $a_1, b_1, c_1, \dots$ with a center $O_1$ and $a_2, b_2, c_2 \dots$ with a center $O_2$ and let $f$ be the projectivity between them, which generates a conic $\kappa$. Now, let us have a projectivity $g$ between the projective plane $\pi$ and another plane $\rho$. In the projectivity $g$, the pencils $a_1, b_1, c_1, \dots$ and $a_2, b_2, c_2, \dots$ are projected onto pencils $a_1', b_1', c_1', \dots$ with a center $O_1'$ and $a_2', b_2', c_2', \dots$ with a center $O_2'$ in the plane $\rho$, respectively.

Now, the inverse projectivity $g^{-1}$ maps the pencils in $\rho$ onto pencils in $\pi$, the projectivity $f$ projects the pencil $a_1, b_1, c_1, \dots$ onto the pencil $a_2, b_2, c_2, \dots$ and the projectivity $g$ maps $\pi$ back onto $\rho$. Thus, the composition of projectivities $g \circ f \circ g^{-1}$ is a projectivity between the pencils of lines $a_1', b_1', c_1', \dots$ with a center $O_1'$ and $a_2', b_2', c_2', \dots$ with a center $O_2'$, which generates a conic $\kappa'$ in the plane $\rho$.

In terms of matrices, let the projectivity $g$ have an associated matrix $\mathbf{T}$ and let the conic $\kappa$ have an associated matrix $\mathbf{A}$. The projectivity $g$ maps for any point $X \in \pi$ the quadratic form

$$X^{\mathrm{T}}\mathbf{A}X \to (\mathbf{T}^{-1}X)^{\mathrm{T}}\mathbf{A}(\mathbf{T}X) = X^{\mathrm{T}}({\mathbf{T}^{-1}}^{\mathrm{T}}\mathbf{A}\mathbf{T})X$$

and the associated matrix of the image $\kappa'$ of the conic $\kappa$ is ${\mathbf{T}^{-1}}^{\mathrm{T}}\mathbf{A}\mathbf{T}$.

## 2.2 Synthetic methods

One can choose different paths to reach the same goal in projective geometry. Naturally, some methods appear faster in one problem and cumbersome in the second. We have already used some of them throughout the text without notion. In this section, we describe two fundamental synthetic methods and combine them with the analytic method. One could also count the principle of duality as a method, which was already defined in Section 1.4.1.

### 2.2.1 The method of projection

The classical synthetic and axiomatic constructions of projective geometry rely on the projections. Projecting ranges of points and pencils of lines by compositions of perspectivities is what we mean by the method of projection, and will become

our basic machinery. For demonstration, see the proof of the following theorem [Coxeter, 2003], [Glaeser et al., 2016, p. 191].

**Theorem 2.2.1.** *Let us have four collinear points $A, B, C$ and $D$. The cross-ratios $(A, B; C, D)$ and $(B, A; D, C)$ are equal.*

*Proof.* (Figure 2.7) We need to find the sequence of perspectivities, which maps the points $A, B, C$, and $D$ to $B, A, D$, and $C$, respectively. Let $O$ be an arbitrary point not on the line $AB$ and draw a distinct line through the point $D$.

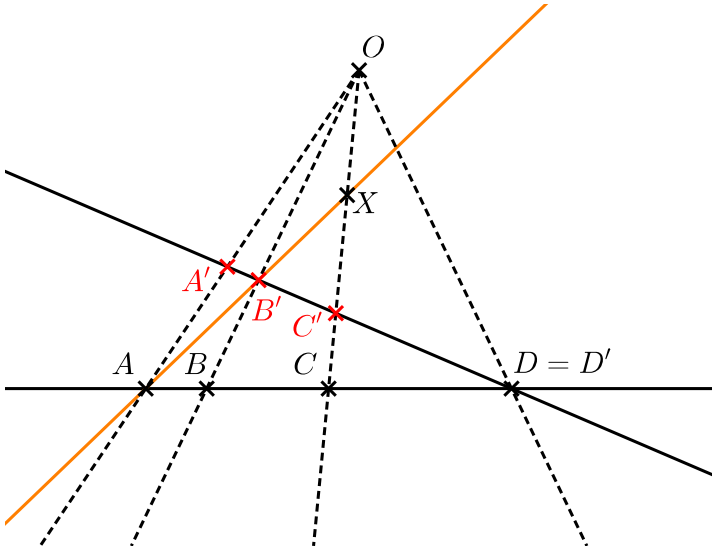



Figure 2.7: Theorem 2.2.1.
Step-by-Step construction: `https://www.geogebra.org/m/djxbrfef`

Assume the following composition of perspectivities:

$$ABCD \overset{(O)}{\barwedge} A'B'C'D$$
$$A'B'C'D \overset{(A)}{\barwedge} OXC'C$$
$$OXC'C \overset{(B')}{\barwedge} BADC$$
$$\text{therefore } ABCD \barwedge BADC$$
$$\text{and } (A, B; C, D) = (B, A; D, C).$$

□

We have already mentioned Pappus's theorem as a part of the fundamental theorem 2.1.2. Let us prove it with the method of projection.

**Theorem 2.2.2** (Pappus's theorem, Figure 2.8)**.** *If the three points $A, B, C$ are on a line and the three points $A', B', C'$ are on another line but not on the first, then the intersecting points of the lines $\overline{AB'}, \overline{CA'}, \overline{BC'}$ with $\overline{BA'}, \overline{AC'}, \overline{CB'}$, respectively, are collinear.*

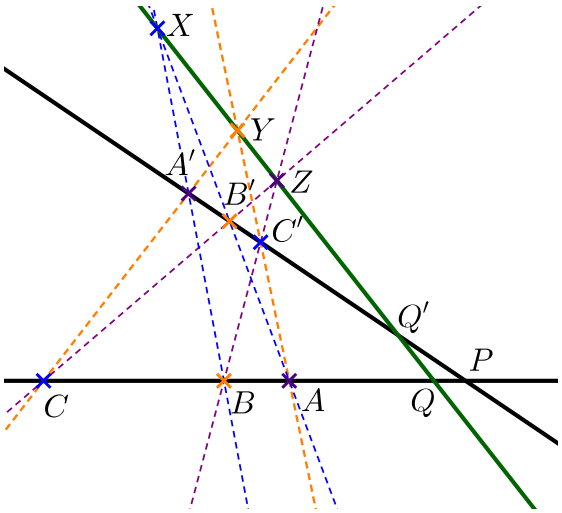



Figure 2.8: Pappus's theorem.
Step-by-Step construction: `https://www.geogebra.org/m/pehvjqpf`

*Proof.* Let $X$ be the point of intersection of the lines $\overline{AB'}$ and $\overline{BA'}$ and $Z$ be the point of intersection of $\overline{BC'}$ and $\overline{CB'}$. Moreover let $Y_1$ be the intersection of $\overline{AC'}$ and $\overline{XZ}$; $Y_2$ the intersection of $\overline{CA'}$ and $\overline{XZ}$; $P$ the intersection of $\overline{AB}$ and $\overline{A'B'}$; $Q$ the intersection of $\overline{AB}$ and $\overline{XZ}$ and $Q'$ the intersection of $\overline{A'B'}$ and $\overline{XZ}$. Now we have the following perspectivities (to demonstrate the method we omit the case when $Q = Q' = P$):

$$ABPQ \overset{(X)}{\barwedge} B'A'PQ'$$
$$ABPQ \overset{(C')}{\barwedge} Y_1ZQ'Q$$
$$B'A'PQ' \overset{(C')}{\barwedge} ZY_2QQ',$$

but from Theorem 2.2.1

$$Y_1ZQ'Q \barwedge ZY_1QQ'.$$

Therefore

$$ZY_1QQ' \barwedge ZY_2QQ'.$$

Assuming that three elements uniquely determine the fourth element of a range, we have $Y_1 = Y_2$, and so the lines $\overline{AC'}$ and $\overline{CA'}$ meet on $\overline{XZ}$. □

Other examples, where the method of projection was used, are also Lemma 1.2.1, in which ranges of points were projected between skew lines and, with the use of conics, von Staudt's constructions in Section 1.3, where we projected a range of points of a conic with its coordinate system onto a range of points on a line and a pencil of lines.

### 2.2.2 The method of selection of an improper hyperplane

The second method consists of selecting the improper line and is closely related to the model of the real projective plane in perspective drawings (Figure 1.18). If

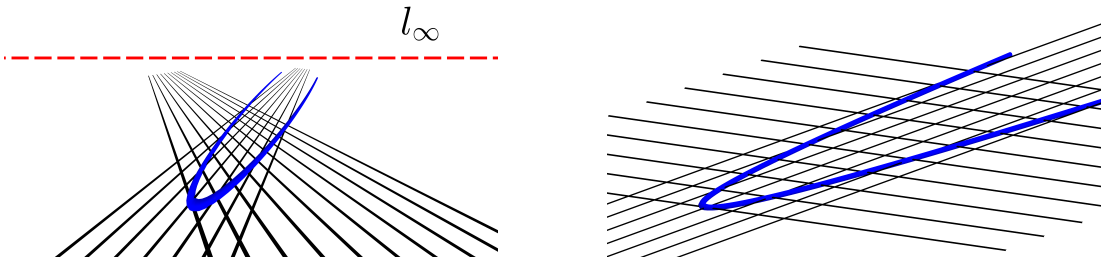



Figure 2.9: (Left) The improper line is mapped onto a finite line. (Right) The improper line is the set of intersections of parallels.

we draw a picture, which holds the "straightness" of lines, on a paper or computer screen, we consider it embedded into an affine plane (there simply exist points at infinity). To think projectively, we have these two choices: We can accept directions of parallel lines to be the infinite points and work in our affine plane of the paper (the horizontal plane in Figure 1.18). The parallelism is preserved and we only formally substitute it by an intersection at infinity. On the other hand, we perform a projective transformation, which maps the improper line onto a finite line (the vertical plane in Figure 1.18). Although the parallelism is distorted, we have the freedom of choice of the improper line. Hence in many cases, the crucial step to solve a problem is to choose the right line as improper. This way we transfer projective theorems to affine ones and vice-versa. The attention must be paid to avoid any non-projective metric notions such as angles, distances or affine ratio during the constructions (but they will obviously induce analogies). We will show, that this method leads to direct intuition mostly based on our experience in elementary geometry, and (yet algorithmic in axiomatization) previous constructions will be clarified. Figure 2.9 demonstrates the idea on two images of the same parabola.

Let us revise the addition and multiplication on a projective scale. The common interpretation (see [Richter-Gebert, 2011, pp. 89–91], [Hlavatý, 1944, pp. 20–24, 88–91]) is given on constructions on a projective line. In Figures 2.10 and 2.11 the projective constructions are on the left side, the right side is the affine analogy after the choice of the improper line.

**Construction 3.** *The construction of the sum of elements $A$ and $B$ on a projective line with the coordinate system $0, 1, \infty$ is performed as follows:*

1. *Choose an arbitrary point $\infty'$ not on the line $\overline{AB}$ and join it with $0$ and $A$ (blue lines).*

2. *Choose an arbitrary distinct line (orange) through $\infty$ and not incident with $\infty'$ or $0$, its intersection with the line $\overline{0\infty'}$ is $0'$, the intersection with the line $\overline{A\infty'}$ is $A'$.*

3. *The intersection of the join $0'$ and $B$ (green line) with $\overline{\infty\infty'} = l_\infty$ (red dashed line) is the point $\infty''$.*

4. *The intersection of the join $\infty''$ and $A'$ (green line) with the line $\overline{0\infty}$ is the sum $A + B$.*

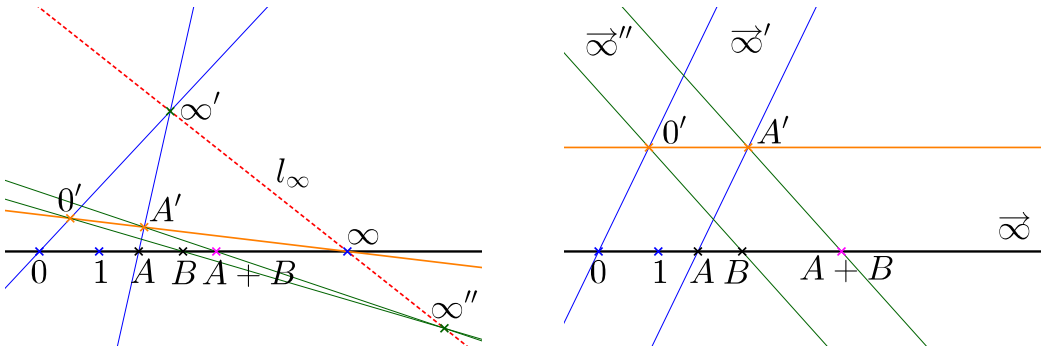



Figure 2.10: (Left) Addition on a projective scale.
Step-by-Step construction: `https://www.geogebra.org/m/t382jehb`
(Right) Addition on an affine scale.

If we consider the line $\overline{\infty'\infty}$ to be the infinite line, it is analogous to the affine situation, and the construction becomes much easier to understand. Translation of the oriented segment $0A$ to an arbitrary parallel line in the "direction" of $\infty'$ and another translation of the segment to the point $B$ in the direction $\infty''$ may be one of the interpretations. The addition is not dependent on the choice of the multiplicative unit element 1, which was not used.

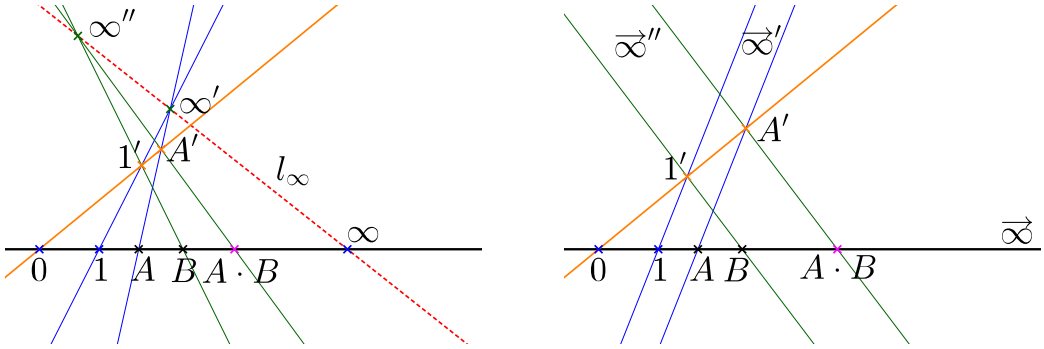



Figure 2.11: (Left) Multiplication on a projective scale.
Step-by-Step construction: `https://www.geogebra.org/m/pcucykzt`
(Right) Multiplication on an affine scale.

**Construction 4.** *The construction of the product of elements $A$ and $B$ on a projective line with the coordinate system $0, 1, \infty$ is performed as follows:*

1. *Choose an arbitrary point $\infty'$ not on the line $\overline{AB}$ and join it with 1 and $A$ (blue lines).*

2. *Choose an arbitrary distinct line (orange) through 0 and not incident with $\infty'$ or $\infty$, its intersection with the line $\overline{1\infty'}$ is $1'$, the intersection with the line $\overline{A\infty'}$ is $A'$.*

3. *The intersection of the join $1'$ and $B$ (green line) with $\overline{\infty\infty'} = l_\infty$ (red dashed line) is the point $\infty''$ .*

4. *The intersection of the join $\infty''$ and $A'$ (green line) with the line $\overline{0\infty}$ is the product $A \cdot B$.*

The Euclidean analogy is apparent from the similar triangles $0B1'$ and $0(A \cdot B)A'$, in the affine setting the affine ratios $(0, B; (A \cdot B))$ and $(0, 1'; A')$ are equal.

The importance of the construction of the harmonic conjugate $D$ of the given point $C$ with respect to a pair $A, B$ (Figure 1.7) was already discussed in Sections 1.3 and 1.4.1. With the well selected improper line (Figure 2.12) the construction becomes evident in the affine and Euclidean plane.

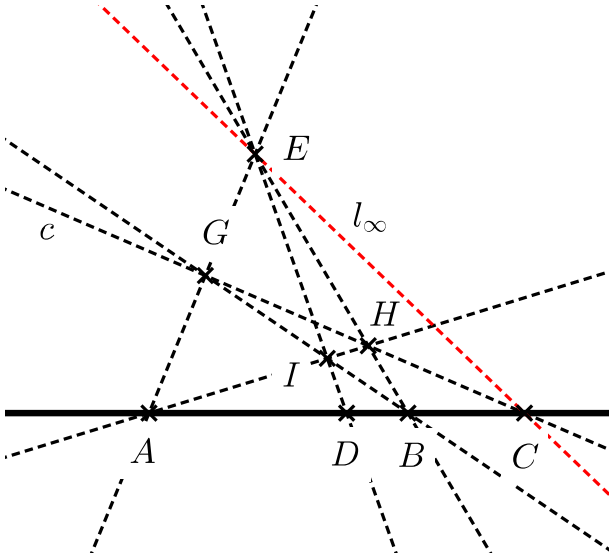



Figure 2.12: Selection of the improper line in harmonic quadruple construction.

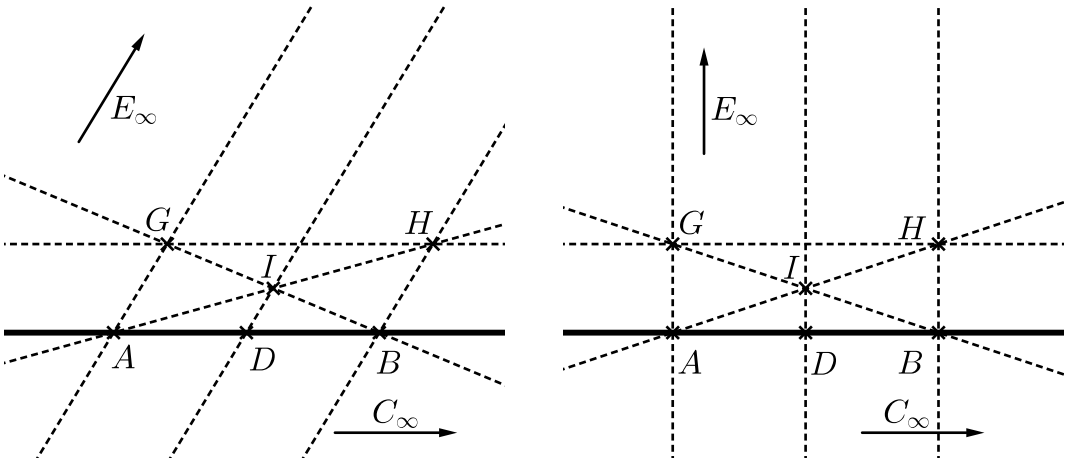



Figure 2.13: Transformations of the harmonic conjugate construction in (Left) the affine setting and (Right) the orthogonal Euclidean setting.

The whole process is an analogy to the parallelogram rule (Figure 2.13) in the affine setting or a simple property of a rectangle in the Euclidean plane.

The harmonic conjugate $D$ of the infinite point $C_\infty$ becomes the center of the segment $AB$.

The following construction is a planar generalization of the harmonic conjugate construction on a line (Construction 1). More precisely, we find the *pole of a line with respect to a triangle*. Poles and polars with respect to a triangle are described for example in [Lachlan, 1893, pp. 61–62].

**Construction 5** (Pole of line with respect to a triangle)**.** *Let us have a triangle $ABC$ and a line $l_\infty$ in its plane, the construction with the use of the three-dimensional space goes as follows (Figure 2.14):*

1. *Choose an arbitrary point $V$ not in the plane $ABC$ and join it with the vertices of the triangle.*
2. *Choose an arbitrary distinct plane $\rho$ through the line $l_\infty$ and find its intersection points $A', B'$ and $C'$ with the lines $\overline{AV}, \overline{BV}$ and $\overline{CV}$, respectively.*
3. *Find the intersection point $O$ of the planes $ABC', BCA'$ and $ACB'$.*
4. *The point $T$ (pole of the line $l_\infty$) is the intersection of the line $\overline{VO}$ and the plane $ABC$.*

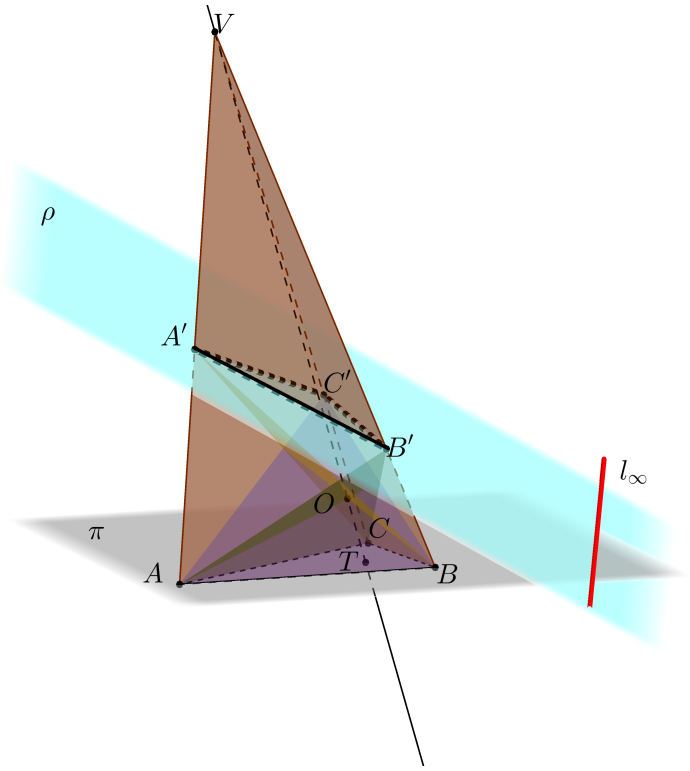



Figure 2.14: A pole and polar with respect to a triangle.
Step-by-Step construction: `https://www.geogebra.org/m/ez3tzaeq`

Again, if we transform our problem into the affine setting with $l_\infty$ being the line at infinity, the point $T$ is nothing else then the centroid of the triangle $ABC$. Figure 2.14 shows the net of the pyramid $ABCV$, in which we can notice three

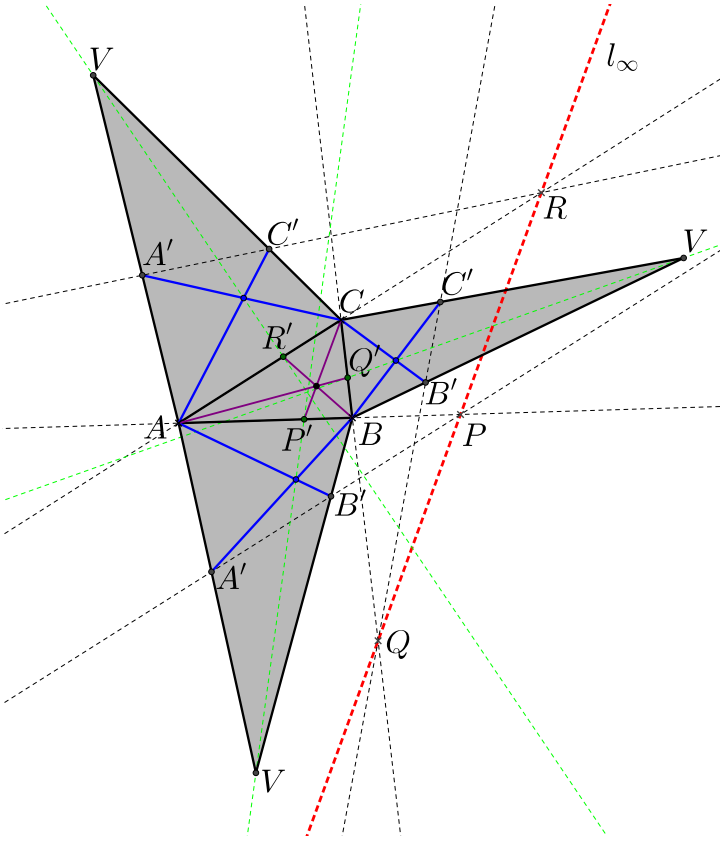



Figure 2.15: The net of the pyramid in Figure 2.14 with constructions of harmonic conjugates.
Step-by-Step construction: `https://www.geogebra.org/m/ez3tzaeq`

hidden constructions of harmonic conjugates on the sides of the triangle $ABC$. The points $P, Q$, and $R$ are the intersections of the sides $\overline{AB}, \overline{BC}$, and $\overline{CA}$ of the triangle with the line $l_\infty$. At last, the whole construction may be simplified to find the harmonic conjugates $P', Q'$, and $R'$ of the points $P, Q$, and $R$ on the improper line (the centers of sides), joining them with the opposite vertices (medians), and find the intersection point (centroid) (Figure 2.16).

We have already shown that a projectivity transforms a conic onto another conic. Now we use the perspective projection to treat conics as circles (Figure 2.17). A center of an Euclidean circle is the intersection of its diameters. A diameter can be found as the join of the tangent points of two parallel tangents, i.e. the polar of an improper pole with respect to the circle (or any regular conic). We use this thought in the following statement: for any conic and a line in the same plane there exists a central projection that maps the conic onto a circle and the line to infinity. This is Poncelet's second principal lemma in [Ostermann and Wanner, 2012, p. 325]. To show this, we use the mechanism of the central projection shown in Figure 1.18. Let a conic $\kappa$ lie in the plane $\pi$, we choose an arbitrary point $E_1$ on the line $l_\infty$, but not on $\kappa$. Let $E_0$ be the pole of the polar $l_\infty$. The point $E_0$ is a polar conjugate of the point $E_1$ and so it lies on its polar $p$. Let $E_2$ be the intersection of $l_\infty$ and $p$. The points $E_0, E_1, E_2$ form a polar triangle of the conic $\kappa$ with one inner and two outer vertices (in the figure $E_0$

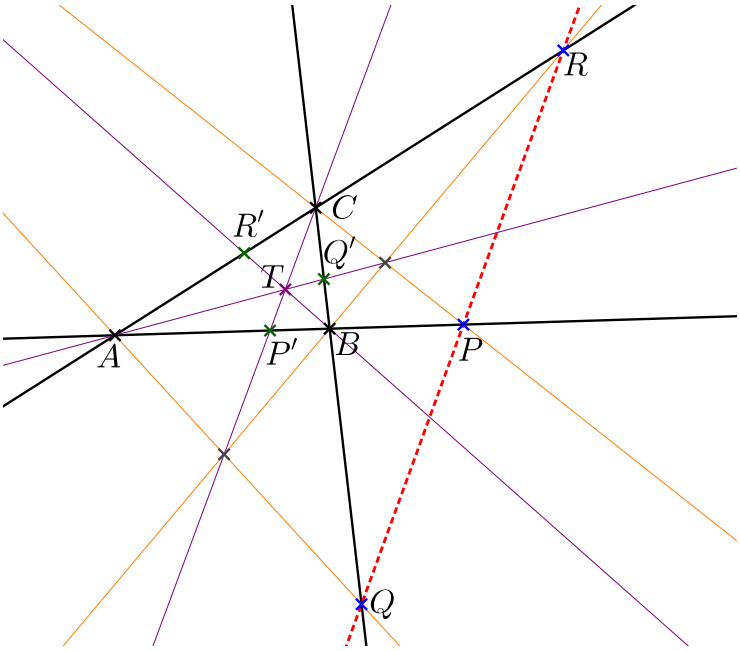



Figure 2.16: The pole $T$ and its polar $l_\infty$ with respect to the triangle $ABC$. The harmonic conjugate $P'$ of $P$ with respect to $AB$ is constructed with the choice of the line $l_\infty$ and the point $C$, which are arbitrary in Figure 1.7. Similarly for the points $Q'$ and $R'$.
Step-by-Step construction: `https://www.geogebra.org/m/shbtu5s3`

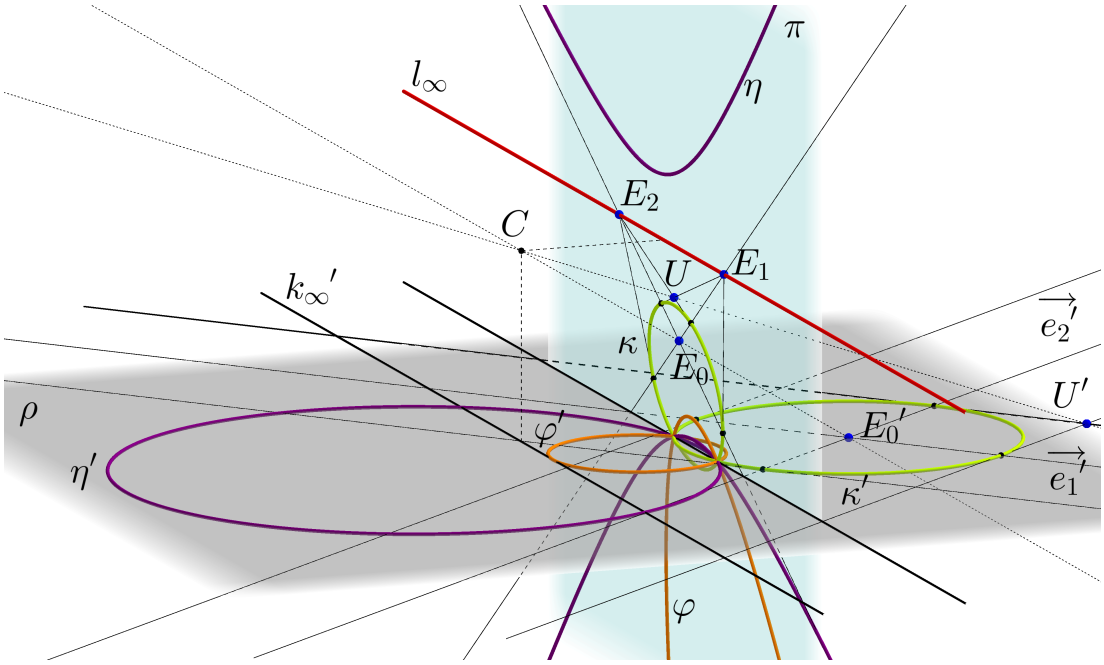



Figure 2.17: A central projection which maps conics onto circles.
Interactive model: `https://www.geogebra.org/m/zt8h5a9u`

is inner and $E_1, E_2$ are outer). Let $U$ be the intersection of the tangents through the outer vertices to $\kappa$ such that $U$ lies inside the triangle $E_0E_1E_2$. According to the construction in Figure 1.18, we can find the images of $E_0E_1E_2$ and $U$ which

form the orthogonal coordinate system and construct the unit circle with the center $O'$. Regular conics with two, one or zero intersections with the improper line are affinely classified into hyperbolas, parabolas or ellipses, respectively. In Figure 2.17, the distinction is clear from the position of the circles $\eta', \varphi'$ and $\kappa'$ and the line $k'_\infty$ (image of the infinite line of the plane $\pi$) in the image plane $\rho$.

A frequent example of the use of a circle is Pascal's theorem and its dual Brianchon's theorem [Ostermann and Wanner, 2012, pp. 327–328], both of them are often used for constructions of conics in synthetic projective geometry.

**Theorem 2.2.3** (Pascal's theorem (Figure 2.18))**.** *Let* $1, 2, 3, 4, 5, 6$ *be the vertices of a hexagon inscribed in a conic. Then the intersection points* $\overline{12} \times \overline{45}, \overline{23} \times \overline{56}$*, and* $\overline{34} \times \overline{61}$ *of pairs of opposite sides are collinear.*

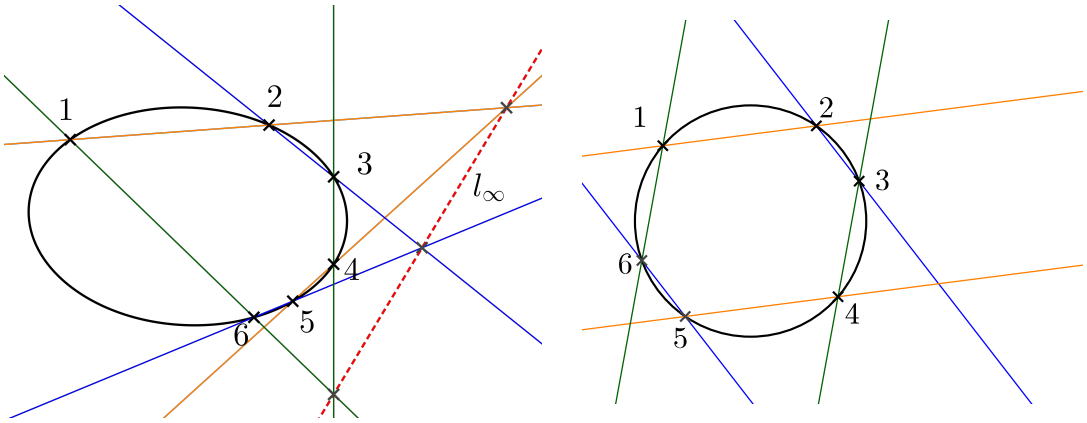



Figure 2.18: (Left) A configuration in Pascal's theorem.
Step-by-Step construction: `https://www.geogebra.org/m/qffpvf5n`
(Right) A transfer to a circle.

*Proof.* Let us have two intersections of sides, e.g. $\overline{12} \times 45$ and $\overline{23} \times \overline{56}$, their joining line is $l_\infty$. We want to prove that the last intersection $\overline{34} \times 61$ lies also on $l_\infty$. We choose $l_\infty$ to be the improper line and perform a transformation of the conic onto a circle. The formulation of this special case of the theorem is restated in Euclidean geometry for the circle with the parallelism to prove that if $\overline{12} \| \overline{45}$ and $\overline{23} \| \overline{56}$, then also $\overline{34} \| \overline{61}$. The four vertices 1245 become the vertices of an inscribed parallelogram and the angles $\angle 234$ and $\angle 561$ are inscribed to the chords 24 and 51 of the same length. Therefore $\angle 234 = \angle 561$, and the sides $\overline{34}$ and $\overline{16}$ are also parallel. One could argue that the inner proof uses Euclidean measures, but the final property stays projective (including the transfer of the improper line). □

**Theorem 2.2.4** (Brianchon's theorem (Figure 2.19))**.** *Let* $1, 2, 3, 4, 5, 6$ *be the sides of a hexagon circumscribing a conic. Then the joins of* $12 \times 34, 23 \times 56$ *and* $34 \times 61$ *of pairs of opposite vertices are concurrent.*

*Proof.* Since it is the dual statement of Pascal's theorem, it holds. To demonstrate the method, after the same transformation and argument as in the proof of Pascal's theorem (using the tangent points), we obtain circumscribed parallelograms with the sides $1245, 2356$, and $3461$. Their diagonals meet in the center $C$ of the circle. □

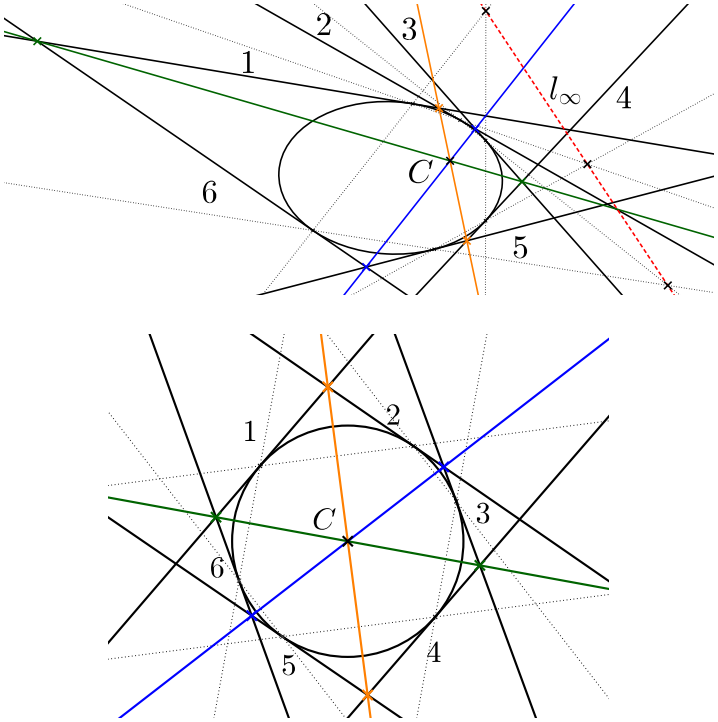



Figure 2.19: (Up) A configuration in Brianchon's theorem.
Step-by-Step construction: `https://www.geogebra.org/m/b8qeqy56`
(Down) A transfer to a circle.

A noteworthy remark is that we can look at Pappus's theorem as a degenerate case of Pascal's theorem. Choosing the line through $X, Y, Z$ to be improper, we obtain a simple theorem in Euclidean geometry (Figure 2.20). On the other hand, this also shows how easily theorems in Euclidean geometry can be obtained as special cases of projective theorems. A comprehensive discussion about projective and Euclidean proofs of Pappus's theorem and also an alternative purely projective proof of Pascal's theorem are in [Richter-Gebert, 2011, pp. 3–31, 183–185].

### Applications in affine and Euclidean geometry

Not only can we use the intuition of affine and Euclidean geometry to solve projective problems, but conversely we can use general projective thinking to solve the problems of the other geometries.

The construction of the harmonic conjugate can be effectively used for solving the following property of a trapezoid, which is usually proven in elementary geometry with the use of areas of triangles or homothetic triangles.

**Theorem 2.2.5** (Figure 2.21)**.** *Let $ABHG$ be a trapezoid, where $\overline{AB} \| \overline{HG}$ and $E$ be the intersection point of $\overline{AG}$ and $\overline{BF}$. The intersection point $I$ of the diagonals $\overline{AH}$ and $\overline{BG}$ lies on the median of the triangle $ABE$ through $E$.*

*Proof.* Instead of the standard argumentation, we conveniently transfer the theorem into the projective extension of the real space. The trapezoid changes into the quadrangle $ABHG$.

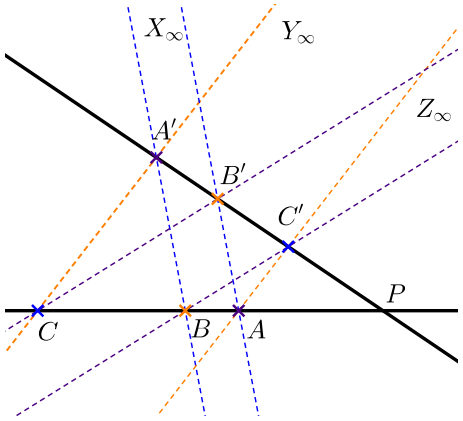



Figure 2.20: A configuration of Pappus's theorem for $P$ not on the line through $X, Y$ and $Z$ deforms into a statement for parallel lines, which can be easily proven with the use of affine ratios or similar triangles.
Step-by-Step construction: `https://www.geogebra.org/m/gqvdrbmz`

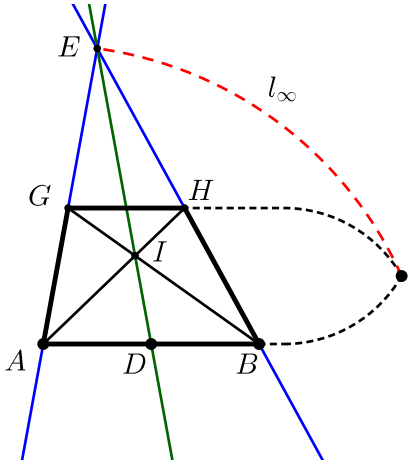



Figure 2.21: A property of a trapezoid. To keep the parallelism, the improper line is depicted as a curved line.

Let $C$ be the intersection of the parallel sides $\overline{AB}$ and $\overline{HG}$. By naming $\overline{CE}$ the infinite line, we obtain the construction of a harmonic conjugate, where $A, B, C$, and $D$ form a harmonic quadruple (see Figure 2.12). Hence the point $I$ lies on the join of $E$ and the midpoint of $AB$. □

In the next theorem, we use projective reasoning in the following Euclidean property of a parabola. Its similarity with the power of a point with respect to a circle is discussed (in terms of the analytic method) to show the relationship between Euclidean and Galilean circles, which appear as parabolas, in [Richter-Gebert, 2011, pp. 463–464]. The analytic formulation and proof is in Section 2.3, Theorem 2.3.1.

**Theorem 2.2.6.** *Let us have a parabola given by its focus $F$ and directrix $d$. Let $P$ be an arbitrary point and $a$ be a secant of the parabola through $P$, which intersects it in points $A$ and $B$. The parallels with the axis $o$ through the points*

*$A, B$, and $P$ intersect the directrix $d$ in points $A', B'$, and $P'$, respectively. The product $|P'A'||P'B'|$ is constant for any position of the secant $a$.*

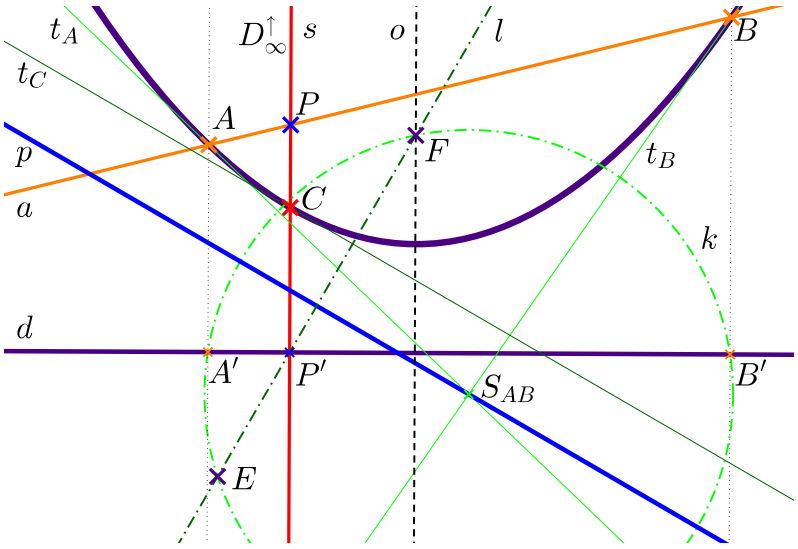



Figure 2.22: The property of a parabola.
Interactive model: `https://www.geogebra.org/m/pfq8egep`

*Proof.* At start, we need the following polar property of conics: the tangents $t_A$ and $t_B$ to the conic through points $A$ and $B$ intersect in the point $S_{AB}$, which lies on the polar $p$ of the point $P$ with respect to the conic. The point $S_{AB}$ is the pole of the secant $a$, and the point $P$ lies on the secant $a$, and hence $P$ and $S_{AB}$ are polar conjugates, and the polar $p$ of the point $P$ with respect to the conic passes through $S_{AB}$ (a complex non-projective argument could be performed with the use of a collineation of the parabola onto a circle, inscribed quadrilaterals, circle inversion and Pythagoras's theorem).

From the focal properties of a parabola, the tangent $t_A$ is the axis of the segment $A'F$ and also $t_B$ is the axis of $B'F$. Thus, the points $A', B'$, and $F$ lie on the circle $k$ with the center $S_{AB}$. If $P$ is an outer point of the parabola, we can use the real tangents through $P$ for the rest of the proof. In a general case, let $s$ be the secant of parabola through $P$ parallel to the axis. The secant $s$ intersects the parabola in one proper point $C$ and one improper point $D$. The images of the points $C$ and $D$ in the parallel projection in the direction of the axis $o$ onto the directrix $d$ merge into the point $P'$. The tangents of the points $C$ and $D$ intersect in the improper point of the polar $p$, and therefore the circle through the double point $P'$ and the focus $F$ with the improper center appears as the line $l = \overline{P'F}$ perpendicular to the polar $p$.

The line $l$ and the circle $k$ have the common point $F$ and the second intersection is labeled $E$. The point $E$ is also the image of the point $F$ in the reflection about the polar $p$. Thus, the points $E, F$, and $P'$ are collinear, and from the power of the point $P$ with respect to the circle $k$, we have $|P'A'||P'B'| = |P'F||P'E|$, while the right side is independent of the choice of the points $A$ and $B$. □

The following problem formulated for circles in a plane is called Monge's theorem. Primarily, the task was to construct a tangent plane of three spheres [Monge,

1799, pp. 53–54], and Monge's solution used tangent cones. This stunning proof with the use of an improper line, or horizon, in perspective is assigned to Martin Rebas in [Bogomolny, Monge's Theorem of three circles and common tangents] (where also alternative proofs are to be found).

**Theorem 2.2.7** (Monge's theorem)**.** *Let $c_1, c_2$ and $c_3$ be three circles of different radii and non-collinear centers lying completely outside each other. Then six external tangents to two of the three circles, taken pairwise, intersect at three collinear points.*

*Proof.* The planar formulation, including inner tangents, is depicted in Figure 2.23.

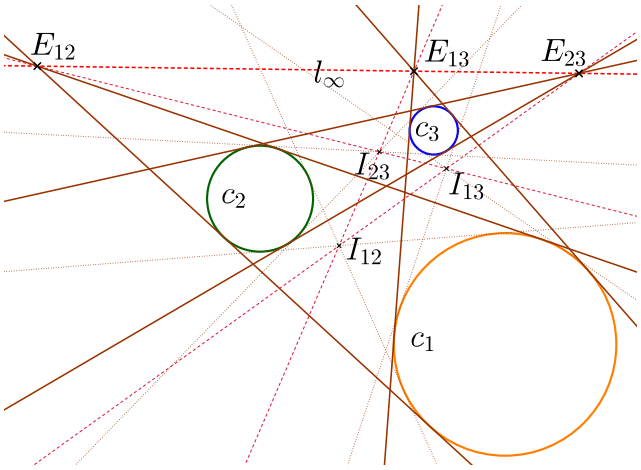



Figure 2.23: Monge's theorem. The points $E_{12}, E_{13}, E_{23}$ are respective intersections of the external tangents. The inner tangents intersect in the points $I_{12}, I_{23}, I_{13}$, from which two are always collinear with one external intersection. Step-by-Step construction: `https://www.geogebra.org/m/uw8qjssc`

Figure 2.24 may be explained as a perspective view on three spheres with a common radius. The apparent contour of a sphere in the linear perspective is a circle (tangent circle of the projecting cone). Now, each two of the three spheres have a common tangent cylinder, which shrinks into a point on a horizon. Hence the horizon is the common line we were looking for.

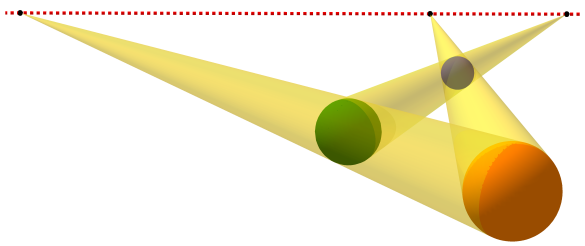


Figure 2.24: A 3D approach to Monge's theorem.

□

## 2.3 Mixing analytic and synthetic methods

### 2.3.1 Analytic methods

Computations with joins and meets, determinants, cross-ratio, linear and quadratic forms ... in homogenous coordinates is what we mean by the analytic method of projective geometry.

For example, Pappus's theroem 2.2.2 could be analytically rewritten as the following exercise in linear algebra:

*Example.* If $[ABC] = 0$ and $[A'B'C'] = 0$ and $A' \times B' \neq k(A \times B)$ for $k \in \mathbb{R} \setminus \{0\}$, show that

$$[(A \times B') \times (A' \times B), (B \times C') \times (B' \times C), (A \times C') \times (A' \times C)] = [XYZ] = 0.$$

The next revision of Theorem 2.2.6 shows an example of the convenient use of the analytic method.

**Theorem 2.3.1.** *Let us have a parabola in $\mathbb{RP}^2$ with the equation $x_1^2 - x_2x_0 = 0$. A secant of the parabola through a proper point $P(p_x, p_y, 1)$ intersects the parabola in two distinct proper points $A(a, a^2, 1)$ a $B(b, b^2, 1)$ for $a, b \in \mathbb{R}$. The product $(p_x - a)(p_x - b)$ is constant for any choice of the secant.*

*Proof.* In other words we want to prove that the given product is independent of the position of the points $A$ and $B$.

The points $A, B$, and $P$ are collinear. Therefore

$$[ABP] = \det \begin{pmatrix} a & b & p_x \\ a^2 & b^2 & p_y \\ 1 & 1 & 1 \end{pmatrix} = (b - a)(-p_x(a + b) + p_y + ab) = 0.$$

The points $A$ and $B$ are distinct hence $(a - b) \neq 0$, and

$$-p_x(a + b) + p_y + ab = 0.$$

Subtracting the right side from the wanted product we obtain

$$(p_x - a)(p_x - b) - 0 = p_x^2 - p_x(a + b) + ab - (-p_x(a + b) + p_y + ab) = p_x^2 - p_y.$$

We have shown that the product $(p_x - a)(p_x - b)$ is independent of the coordinates $a$ a $b$ and also the points $A$ and $B$. □

To return to the synthetic formulation, it is easy to show that all parabolas are similar (they only depend on the distance between the focus and directrix), and by an isometry we can place the vertex of our parabola into the origin and the axis of parabola onto the $y$-axis. Hence we can, without loss of generality, use the assumption of the specific equation of a parabola.

As a corollary, if we choose the coordinate $p_x = 0$, the point $P$ lies on the $y$-axis in the Euclidean setting, and the product $ab = -p_y$. Thus, the joins of points $A$ and $B$ with coordinates $A(-a, a^2, 1)$ and $B(b, b^2, 1)$ for $1 < a, b \in \mathbb{Z}$ intersect the $y$-axis in all positive composite numbers. This construction of the prime sieve was found by Matijasevich [1971] and the idea of visualization by Boris Stechkin (Figure 2.25).

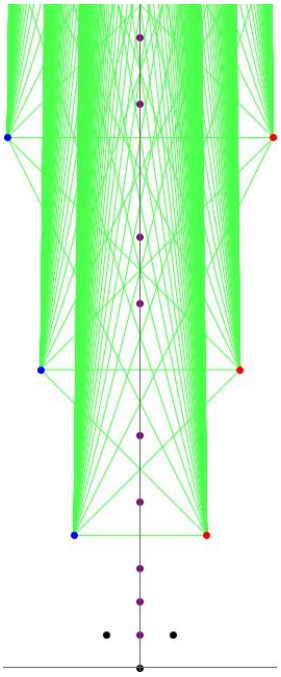


Figure 2.25: Matijasevich-Stechkin's parabola. Points $A$ are on the left side, points $B$ on the right side. The purple points correspond to prime numbers.

### 2.3.2 Mixed methods

The following two theorems use the affine ratio (or the ratio of oriented lengths), which is invariant in affine geometry. Let us see their projective generalizations when we choose an improper line and their proofs with the use of the synthetic method of projection and the analytic expression of the cross-ratio with determinants. A different approach to generalization via oriented areas of triangles is in [Richter-Gebert, 2011, pp. 277–282].

**Theorem 2.3.2** (Menelaus's theorem, affine)**.** *If in a triangle $ABC$ with points $K \in \overline{BC}, L \in \overline{CA}$, and $M \in \overline{AB}$, the points $K, L, M$ are collinear, then we have*

$$(A, B; M) \cdot (B, C; K) \cdot (C, A; L) = 1.$$

In Section 1.4.3 the relation of the cross-ratio and the affine ratio if we work with proper points were explained. Hence if we choose an arbitrary line $l_\infty$, which does not pass through the given points $A, B, C, K, L, M$, we can generalize the theorem with respect to this line.

**Theorem 2.3.3** (Menelaus's theorem, projective, Figure 2.26)**.** *Let us have a triangle $ABC$ with points $K \in \overline{BC}, L \in \overline{CA}$, and $M \in \overline{AB}$ and a line $l_\infty$ such that $A, B, C, K, L, M \notin l_\infty$. The points $X', Y'$, and $Z'$ are the intersecting points of the sides $\overline{AB}, \overline{BC}$, and $\overline{CA}$ with $l_\infty$, respectively. If the points $K, L, M$ are collinear, then we have*

$$(A, B; M, X) \cdot (B, C; K, Y) \cdot (C, A; L, Z) = 1.$$

*Proof.* Let $W$ be the point of intersection of $\overline{KL}$ and $l_\infty$, and $D$ be the point of

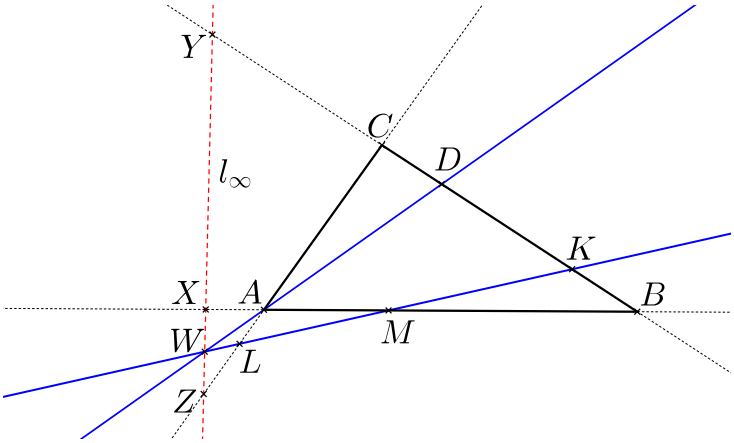



Figure 2.26: Menelaus's theorem.

intersection of $\overline{AW}$ and $\overline{BC}$. We have the following perspectivities

$$ABMX \overset{(W)}{\barwedge} DBKY$$
$$CALZ \overset{(W)}{\barwedge} CDKY,$$

hence

$$\begin{aligned}(A,B;M,X)\cdot(B,C;K,Y)\cdot(C,A;L,Z) &= \\ (D,B;K,Y)\cdot(B,C;K,Y)\cdot(C,D;K,Y) &= \\ \frac{[DK][BY]}{[BK][DY]}\,\frac{[BK][CY]}{[CK][BY]}\,\frac{[CK][DY]}{[DK][CY]} &= 1.\end{aligned}$$

□

The dual of Menelaus's theorem is Ceva's theorem.

**Theorem 2.3.4** (Ceva's theorem, affine)**.** *If in a triangle $ABC$ with points $K \in \overline{BC}, L \in \overline{CA}$, and $M \in \overline{AB}$, the lines $\overline{AK}, \overline{BL}, \overline{CM}$ are concurrent, then we have*

$$(A,B;M)\cdot(B,C;K)\cdot(C,A;L) = -1.$$

The projective interpretation with respect to a line is:

**Theorem 2.3.5** (Ceva's theorem, projective, Figure 2.27)**.** *Let us have a triangle $ABC$ with points $K \in \overline{BC}, L \in \overline{CA}$, and $M \in \overline{AB}$ and a line $l_\infty$ such that $A, B, C, K, L, M \notin l_\infty$. The points $X', Y'$, and $Z'$ are the intersecting points of the sides $\overline{AB}, \overline{BC}$, and $\overline{CA}$ with $l_\infty$, respectively. If the lines $\overline{AK}, \overline{BL}, \overline{CM}$ are concurrent, then we have*

$$(A,B;M,X)\cdot(B,C;K,Y)\cdot(C,A;L,Z) = -1.$$

*Proof.* Let $S$ be the common point of $\overline{AK}, \overline{BL}, \overline{CM}$. The intersection of $\overline{AC}$ and $\overline{SX}$ is the point $P$, and the intersection of $\overline{AC}$ and $\overline{SY}$ is the point $Q$. The point

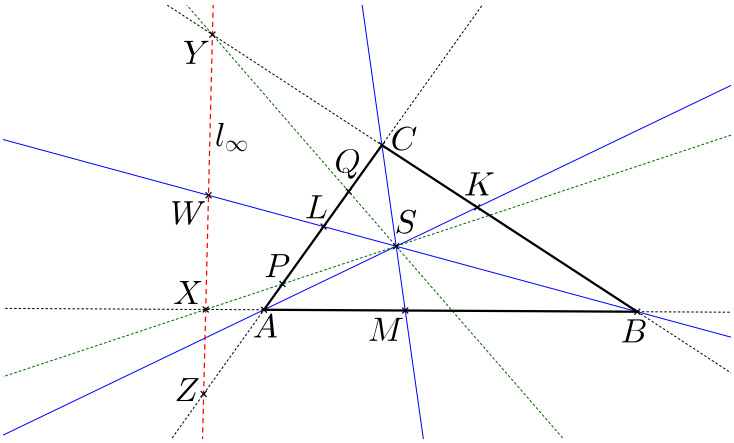



Figure 2.27: Ceva's theorem.

$S$ lies at most on two of the sides $\overline{AB}, \overline{BC}, \overline{AC}$. Without loss of generality, let it not lie on $\overline{AC}$. We have the following perspectivities

$$ABMX \overset{(S)}{\barwedge} ALCP$$
$$BCKY \overset{(S)}{\barwedge} LCAQ.$$

Hence

$$(A,B;M,X)\cdot(B,C;K,Y)\cdot(C,A;L,Z) = (A,L;C,P)\cdot(L,C;A.Q)\cdot(C,A;L,Z).$$

After the expansion, we multiply it by one

$$\frac{[AC][LP]}{[LC][AP]}\frac{[LA][CQ]}{[CA][LQ]}\frac{[CL][AZ]}{[AL][CZ]}\left(\frac{[LZ]}{[LZ]}\right) = -\frac{[LP][AZ]}{[AP][LZ]}\frac{[CQ][LZ]}{[LQ][CZ]} = -\frac{(L,A;P,Z)}{(L,C;Q,Z)},$$

and since

$$LAPZ \overset{(X)}{\barwedge} LBSW$$
$$LCQZ \overset{(Y)}{\barwedge} LBSW,$$

we have

$$-\frac{(L,A;P,Z)}{(L,C;Q,Z)} = -\frac{(L,B;S,W)}{(L,B;S,W)} = -1.$$

□

The synthetic constructions enriched with the analytic method or vice-versa can be well used for demonstrations in computer graphics. Let us show how general a projective coordinate system and its element can be in von Staudt's

constructions from Figures 2.10 and 2.11 when, instead of one point, we use a set of points.

The following models are created in *Wolfram Mathematica 11* (the full code is in Attachment A.5). A predefined point model of a 3D cow in the Euclidean coordinates is in Figure 2.28. The set of its proper points is represented by a matrix $\mathbf{C}_0$, to which we append the last coordinate 1 (Figure 2.29), and we label the cow as 0-cow.

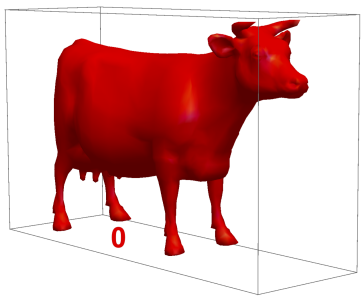



Figure 2.28: A 3D model of a 0-cow.

```
{{0.107994, 0.0846667, -0.218938, 1}, {0.11579, 0.0886993, -0.228472, 1}, {0.107198, 0.0955648, -0.232789, 1},
  {0.12329, 0.0940333, -0.241455, 1}, ··· 2895 ···, {-0.264289, -0.0454424, -0.129848, 1},
  {-0.259885, -0.043327, -0.131191, 1}, {-0.222232, -0.0354683, -0.135835, 1}, {-0.265862, -0.0401731, -0.136741, 1}}[1]
large output | show less | show more | show all | set size limit...
```

Figure 2.29: The matrix of points of the 0-cow in the homogenous coordinates.

Now we perform a projective collineation with the associated matrix $\mathbf{P}$

$$\mathbf{P} = \begin{pmatrix} 2 & 0 & 0 & 5 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

We call the image of the 0-cow the $\infty$-cow with the matrix $\mathbf{C}_\infty = \mathbf{P}\mathbf{C}_0$ (Figure 2.30). Now we create a 1-cow as a linear combination of the 0-cow and $\infty$-

```
{{4.7076, 4.66549, -0.395197, 1.}, {4.68868, 4.64012, -0.409525, 1.}, {4.70954, 4.68853, -0.420501, 1.},
  {4.67073, 4.61864, -0.429907, 1.}, {4.68747, 4.65321, -0.438676, 1.}, ··· 2894 ···, {6.07769, 6.67261, -0.352986, 1.},
  {6.05342, 6.63863, -0.354515, 1.}, {5.85719, 6.33745, -0.349294, 1.}, {6.08643, 6.70127, -0.372521, 1.}}[1]
large output | show less | show more | show all | set size limit...
```

Figure 2.30: The matrix of points of the $\infty$-cow in the homogenous coordinates.

cow. The 1-cow is represented by a matrix $\mathbf{C}_1$

$$\mathbf{C}_1 = k_1\mathbf{C}_\infty + k_0\mathbf{C}_0, (k_1, k_0) \in \mathbb{R} \setminus \{(0,0)\}.$$




Figure 2.31: A projective coordinate system 0,1,∞ and the element with the coordinate $k = 0.5$.

We only visualize finite points, and hence we may assume $k_0 \neq 0$. Therefore we substitute $k = \frac{k_1}{k_0}$, and each real number $k$ represents one cow of the system with the projective coordinates $(k, 1)$. For $k = 1$, we obtain the 1-cow.[11] The projective coordinate system with the elements 0-cow, 1-cow and $\infty$-cow is visualized in Figure 2.31.

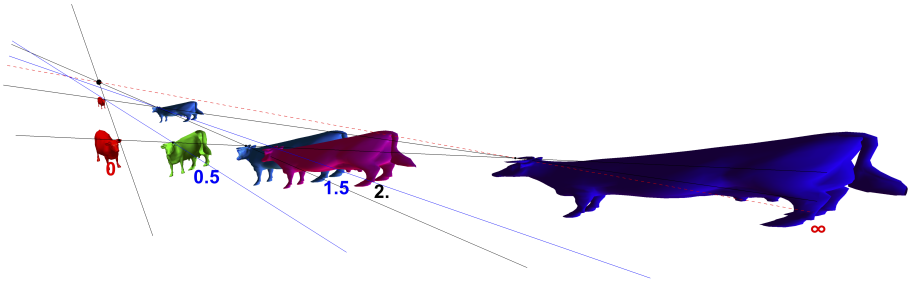



Figure 2.32: The sum of cows corresponding to the numbers 0.5 and 1.5.

Algebraic operations can be synthetically reconstructed (instead of points, we construct cows) according to Constructions 3 (Figure 2.32) and 4 (Figure 2.33). Step-by-step animated constructions of the subtraction and division are to be found in the attachment.

[11] In the code, for the sake of visualization, we choose the unit element with coordinates $(\frac{1}{3}, 1)$ and rescale it with another coefficient.

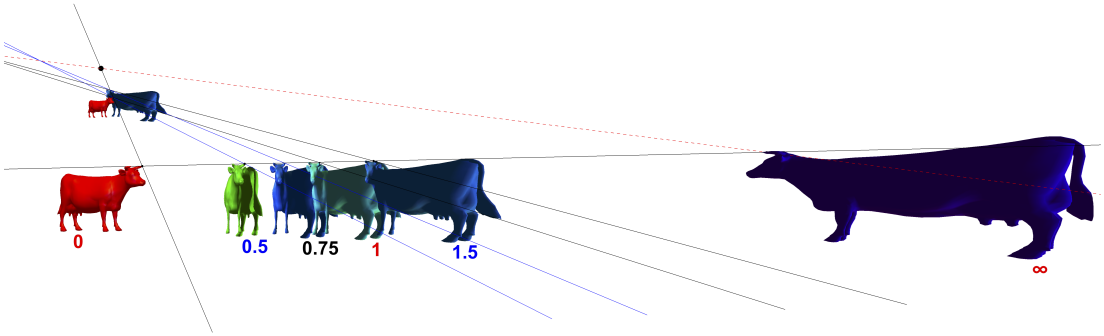



Figure 2.33: The product of cows corresponding to the numbers 0.5 and 1.5.

# 3. A method of visualization of the four-dimensional space

## 3.1 Introduction

On the following lines we introduce a visual method of examination of the four-dimensional space. Our goal is to develop instructive synthetic techniques with the use of descriptive geometry and interactive graphics to improve our spatial perception of four-dimensional objects. Afterwards, the given method of visualization is used for the synthetic classification of regular quadrics as sections of four-dimensional cones in Section 4.3.

Understanding of the idea of space and its dimensions has lain in the interest of philosophers, scientists, and artists through the ages. Four-dimensional and multidimensional geometry has evolved from being completely absent for the ancient world to acceptance in synthetic and analytic geometry in the golden era of geometry in the 19th century. General understanding of higher dimensions, based on analogies of the one, two, and three-dimensional space, was enhanced by Hinton's essay What is the fourth dimension? in [Hinton and Rucker , ed.], his later popularizing textbook [Hinton, 1904], and provoked by Abbott's masterpiece novel — Flatland [Abbott and Jann , ed.]. Especially Abbott's book was the author's greatest inspiration for the following text. A brief history of multidimensional geometry and, more importantly, synthetically built four-dimensional geometry can be found in [Manning, 1956]. Though this chapter presents a synthetic method of visualization, the analytical point of view of linear geometry is essential for verifications. The analytical approach and differential geometry of four-dimensional space is described in [Forsyth, 1930]. A comprehensive and valuable piece on visualization of points, lines and planes of the four-dimensional space onto planes is in [Lindgren and Slaby, 1968]. Short notions of four-dimensional descriptive geometry are also in [Kruppa, 1936], [Hohenberg and Tschupik, 1960, pp. 314–317]. Stachel [1990] proves the analogy of right angle theorem in the orthogonal projection of the four-dimensional space onto a plane. The principle of projections onto planes may be easily generalized for higher dimensions. For example, Weiss [1997] also visualizes $n$-dimensional Euclidean space onto a plane. On the other hand, the more dimensions we want to project, the more projecting planes we need to distinguish the visualized object uniquely. Consequently more copies of the same point in different projections are obtained. It seems to be unbearable to carry three or more such images of complex objects such as polychora[12] in mind and perform operations with them. To mentally reconstruct an original image, we would like to have the least number of its projections with as much information as possible. Thus, our analogical idea is to project the four-dimensional space onto two three-dimensional spaces.[13]

To recognize the position and visual properties of an object in the four-dimensional space, methods similar to those in three dimensions can be used.

[12] A polychoron is a four-dimensional polytope, generalization of a polyhedra.

[13] In this sense, for a visualization of a space of $n$-th dimension we need two $(n-1)$-dimensional spaces, which seems to be a dead end ..., at least for now.

To comprehend spatial properties of a three-dimensional solid drawn on a two-dimensional paper, different strategies may be chosen. Various projections give us some basic information about the solid. However, if we also draw projections of intersections with planes or the shadow cast by the solid, we have much better understanding of its spatiality. In four dimensions we proceed analogically and construct visualizations of solids, their sections, and shadows, up to one small difference — we cannot really imagine them as a whole.

Four-dimensional solids, such as notoriously known tesseract, are often depicted in central or orthogonal projections onto a plane and have become a part of common mathematical knowledge. On evolution of models of four-dimensional objects see [Volkert, 2017]. Orthogonal projections are used in [Miyazaki, 1988], where a rolling four-dimensional die is visualized. In contrast to parallel projections, in [Séquin, 2002] physical three-dimensional images of a regular polychora in a perspective projection are constructed. The properties of polychora are derived as natural analogies to the three-dimensional polyhedra. A famous treatise on regular polytopes as inductive dimensional analogies is in [Coxeter, 1973, Chapter VII]. Nowadays, various models of polychora, in both parallel and central projections, may also be found all over the Internet. A very good example with an extensive encyclopedia is [eus]. Interesting method of a distorted visualization of four-dimensional objects on an example of a tesseract is proposed in [Casas, 1984]. Casas developed a polar perspective — method of visualization of the four-dimensional space created by continuous compositions of perspective projections onto spheres, which are later flattened. Worth mentioning is also the investigation of polychora after decomposition into their three-dimensional net. A popular depiction of such a net is Dalí's Crucifixion (Corpus Hypercubus) (see [Kemp, 1998]). Methods of visualization of four-dimensional surfaces with computer graphics are in [Zachariáš and Velichová, 2000] and incorporating the use of offset curves in [Hoffmann and Zhou, 1991, Zhou, 1991].

Abbott in his Flatland [Abbott and Jann , ed.] describes the process when a three-dimensional sphere appears as its circle sections in the flat two-dimensional world. Several approaches of visualizing cuts of four-dimensional objects by three-dimensional spaces have been developed in computer graphics. A regular pentachoron is also the subject of investigation in [Kageyama, 2016]. The pentachoron is given by analytic coordinates of its vertices, while multiple slices with parallel hyperplanes are computed and visualized in perspective projection, and then they are placed on a parabolic curve. A similar process is applied to rotated positions of the polychoron for even better performance. Kageyama also gives extensive literature on methods of investigation of four-dimensional objects in computer vision. A visualization of series of three-dimensional slices of time-varying data, considered to be a four-dimensional data field is shown in [Woodring et al., 2003]. All these computer graphics methods use analytic representations and computations. In this thesis, we proceed synthetically. Since we can properly visualize an object and its cutting hyperplane in the double orthogonal projection, we construct images of both — the object and its section.

There is no better motivation to investigate shadows of "the upper world" then Plato's Allegory of the Cave: *"He will require to get accustomed to the sight of the upper world. And first he will see the shadows best, next the reflection of men and other objects in the water, and then the objects themselves; next he*

*will gaze upon the light of the moon and the stars; and he will see the sky and the stars by night, better then the sun, or the light of the sun, by day?"* [Plato et al., 1897, p. 320]. Numerical descriptions and projections of shadows of four-dimensional regular polytopes in special positions are described in [Chilton, 1971, 1980]. An exquisite interpretation of models of three-dimensional shadows and projections of a four-dimensional hypercube is in [Segerman, 2016, Chapter 3]. [Hanson and Heng, 1991] conclude their work on the four-dimensional lighting with a provocative question: *"Can humans acquire facility with four dimensions with enough practice?"* Although we cannot answer the question (we simply hope so), we add another training method. We derive visualizing techniques of constructions of shadows in parallel and central lightings as the analogy to classical methods of three-dimensional descriptive geometry.

With such an amount of information about the fourth dimension, and two hundred years after Gaspard Monge, we use his method — Monge's projection, described in [Monge, 1799]. Not only do we depict four-dimensional objects, but also, with the possibilities of interactive 3D software (e.g. *GeoGebra 5*), we can manipulate them and study them empirically without any analytic representation. Later on, we use the notation 3-space or 4-space for three or four-dimensional real space.

Monge's projection consists of a double orthogonal projection of the 3-space (with the system of orthogonal axes $x, y, z$) onto two orthogonal planes — $\pi(x, y)$, $\nu(x, z)$. One of the planes is chosen to be the drawing plane (a paper, computer screen), and the second plane is rotated around their intersecting line — the $x$-axis, called the *reference line*, onto the drawing plane such that positive values of the $y$-axis and $z$-axis have opposite orientation (usually $y$ downwards, $z$ upwards). Thus each object has two conjugated images (horizontal and vertical projection), and the conjugated points lie on the *line of recall* or *ordinal line*, which is perpendicular to the reference line in the drawing plane. The conjugated images in planes are affine projections of the original object in the 3-space. Basic constructions in Monge's projection can be found in any textbook on descriptive geometry or technical drawing.

Lately, an analogy of Monge's projection has been well described in [Şerbănoiu and Şerbănoiu, 2017] for projecting four-dimensional objects onto planes or six three-dimensional spaces. We use the same methods of descriptive geometry in analogy of Monge's projection of a four-dimensional space given by the system of orthogonal axes $x, y, z, w$, but instead of making visualizations in many projections, we use (in general cases) only projections onto two orthogonal 3-spaces $\Xi(x, y, z)$, $\Omega(x, y, w)$. Again, one of the 3-spaces is chosen to be the *modeling 3-space*, and the second is rotated around their intersecting plane[14] — $\pi(x, y)$, called the *reference plane*, onto the modeling 3-space such that positive values of the $z$-axis and $w$-axis have the opposite orientation (we use $z$ downwards, $w$ upwards). Analogically each object has two conjugated images, and the conjugated points lie on the line of recall, which is perpendicular to the reference plane. The conjugated images in the 3-spaces $\Xi(x, y, z)$, $\Omega(x, y, w)$ are affine projections of the original object in the 4-space. Since we represent our geometric 3-space onto

[14]This step has no real visualization for us. Imagine a two dimensional creature trying to rotate something around a line. It is impossible for the creature to do the rotation, but it is still possible to draw the rotated result in its two dimensional plane.

a two-dimensional paper, we must certainly be aware of consequences of such a projection. For simplicity's sake, we assume that we have three-dimensional tools for measuring angles and distances, and we can make three-dimensional transformations in the modeling 3-space.[15] Clearly, we can project our modeling 3-space with any type of parallel (as in this text) or even central projection, both of which give us a realistic picture of three-dimensional projections of the original object, and we can also perform planar constructions to describe its spatial properties.

The structure of the chapter is as follows. We start with incidence properties of points, lines, planes, and 3-spaces, and their special cases of perpendicularity. Metric properties are discussed afterwards. Then, we use the described constructions to build a tesseract and pentachoron, and show constructions of section and shadow of a cubic pyramid. In the end, with the model of a pentachoron, we construct the pole of an arbitrary polar plane with respect to a three-dimensional tetrahedron with the use of four-dimensional methods. Although we tried to keep the structure methodical, we would like to put a different approach forward. For example, it is easier for us to imagine a cube in the 3-space than infinite planes in special positions. Therefore we suggest that the kind reader investigates the tesseract from Section 3.4.2 and the attached GeoGebra model at first. It may also be essential to follow consecutive properties of lines, planes, and 3-spaces on the tesseract model.

## 3.2 Projecting points, lines, planes, 3-spaces

A point $A[x_A, y_A, z_A, w_A]$ in 4-space is represented by two orthogonal projections (Figure 3.1). The point $A_3[x_A, y_A, z_A, 0]$, the $\Xi$-*image*, is the orthogonal projection of $A$ onto $\Xi(x, y, z)$. The line $\overline{AA_3}$, which projects $A$ onto $A_3$, is called the $\Xi$-*projecting line* of the point $A$. The point $A_4[x_A, y_A, 0, w_A]$, the $\Omega$-*image*, is the orthogonal projection of $A$ onto $\Omega(x, y, w)$ by $\Omega$-*projecting line*. Furthermore, we

[15] The supplementary interactive models in *GeoGebra 5* give such a possibility.

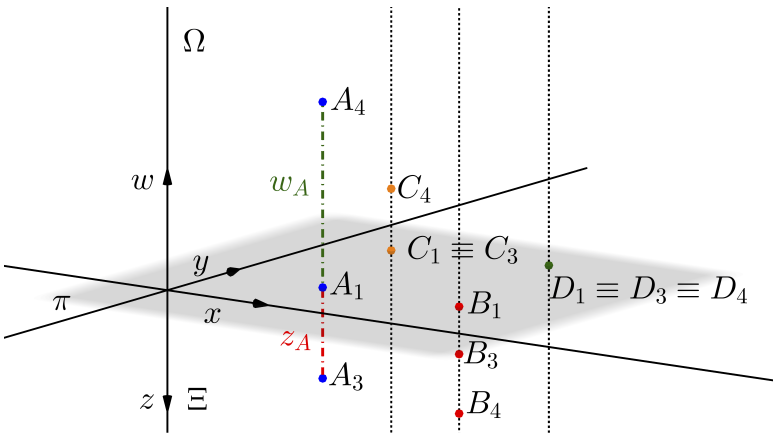



Figure 3.1: The $\Xi, \Omega$, and $\pi$-images of points $A, B, C$ and $D$. The point $A$ is in a general position with positive $z_A$ and $w_A$ coordinates. The point $B$ has a negative $w_B$ coordinate. The point $C$ has the coordinate $z_C = 0$. The point $D$ lies in $\pi(x, y)$ and has the coordinates $z_D = w_D = 0$.
Interactive model: `https://www.geogebra.org/m/cbezmk9x`

denote $A_1[x_A, y_A, 0, 0]$ the $\pi$-*image* of $A$, which is the orthogonal projection of $A_3$ and $A_4$ onto the reference plane $\pi(x, y)$, and hence $A_1$ is the point of intersection of the line of recall $\overline{A_3A_4}$ and the plane $\pi(x, y)$ in the modeling 3-space.

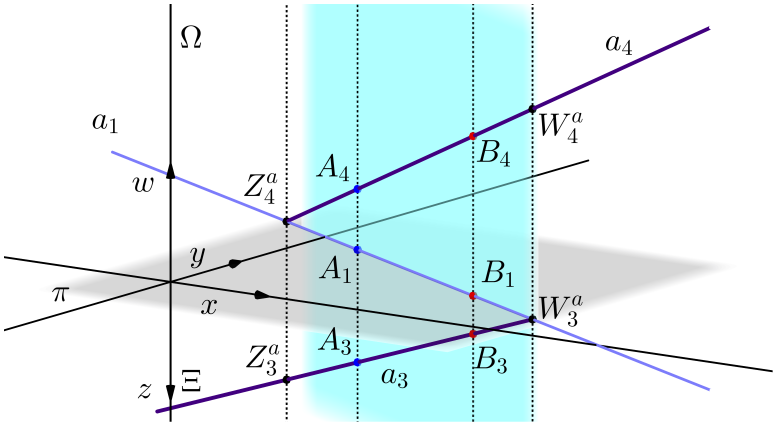



Figure 3.2: A projection of a line $a = \overline{AB}$ in a general position. The $\Xi, \Omega$, and $\pi$-images lie in the plane of recall of the line $a$. The point $Z$ is the $\Xi$-trace and the point $W$ is the $\Omega$-trace of the line $a$.
Interactive model: `https://www.geogebra.org/m/nf3cnwtb`

A line $a = \overline{AB}$ is represented by its orthogonal projections $a_3 = \overline{A_3B_3}$ onto $\Xi(x, y, z)$ and $a_4 = \overline{A_4B_4}$ onto $\Omega(x, y, w)$ (Figure 3.2), which are either lines, or they may become a point and line in special positions. The line $a_3$ is the line of intersection of the 3-space $\Xi(x, y, z)$ and the $\Xi$-*projecting plane* of the line $a$. The same holds for the line $a_4$ as the intersection line of the 3-space $\Omega(x, y, w)$ and the $\Omega$-*projecting plane* of the line $a$. The orthogonal projections of the lines $a_3$ and $a_4$ onto the reference plane $\pi(x, y)$ overlap in the line $a_1 = \overline{A_1B_1}$ or a point if the original line $a$ is perpendicular to $\pi(x, y)$. The line $a_1$ is the line of intersection of its *plane of recall* — the plane of lines of recall of points on the given line $a$ (such lines are $\overline{A_3A_4}$ and $\overline{B_3B_4}$) and the reference plane $\pi(x, y)$. In the modeling 3-space, the plane of recall is perpendicular to the reference plane $\pi(x, y)$. The point $Z^a$ is the point of intersection of the line $a$ and the 3-space $\Xi(x, y, z)$, and it is called the $\Xi$-*trace* of the line $a$. The point $W^a$ is the point of intersection of the line $a$ and the 3-space $\Omega(x, y, w)$, and it is called the $\Omega$-*trace* of the line $a$. If $Z^a \equiv W^a$, the line $a$ intersects the reference plane $\pi(x, y)$, otherwise $a$ and $\pi(x, y)$ are parallel or skew.

Let us describe some special positions of lines with respect to the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$ (Figure 3.3). By a special position, we mean the position of a line parallel, perpendicular, or lying in any of the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$.[16] Spatial relations in the 4-space are well described in [Lindgren and Slaby, 1968].

> A line completely perpendicular to the 3-space $\Omega(x, y, w)$ is also perpendicular to the reference plane $\pi(x, y)$. Therefore its $\Xi$-image in the 3-space $\Xi(x, y, z)$ is a line perpendicular to $\pi(x, y)$, while the $\Omega$-image in $\Omega(x, y, w)$ is a point. Its plane of recall is represented by a line in the modeling 3-space.

[16]Since there are obviously many special positions, we will visualize only those, which are essential for this thesis. On the other hand, we would like to encourage the reader to make constructions of her or his own as a valuable exercise.

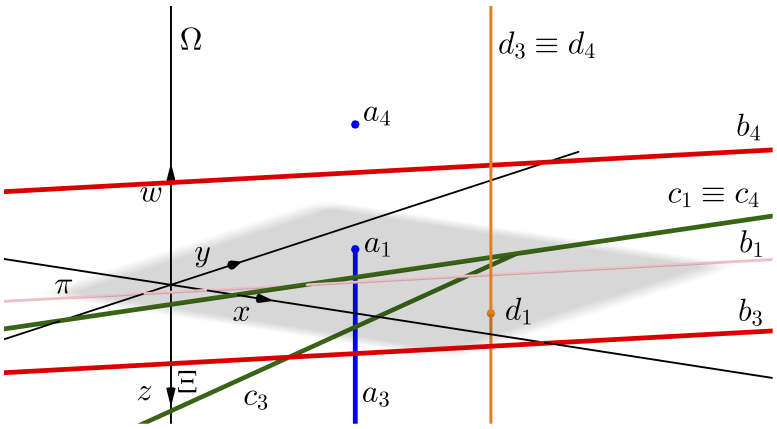



Figure 3.3: Projections of lines $a, b, c$, and $d$. The line $a$ is perpendicular to the 3-space $\Omega(x, y, w)$. The $\Xi$-image $a_3$ is perpendicular to the reference plane $\pi(x, y)$ and the $\Omega$-image $a_4$ becomes a point. The line $b$ is parallel to the reference plane $\pi(x, y)$, and its $\Xi$ and $\Omega$-images are lines parallel to $\pi(x, y)$. The line $c$ lies in the 3-space $\Xi(x, y, z)$, and hence its $\Omega$-image $c_4$ lies in the reference plane $\pi(x, y)$. The line $d$ is perpendicular to the plane $\pi(x, y)$ but not completely perpendicular to any of the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$, and therefore its $\Xi$ and $\Omega$-images overlap in a line perpendicular to the reference plane $\pi(x, y)$.
Interactive model: `https://www.geogebra.org/m/vq8uceur`

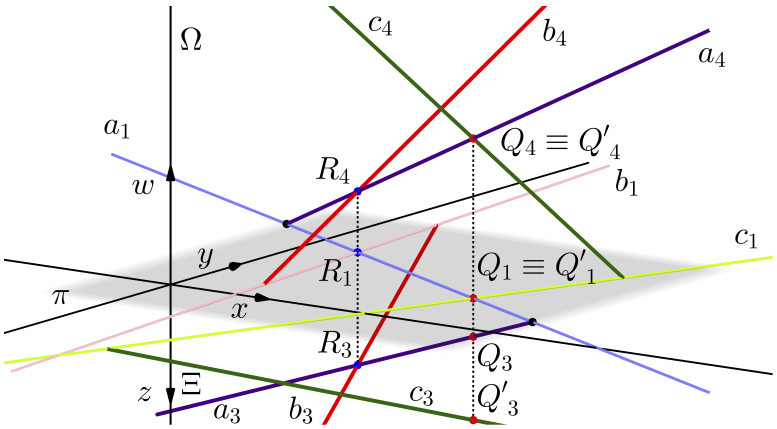



Figure 3.4: Projections of lines $a, b$, and $c$. The lines $a$ and $b$ intersect in a point $R$. Thus the points $R_3$ and $R_4$ are intersecting points of $a_3, b_3$ and $a_4, b_4$, respectively. The lines $a$ and $c$ are skew. The $\Omega$-images of points $Q$ and $Q'$ overlap in the point $Q_4 \equiv Q'_4$ — the intersection of $a_4$ and $c_4$, but the points $Q_3$ on $a_3$ and $Q'_3$ on $c_3$ are distinct.
Interactive model: `https://www.geogebra.org/m/yxekpdzk`

A line perpendicular to the reference plane $\pi(x, y)$ but not completely perpendicular to any of the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$ is represented by both images overlapping in one line perpendicular to the reference plane $\pi(x, y)$ in the modeling 3-space.

A line parallel to the reference plane $\pi(x, y)$ is represented in both 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$ by lines parallel to $\pi(x, y)$ in the modeling 3-space.

The $\Omega$-image of a line lying in the 3-space $\Xi(x, y, z)$ lies in $\pi(x, y)$, while

the Ξ-image overlaps the original line.

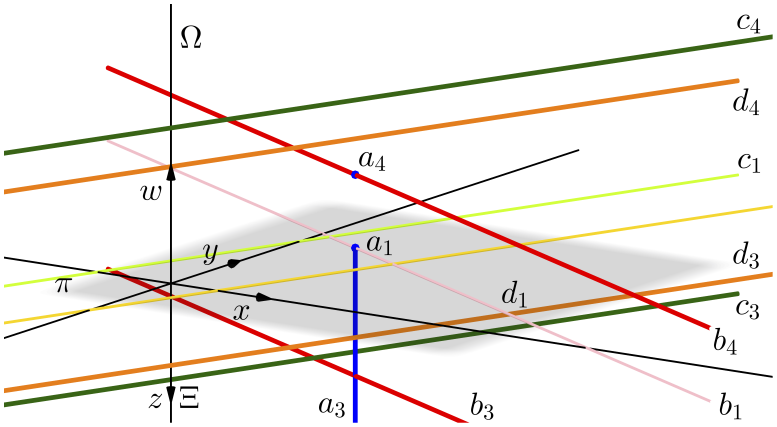



Figure 3.5: Projections of lines $a, b, c$, and $d$. The lines $a$ and $b$ are perpendicular, where $a$ is also perpendicular to $\Omega(x, y, w)$. The Ξ-images $a_3$ and $b_3$ are perpendicular to each other in the modeling 3-space. The lines $c$ and $d$ are parallel, and hence their Ξ and Ω images are also parallel.
Interactive model: `https://www.geogebra.org/m/hy2z8ffd`

Two distinct lines are intersecting, skew (Figure 3.4), or parallel (Figure 3.5). The Ξ-images of two perpendicular lines $a$ and $b$, from which one is parallel to the 3-space $\Xi(x, y, z)$ are perpendicular lines, or one of them is a point. The same holds for the 3-space $\Omega(x, y, w)$. This property of orthogonal projections is called the right-angle theorem, for details in projections onto a plane see [Stachel, 1990]. Orthogonal projections preserve parallelism, and so the images of the parallel lines are parallel, or they become one line, or two distinct points.

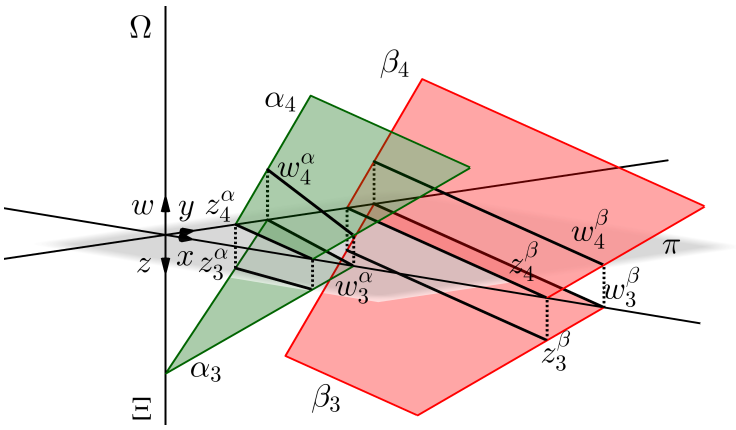



Figure 3.6: Projections of planes $\alpha$ and $\beta$. The line $z^\alpha$ is the Ξ-trace and the line $w^\alpha$ is the Ω-trace of the plane $\alpha$. The traces are not parallel in the modeling space, and hence the plane $\alpha$ intersects the reference plane $\pi(x, y)$ in one point. The plane $\beta$ is parallel to the reference plane $\pi(x, y)$ in one direction. Its traces $w^\beta$ and $z^\beta$ are parallel lines.
Interactive Model: `https://www.geogebra.org/m/vaduhxuz`

A plane $\alpha$ is represented by its conjugated images (Figure 3.6), which are the orthogonal projections $\alpha_3$ onto $\Xi(x, y, z)$ and $\alpha_4$ onto $\Omega(x, y, w)$. The images are

either planes, or they are a line and plane, or two lines in special positions. The plane $\alpha_3$ is the intersection of its $\Xi$-*projecting 3-space* and the 3-space $\Xi(x, y, z)$. Similarly, the plane $\alpha_4$ is orthogonally projected by its $\Omega$-*projecting 3-space* into the 3-space $\Omega(x, y, w)$. The orthogonal projections of the planes $\alpha_3$ and $\alpha_4$ onto the reference plane $\pi(x, y)$ overlap into the whole reference plane $\alpha_1 \equiv \pi(x, y)$, or they overlap in a line, or point if the original plane $\alpha$ is perpendicular to the reference plane $\pi(x, y)$. In general position, the *3-space of recall* of the plane $\alpha$ generated by the lines of recall of its points is the whole modeling space. The trace $z^\alpha$, the $\Xi$-trace, is the line of intersection of the 3-space $\Xi(x, y, z)$ and the plane $\alpha$. The trace $w^\alpha$, the $\Omega$-trace, is the line of intersection of the 3-space $\Omega(x, y, w)$ and the plane $\alpha$. If $w_3^\alpha \equiv z_4^\alpha$, the plane $\alpha$ and the reference plane $\pi(x, y)$ intersect in a common line. Otherwise they intersect in one point, or they are parallel.

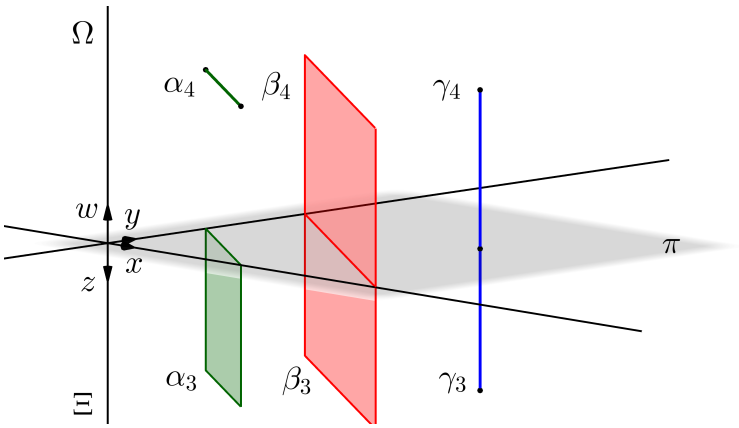



Figure 3.7: Projections of planes $\alpha, \beta$, and $\gamma$ perpendicular to the reference plane $\pi(x, y)$. The plane $\alpha$ is perpendicular to the 3-space $\Omega(x, y, w)$ and parallel to the 3-space $\Xi(x, y, z)$. Its $\Omega$-image $\alpha_4$ appears as a line parallel to the reference plane $\pi(x, y)$. The plane $\beta$ is perpendicular to the reference plane $\pi(x, y)$ with a common intersection line. The plane $\gamma$ is perpendicular to the reference plane $\pi(x, y)$ and intersects it in one point. The whole plane appears as the overlapping lines $\gamma_3 \equiv \gamma_4$ in the modeling space.
Interactive Model: `https://www.geogebra.org/m/v5a9q8wy`

Let us discuss some special positions of planes with respect to the modeling space (Figures 3.7 and 3.8):

A plane perpendicular to the 3-space $\Omega(x, y, w)$ is also perpendicular to the reference plane $\pi(x, y)$. Therefore its $\Xi$-image in the 3-space $\Xi(x, y, z)$ is a plane perpendicular to $\pi(x, y)$, while the $\Omega$-image in $\Omega(x, y, w)$ is a line parallel to the reference plane $\pi(x, y)$. Its 3-space of recall is represented by a plane in the modeling 3-space.

A plane completely perpendicular to the reference plane $\pi(x, y)$ is represented by both images overlapping in one line perpendicular to the reference plane $\pi(x, y)$ in the modeling 3-space. An example of such plane is the plane spanned by the $z$ and $w$ axis through the origin.

The $\Omega$-image of a plane parallel to the 3-space $\Xi(x, y, z)$ is a plane or line (if it is also perpendicular to $\Omega(x, y, w)$) parallel to the reference plane $\pi(x, y)$.

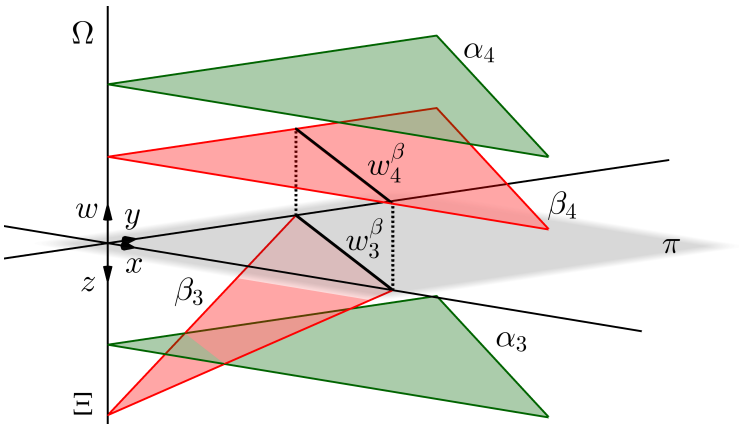



Figure 3.8: Projections of planes $\alpha$ and $\beta$ parallel to the 3-space $\Xi(x, y, z)$. The plane $\alpha$ is parallel to the both 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$. Both images $\alpha_3$ and $\alpha_4$ are parallel to the reference plane $\pi(x, y)$. The plane $\beta$ is parallel to the 3-space $\Xi(x, y, z)$ but intersects the 3-space $\Omega(x, y, w)$, and hence only the $\Omega$-image $\beta_4$ is parallel to the reference plane $\pi(x, y)$.
Interactive Model: `https://www.geogebra.org/m/xhzwvkfu`

A plane parallel to the reference plane $\pi(x, y)$ in two distinct directions is represented in both 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$ by planes parallel to $\pi(x, y)$ in the modeling 3-space.

The $\Omega$-image of a plane lying in the 3-space $\Xi(x, y, z)$ lies in the reference plane $\pi(x, y)$, while the $\Xi$-image overlaps the original plane.

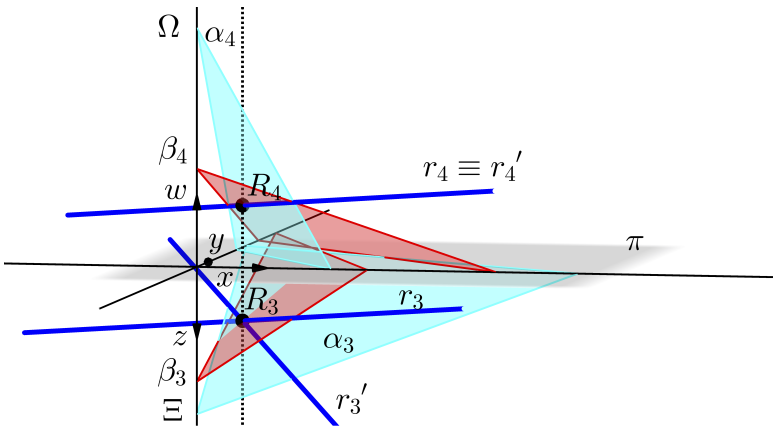



Figure 3.9: Projections of planes $\alpha$ and $\beta$ intersecting in a point $R$. The intersecting point $R$ lies on a line $r$ in the plane $\beta$ and also on a line $r'$ in the plane $\alpha$. The $\Omega$-images of the lines $r$ and $r'$ are chosen as the lines overlapping in the line of intersection of $\alpha_4$ and $\beta_4$, but their $\Xi$-images $r_3$ and $r_3'$ are intersecting lines with the common point $R_3$. The $\Omega$-image $R_4$ lies on the line $r_4 \equiv r_4'$ and on the reference line of the point $R$ through $R_3$.
Interactive Model: `https://www.geogebra.org/m/ef8z57tv`

Two distinct planes in 4-space intersect in a point (Figure 3.9), in a line (Figure 3.10), or they are parallel (Figures 3.11 and 3.12). To determine the position of two planes $\alpha$ and $\beta$, we find the intersections of their $\Xi$-images and

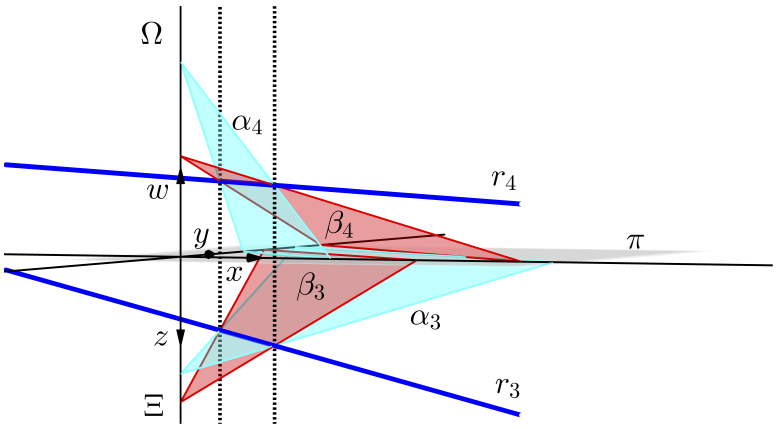



Figure 3.10: Projections of planes $\alpha$ and $\beta$ intersecting in a line $r$. The images $r_3$ and $r_4$ of the line of intersection of the planes $\alpha$ and $\beta$ are the lines of intersection of the images $\alpha_3, \beta_3$ and $\alpha_4, \beta_4$ of the planes $\alpha$ and $\beta$.
Interactive Model: `https://www.geogebra.org/m/vuchdkpe`

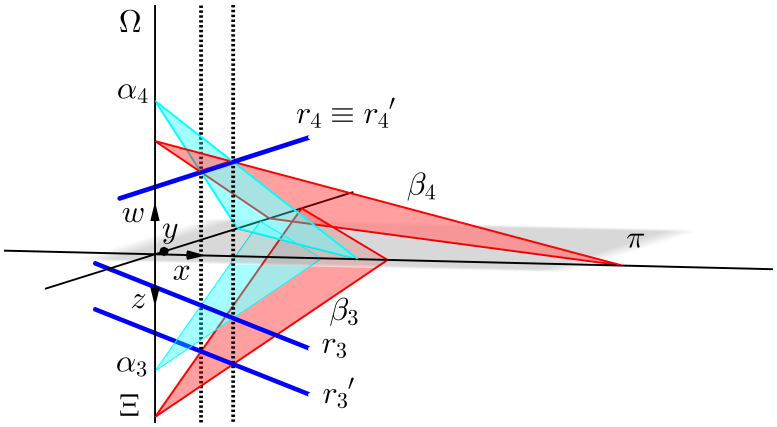



Figure 3.11: Projections of planes $\alpha$ and $\beta$ parallel in one direction. The Ω-images $\alpha_4$ and $\beta_4$ intersect in the overlapping Ω-images $r_4 \equiv r_4'$ of the lines $r$ and $r'$. The Ξ-images $r_3$ in $\alpha_3$ and $r_3'$ in $\beta_3$ are parallel lines.
Interactive Model: `https://www.geogebra.org/m/xnpggwg5`

Ω-images. Let $r_4$ be the intersection line of $\alpha_4$ and $\beta_4$. The intersections of the plane of recall of the line $r_4$ with the Ξ-images $\alpha_3$ and $\beta_3$ are two distinct intersecting or parallel lines $r_3$ and $r_3'$ (hence $r_4 \equiv r_4'$ is overlapping Ω-image of lines $r$ and $r'$) or coincident lines. If $r_3$ and $r_3'$ are intersecting lines, the planes $\alpha$ and $\beta$ intersect in one point $R$, where $R_3$ is the point of intersection of $r_3$ and $r_3'$, and $R_4$ lies on the line $r_4 \equiv r_4'$ and its line of recall through $R_3$. If $r_3$ and $r_3'$ are parallel, the planes $\alpha$ and $\beta$ are parallel (they have no intersection and one common direction). If $r_3 \equiv r_3'$, the planes $\alpha$ and $\beta$ intersect in the line $r$. Finally, if $\alpha_4$ and $\beta_4$ and also $\alpha_3$ and $\beta_3$ do not intersect, the planes $\alpha$ and $\beta$ are parallel.

The following property of an orthogonal projection is useful for constructions with special planes and lines of 3-spaces (Figures 3.13, 3.14, 3.15). The Ξ-images of a line $k$ perpendicular to a plane $\alpha$, from which one is parallel to the 3-space $\Xi(x, y, z)$, are also perpendicular, or the image $k_3$ of the line is a point, or the image $\alpha_3$ of the plane is a line perpendicular to the image $k_3$. The same holds for the Ω-images and the space $\Omega(x, y, w)$.

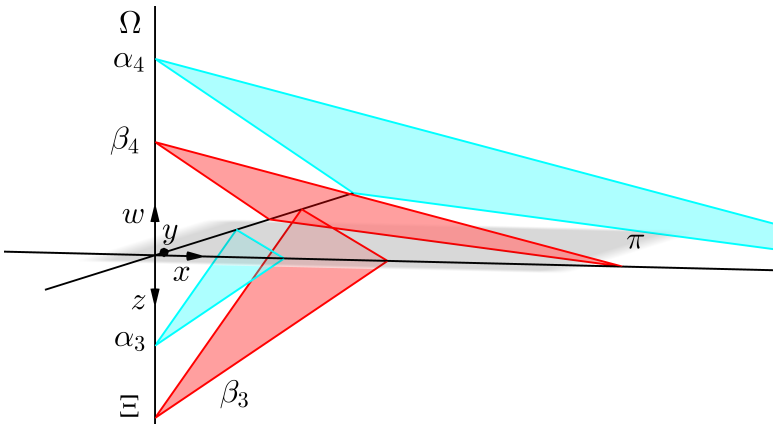



Figure 3.12: Projections of planes $\alpha$ and $\beta$ parallel in two directions. The Ξ-images $\alpha_3, \beta_3$ and also the Ω-images $\alpha_4, \beta_4$ of the planes are parallel.
Interactive model: `https://www.geogebra.org/m/zhzyfhgv`

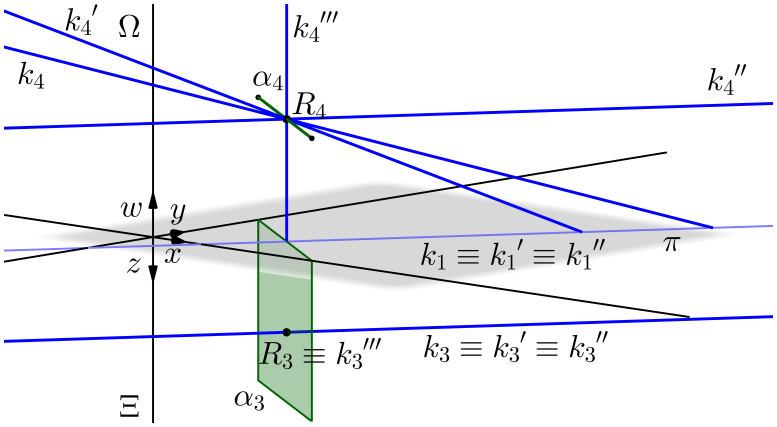



Figure 3.13: Projections of all the different positions of lines $k, k', k''$, and $k'''$ perpendicular to a plane $\alpha$. The plane $\alpha$ is parallel to the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$. The line $k''$ is parallel to $\Omega(x, y, w)$.
Interactive model: `https://www.geogebra.org/m/dw28hjqm`

The orthogonal projection of a 3-space onto a non-orthogonal 3-space is the whole 3-space. The orthogonal projection of a 3-space onto an orthogonal 3-space is a plane. We represent a 3-space $\Sigma$ by its intersections with $\Xi(x, y, z)$ and $\Omega(x, y, w)$ (Figure 3.16). The trace $\xi^\Sigma$, the Ξ-trace, is the plane of intersection of the 3-space $\Xi(x, y, z)$ and the 3-space $\Sigma$. The trace $\omega^\Sigma$, the Ω-trace, is the plane of intersection of the 3-space $\Omega(x, y, w)$ and the 3-space $\Sigma$. If the 3-space $\Sigma$ is not parallel to $\Xi(x, y, z)$ or $\Omega(x, y, w)$, then it intersects $\pi(x, y)$ in a line, which is common for both traces of $\Sigma$. Planes of the 3-space $\Sigma$ which are parallel to the $\Xi(x, y, z)$ or $\Omega(x, y, w)$ are called Ξ-*planes* or Ω-*planes of* $\Sigma$, respectively. Lines of the 3-space $\Sigma$, perpendicular to its Ξ-plane or Ω-plane, are called the Ξ-*slope line* or Ω-*slope line*, respectively.

Let us have one image, for instance $A_4$, of a point $A$, lying in a 3-space $\Sigma$ given by its traces. We can conveniently use the Ξ or Ω-planes through the point $A$ to determine the second conjugated image. Let $\eta_4^\Sigma$ be the Ξ-plane of $\Sigma$ through the point $A$. The intersection of $\eta_4^\Sigma$ and the Ω-trace $\omega_4^\Sigma$ is the Ω-image of the Ω-trace of $\eta^\Sigma$. The Ξ-image of the Ω-trace of $\eta^\Sigma$ is the intersection of its plane of recall

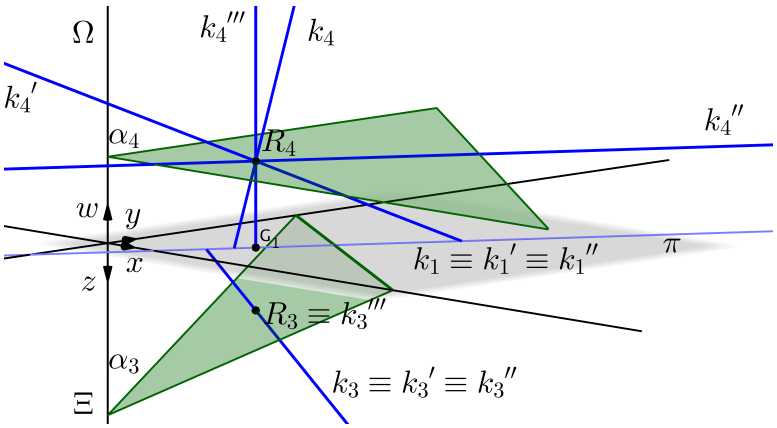



Figure 3.14: Projections of all the different positions of lines $k, k', k''$, and $k'''$ perpendicular to a plane $\alpha$. The plane $\alpha$ is parallel to the 3-space $\Xi(x, y, z)$ and intersects the 3-space $\Omega(x, y, w)$ in a line. The line $k'''$ is parallel to $\Omega(x, y, w)$.
Interactive model: `https://www.geogebra.org/m/cuwyutca`

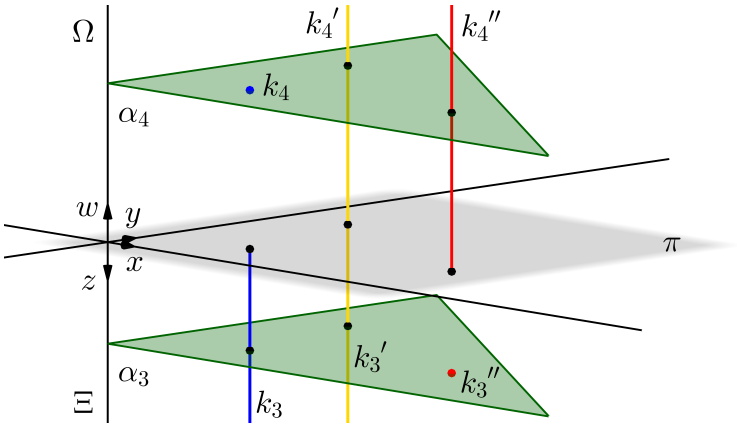



Figure 3.15: Projections of all the different positions of lines $k, k', k''$, and $k'''$ perpendicular to a plane $\alpha$. The plane $\alpha$ is parallel to the 3-space $\Xi(x, y, z)$ and perpendicular to the 3-space $\Omega(x, y, w)$. All the lines $k, k', k''$, and $k'''$ are parallel to $\Omega(x, y, w)$.
Interactive model: `https://www.geogebra.org/m/ckwb9kvs`

and the reference plane $\pi(x, y)$. The Ξ-image $\eta_3^\Sigma$ of the plane $\eta^\Sigma$ passes through the Ξ-image of its Ω-trace and is parallel to the Ξ-trace of the 3-space $\Sigma$.

Two distinct 3-spaces are parallel, if their Ξ-traces and Ω-traces are parallel. Otherwise, at least one pair of traces are intersecting planes, and the 3-spaces intersect in a plane (Figure 3.17). Let us find the plane of intersection $\rho$ of two 3-spaces $\Sigma$ and $\Gamma$ given by their traces. At first, we find the intersecting line of one pair of the traces, for example $\omega_4^\Sigma$ and $\omega_4^\Gamma$. This line is the Ω-image of the Ω-trace of the plane $\rho$, and its Ξ-image is in the reference plane $\pi(x, y)$. If the Ξ-traces $\xi_3^\Sigma$ and $\xi_3^\Gamma$ are parallel, then the Ξ-image $\rho_3$ is the parallel plane through the Ξ-image of the Ω-trace of the plane $\rho$. In case of $\xi_3^\Sigma$ and $\xi_3^\Gamma$ are intersecting planes, the join of the line of their intersection, the Ξ-image of the Ξ-trace of the plane $\rho$, with the Ξ-image of the Ω-trace of the plane $\rho$, is the Ξ-image $\rho_3$. The existence of the plane $\rho_3$ can be concluded from the following fact: Since the Ω and Ξ-traces of the plane $\rho$ lie in the Ξ and Ω-traces of both 3-spaces $\Sigma$

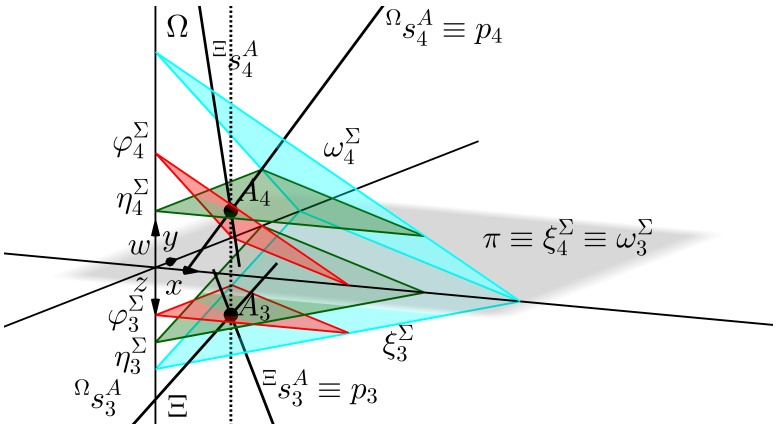



Figure 3.16: Projections of a 3-space $\Sigma$ represented by its traces $\xi^\Sigma$ and $\omega^\Sigma$. The plane $\eta$ is the $\Xi$-plane and the plane $\varphi$ is the $\Omega$-plane of the 3-space $\Sigma$. The point $A$ lies in the 3-space $\Sigma$ and also in the planes $\eta$ and $\varphi$. The lines $^{\Xi}s^A$ and $^{\Omega}s^A$ are the $\Xi$ and $\Omega$-slope lines of the 3-space $\Sigma$ through the point $A$. The line $p$ through the point $A$ is perpendicular to the 3-space $\Sigma$.
Interactive model: `https://www.geogebra.org/m/sy6egnna`

and $\Gamma$, they all intersect in one point on the reference plane $\pi(x, y)$, or the $\Omega$ and $\Xi$-traces of the plane $\rho$ are parallel. Therefore the $\Xi$-image of the $\Omega$-trace and the $\Xi$-image of the $\Xi$-trace of the plane $\rho$ generate the plane $\rho_3$.

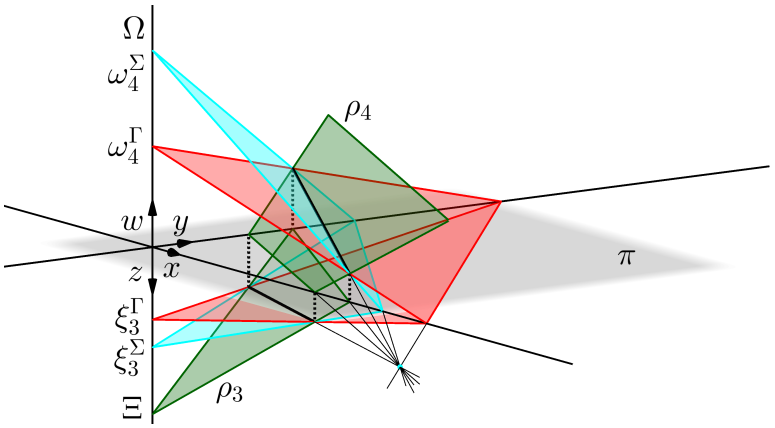



Figure 3.17: Projections of 3-spaces $\Sigma$ and $\Gamma$ intersecting in a plane $\rho$. The intersecting lines of the 3-spaces $\Sigma$ and $\Gamma$ with the reference plane $\pi(x, y)$ and also the $\Xi$ and $\Omega$-traces of $\rho$ meet in one point of the modeling space.
Interactive model: `https://www.geogebra.org/m/rcjb6wrv`

The $\Xi$ and $\Omega$-images of a line $p$ perpendicular to a 3-space $\Sigma$ are also perpendicular to its traces $\xi^\Sigma$ and $\omega^\Sigma$. From the discussion on perpendicularity of lines and planes in special positions, we conclude that $p_3$ is perpendicular to $\xi_3^\Sigma$ and $p_4$ is perpendicular to $\omega_4^\Sigma$.

## 3.3 Measuring lengths and angles

Elementary constructions for measuring lengths and angles in four dimensions are described in this section. If an object lies in the 3-space $\Xi(x, y, z)$, $\Omega(x, y, w)$,

or in the parallel 3-space to either one of them, then its true shape appears in the modeling 3-space. Otherwise, we need to rotate projecting planes of lines, projecting 3-spaces of planes, or to rotate the whole 3-space around its trace plane, onto the modeling 3-space. After such a rotation we obtain any three-dimensional object of any 3-space in its true size in the modeling 3-space.[17]

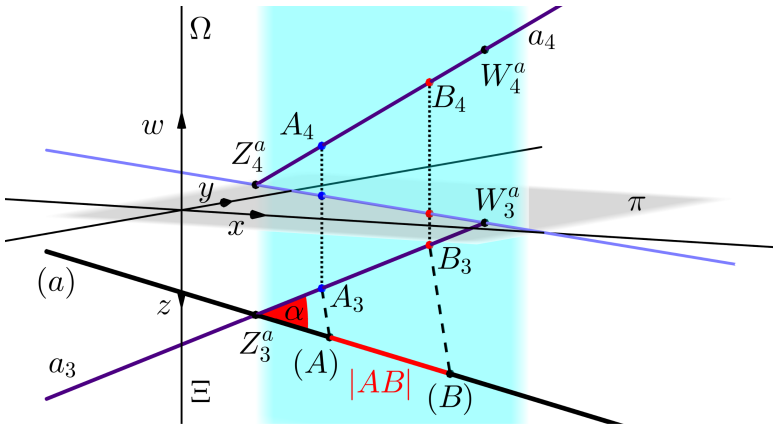



Figure 3.18: The construction of the true length $|AB|$ of a segment $AB$ and the true angle $\alpha$ between a line $a$ and the 3-space $\Xi(x, y, z)$. As an arbitrary plane for the rotation, the projecting plane of the line $a$ is chosen.
Step-by-Step construction: `https://www.geogebra.org/m/svuxv8ne`

Let us have a segment $AB$ given by its two conjugated images $A_3B_3$ and $A_4B_4$ (Figure 3.18). To measure the length of the segment we rotate the Ξ-projecting plane of the line $a = \overline{AB}$ (or the right-angle trapezoid $A_3B_3BA$) onto the modeling 3-space. We choose an arbitrary plane containing the Ξ-image $a_3$ and draw the perpendicular lines $a'$ and $b'$ in this plane to $a_3$ through $A_3$ and $B_3$, respectively. The length of the segment $A_3A$ is the $w$-coordinate of the point $A$, or simply the distance between $A_4$ and $\pi(x, y)$. The last step is to construct the right-angle trapezoid $A_3B_3(A)(B)$, where the points $(A)$, $(B)$ lie on $a'$, $b'$ and $|A_3(A)| = w_A$, $|B_3(B)| = w_B$. Another remark of this construction is the visualization of the true angle between the line $a$ and the 3-space $\Xi(x, y, z)$ since it is also the angle $\sphericalangle(a, a') = \sphericalangle(a', (a))$, where $(a) = \overline{(A)(B)}$. We can conveniently use the Ξ-trace of the line $a$ with zero $w$-coordinate.

The true size of a planar object appears after a rotation of the projecting space of its plane. Let us have a plane $\alpha$ given by its conjugated images and a point $A$ in $\alpha$ (Figure 3.19). We rotate the Ω-image $\alpha_4$ onto $(\alpha)$. The point $(A)$ lies on the perpendicular line to the Ω-image $\alpha_4$ through the point $A_4$. The length of the segment $A_4(A)$ is equal to $z_A$. The Ω-trace $w^\alpha$ of the plane $\alpha$ is the fixed line of the rotation. The plane $(\alpha)$ is generated by the point $(A)$ and the line $w_4$. There is an axial affinity between two planes $\alpha_4 \to (\alpha)$ in the modeling 3-space. The axis of the affinity is $w_4$ and the direction of the affinity is perpendicular to $\alpha_4$. The affinity can be easily used for the rotated images of other points, if needed.[18]

[17] Even if we cannot visualize the process of rotation of a 3-space around a plane in the 4-space, we can create the resulting model in the modeling 3-space.

[18] In our two-dimensional figures, there is also a planar affinity that maps the plane $\alpha$ to the plane $(\alpha)$. The axis is the same as in the spatial affinity between two planes, and the direction is the projected spatial direction onto our drawing plane.

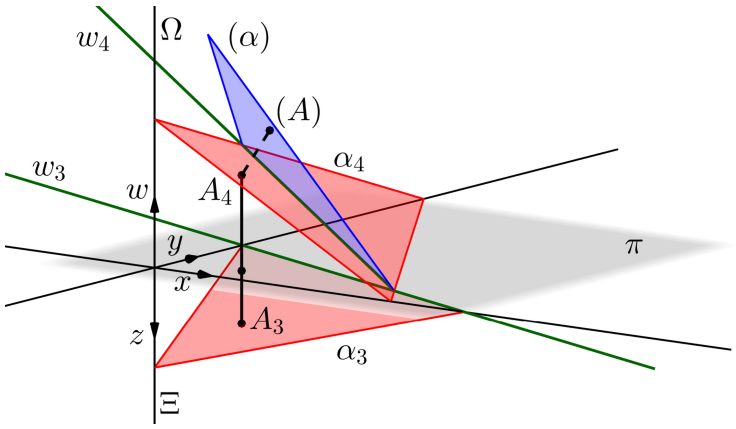



Figure 3.19: A rotation of the Ω-projecting 3-space of a plane $\alpha$.
Step-by-Step construction: `https://www.geogebra.org/m/zyzdcs4e`

Let us have a 3-space $\Sigma$ in a general position with respect to $\Xi(x, y, z)$ and $\Omega(x, y, w)$ and a point $A$ in $\Sigma$ (Figure 3.20). A three-dimensional object in $\Sigma$ is orthogonally projected onto the modeling 3-space. To visualize the object in its true shape, we rotate the 3-space $\Sigma$ around its Ω-trace $\omega^\Sigma$ onto the modeling 3-space. The plane $\omega^\Sigma$ is the plane of fixed points of the rotation. Let $W^s$ be the point of intersection of the $\omega^\Sigma$ and the Ω-slope line $s$ through $A$. The triangle $AA_4W^s$ with the right angle in $A_4$ can be constructed in the true size by rotating the Ω-projecting plane of $s$, as above. Afterwards, the segment $W_4^s(A)$ is rotated to $W_4^sA_0$, where $A_0$ lies on the line $s_4 = \overline{W_4^sA_4}$. The image of any other point of $\Sigma$ can be constructed using the affinity between $\Sigma_4$ and $\Sigma_0$ in the modeling 3-space, with the axial plane $\omega_4^\Sigma$ and the direction of the Ω-slope line $s_4$. The same process can be applied for the rotation around the Ξ-trace of $\Sigma$.

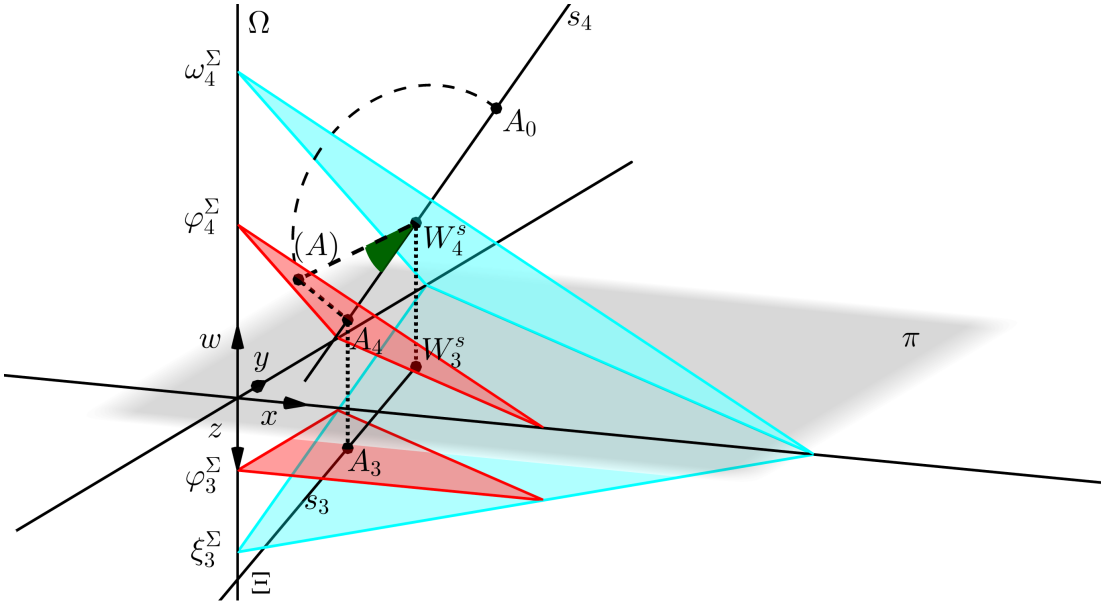



Figure 3.20: A rotation of a 3-space $\Sigma$ onto $\Omega(x, y, w)$.
Step-by-Step construction: `https://www.geogebra.org/m/yvmfvsnq`

## 3.4 Projecting solids

We show double orthogonal projections of a tesseract and pentachoron into the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$ in the following constructions. With the use of interactive software, properties of the projections (with respect to the modeling 3-space) can be analyzed.

### 3.4.1 A regular pentachoron

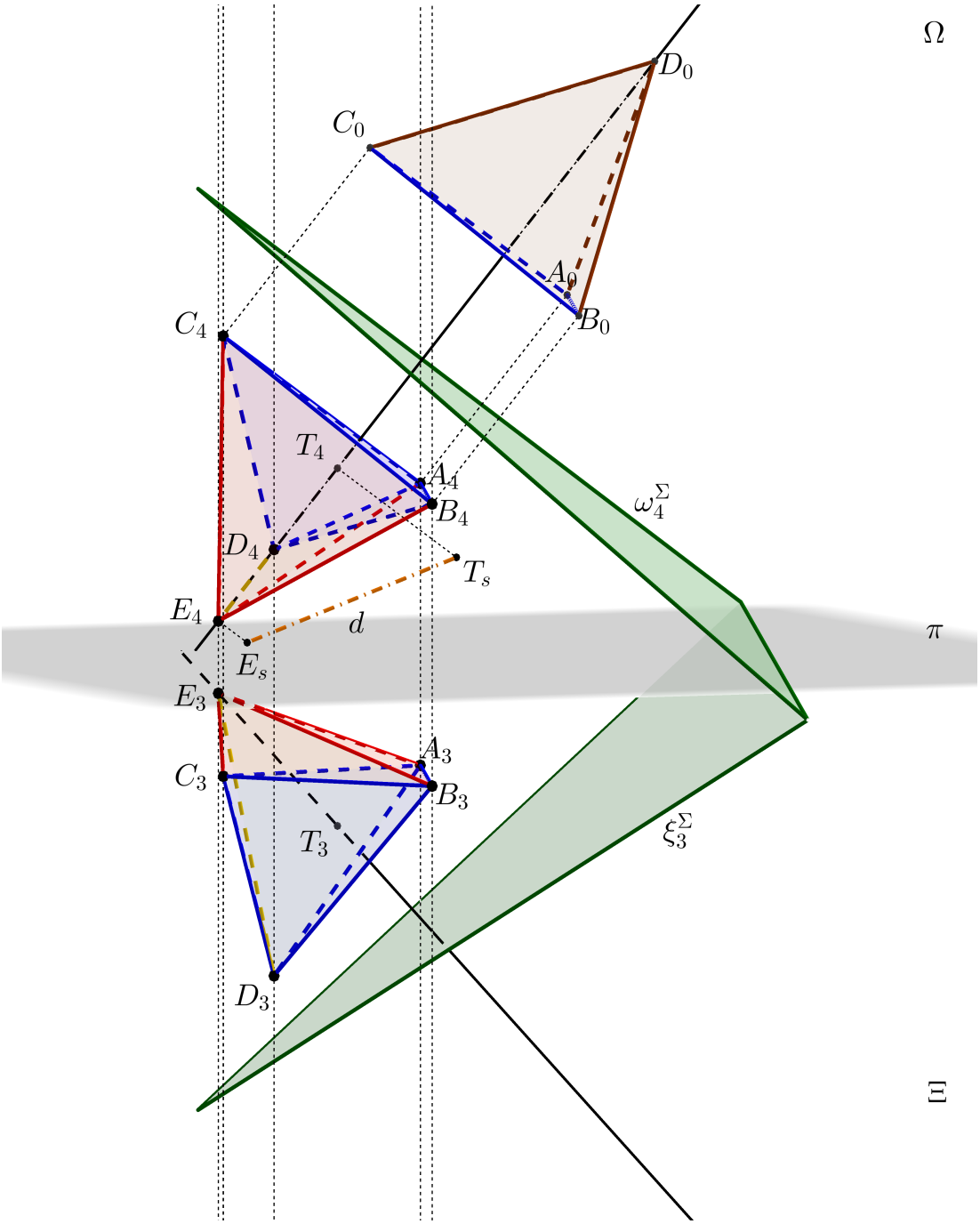



Figure 3.21: A construction of a regular pentachoron.
Step-by-Step construction: `https://www.geogebra.org/m/g895vgwq`

A regular pentachoron is the four-dimensional generalization of a regular tetrahedron. It is a simplex with five vertices, which consists of five regular tetrahedral cells. In three dimensions, a regular tetrahedron is easily created, when we join the vertices of an equilateral triangle with the fourth vertex, which is the endpoint of the altitude perpendicular to the triangle in its center. Imagine, for illustration, that we want to introduce the tetrahedron to a geometrically educated two-dimensional creature. It will not be able to visualize the tetrahedron in mind, but still, we can draw its parallel projections onto a plane. If we draw the top and front views and merge them into one picture, we obtain Monge's projection. Rather than trying to persuade the creature to imagine the missing dimension, we show projections of the tetrahedron, which describe it uniquely. After all, together with the creature, we can use basic descriptive geometry techniques (such as a rotation around a line) to investigate incidence and metric properties in the 3-space on the two-dimensional drawing paper. The construction of the tetrahedron would start with the base equilateral triangle. Both horizontal and vertical images of the triangle would appear distorted in an affinity implied by the parallel projection. To see the triangle in its true shape, we would rotate it into the vertical projecting plane around the line of its intersection with the plane of the triangle. For the creature, this would seem like the axial affinity in the drawing plane, while the rotation around a line is impossible. Apparently, the planar creature would understand the concept of planar triangle and also the relation of perpendicularity between two lines. In Monge's projection, to construct a line perpendicular to a plane, we only need perpendicularity of lines in the drawing plane. For measurements we also use a rotation, and hence we can continue to construct the altitude and the fourth vertex. After connecting all the vertices of the triangle with the fourth vertex, both projections would seem like four connected triangles (faces) for the creature not understanding the "insideness". However, it can measure and further investigate the polyhedron. Analogically we project the two-dimensional creature into ourselves, while we generalize the construction for the regular pentachoron. In the double orthogonal projection, the four-dimensional pentachoron is projected onto two mutually perpendicular 3-spaces. Instead of the drawing plane, we merge these spaces into a modeling 3-space. At first, we select a 3-space of the tetrahedral base. The orthogonal projections of this tetrahedron are affinely distorted, and therefore, we find the tetrahedron in its true shape after a rotation of its 3-space around the plane of intersection with one of the spaces of projection onto the modeling space. This rotation around the plane in the 4-space appears as an affinity in the modeling 3-space. After the construction of the rotated regular tetrahedral base in its true shape, we find its original image with the use of the spatial axial affinity. Then we construct the altitude of the pentachoron between the fifth vertex and the constructed base tetrahedron. The length of the altitude may be computed analytically or by inductive analogy from the regular tetrahedron. For construction of a line perpendicular to a 3-space in the double orthogonal projection, we only need a perpendicularity of lines and planes. The length of the altitude appears distorted, but, again, we can measure the true length of the rotated image. After the construction of the fifth vertex, we connect all the vertices of the pentachoron, and the resulting images appear as compositions of five distorted tetrahedra.

Let us have a regular pentachoron $ABCDE$ with the cell $ABCD$ in the 3-

space $\Sigma$. In our example, the 3-space $\Sigma$ is given by its traces $\xi^\Sigma$ and $\omega^\Sigma$. The $\Omega$-image $A_4$ of the point $A$ is given. The face $ABC$ is parallel to the $\Omega$-trace with the edge $AB$ parallel to the reference plane $\pi(x, y)$.

We start with the construction of the tetrahedron $ABCD$. For the construction of the $\Xi$-images of the points $A$ and $B$ we use the $\Omega$-plane of the 3-space $\Sigma$ (not shown in the figure). The tetrahedron appears in its true shape $A_0B_0C_0D_0$ in the image $\Sigma_0$ of the space $\Sigma$ after its rotation around the $\Omega$-trace $\omega^\Sigma$ onto the modeling space (Figure 3.20). With the edge $A_0B_0$ we can finish the construction of the regular tetrahedron $A_0B_0C_0D_0$. In the affinity between $\Sigma_4$ and $\Sigma_0$ we construct the $\Omega$-images $C_4$ and $D_4$. On the lines of recall and the $\Omega$-planes of the 3-space $\Sigma$ we find the $\Xi$-images $C_3$ and $D_3$.

Further on, we need to find the projections of the point $E$. Parallel projections preserve centers of segments, and therefore also the centroid of a triangle or tetrahedron. Since the pentachoron $ABCDE$ is regular, the vertex $E$ lies on the perpendicular line to the 3-space $\Sigma$ through the centroid $T$ of the tetrahedron $ABCD$, and the true length of the segment $TE$ can be easily computed as $d = |TE| = \frac{|AB|}{2}\sqrt{\frac{5}{2}}$. Therefore the points $E_3$ and $E_4$ lie on the lines through $T_3$ perpendicular to $\xi_3^\Sigma$ and through $T_4$ perpendicular to $\omega_4^\Sigma$, respectively.[19] The segment $TE$ appears in its true shape $T_sE_s$, with the length $|T_sE_s| = d$, after some rotation of the $\Omega$-projecting plane of $TE$ around the $\Omega$-image $T_4E_4$ onto the modeling space (Figure 3.18). Hence the point $E_4$ is found as the original point of the point $E_s$. The $\Xi$-image $E_3$ lies on the line of recall of the point $E$.

Connecting the tetrahedra $ABCD, ABCE, ABDE, ACDE$, and $BCDE$ in their $\Xi$ and $\Omega$-images, the pentachoron $ABCDE$ is constructed.

### 3.4.2 A construction of a tesseract

Let us construct a tesseract $ABCDEFGHIJKLMNOP$ with the cube $ABCDIJKL$ in a 3-space $\Sigma$ given by its $\Xi$ and $\Omega$-traces. Moreover, let the face $ABCD$ lie in the $\Omega$-trace $\omega^\Sigma$, and let the side $AB$ lie on a line parallel to $\pi(x, y)$ (Figure 3.22).

Since the face $ABCD$ lies in $\omega^\Sigma$, it lies also in the 3-space $\Omega(x, y, z)$, and its $\Omega$-image $A_4B_4C_4D_4$ is a square with the side $a = |AB| = |A_4B_4|$ in its true shape in the modeling 3-space. For the same reason, the $\Xi$-projection $A_3B_3C_3D_3 \equiv A_1B_1C_1D_1$ lies in the reference plane $\pi(x, y)$.

To construct the cell $ABCDEFGH$, we construct the point $E$, which lies on the perpendicular line $p$ to the 3-space $\Sigma$ through $A$. The $\Xi$-image $p_3$ is perpendicular to $\xi_3^\Sigma$ and $\Omega$-image $p_4$ is perpendicular to $\omega_4^\Sigma$ in the modeling space. The length of the edge $AE$ in its true size is $|AE| = a$. Following the construction in Figure 3.18, we rotate the $\Omega$-projecting plane of the line $p$ onto the 3-space $\Omega(x, y, w)$ and find $(p)$. The $\Omega$-trace of the line $p$ is the point $A$, and hence $A_4 \equiv (A)$. To find the second point of $(p)$, we construct the $\Xi$-trace $Z^p$ and its image $(Z^p)$ in the rotation (not depicted in Figure 3.22). The point $(E)$ lies on $(p)$ and $|A_4(E)| = a$. The orthogonal projection of $(E)$ onto $p_4$ is the point $E_4$. The face $EFGH$ is parallel to the face $ABCD$. The $\Omega$-image $E_4F_4G_4H_4$ is a

[19] The centroids $T_3$ of the tetrahedron $A_3B_3C_3D_3$ and $T_4$ of the tetrahedron $A_4B_4C_4D_4$ are constructed in the GeoGebra model automatically, even though their synthetic construction is simple.

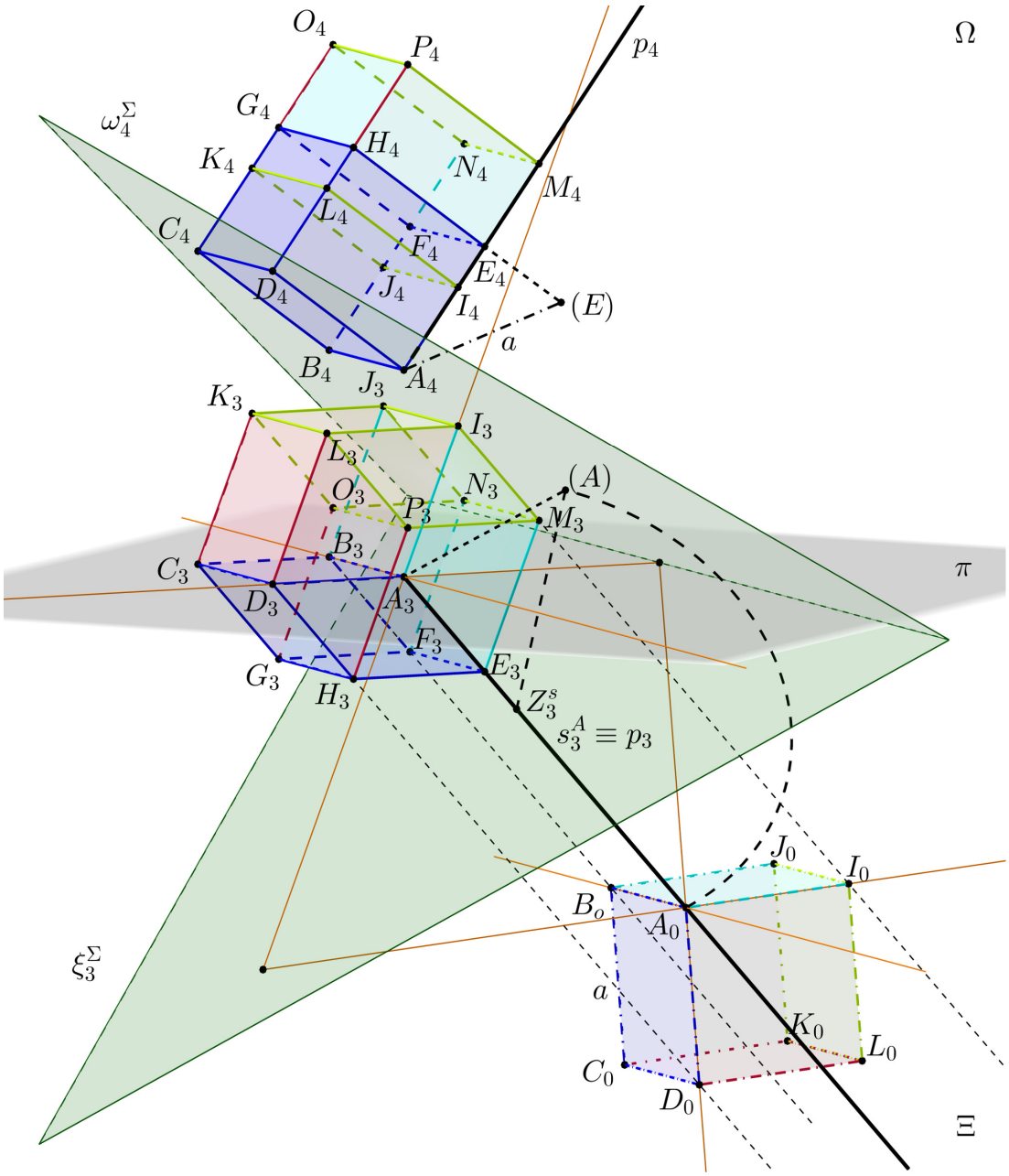



Figure 3.22: A construction of a tesseract.
Step-by-Step construction: `https://www.geogebra.org/m/kpgvsbhf`

square parallel to $A_4B_4C_4D_4$. The $\Xi$-image $E_3$ of the vertex $E$ is the intersection of the line of recall through the point $E_4$ with $p_3$. The $\Xi$-image $E_3F_3G_3H_3$ is a parallelogram parallel to $A_3B_3C_3D_3$.

The next step is to construct the cell $ABCDIJKL$ in the 3-space $\Sigma$. To do so, we rotate the 3-space $\Sigma$ around its trace $\xi^\Sigma$ onto the 3-space $\Xi(x, y, z)$. In the rotated 3-space $\Sigma_0$, we construct the cube $ABCDIJKL$, which appears in its true shape as $A_0B_0C_0D_0I_0J_0K_0L_0$. Following the construction in Figure 3.20, the point $A_0$ lies on the $\Xi$-image of the slope line $s_3^A$ with the $\Xi$-trace $Z^s$. The distance $|Z_3^sA_0|$ equals the distance $|Z^sA|$, which is constructed as the true length $|Z_3^s(A)|$ in the right-angle triangle $A_3(A)Z_3^s$ by a rotation of the $\Xi$-projecting plane of the line $s^A$ onto $\Xi(x, y, z)$. To complete the $\Xi$-image of the cube $ABCDIJKL$, it is

enough to find the point $I_0$ and its original point $I_3$. For a better demonstration, the rotated image of the whole cube is constructed. The points $B_0, C_0$, and $D_0$ lie on their $\Xi$-slope lines through $B_3, C_3$, and $D_3$, respectively. In the figure, the edge $AB$ is parallel to $\xi^{\Sigma}$. Thus, also $A_0B_0$ is parallel to $A_3B_3$. The point $D_0$ is constructed in the affinity between the $\Xi$-image of $\Sigma_3$ and the rotated image $\Sigma_0$. The fixed point of the line $A_3D_3$ is its intersection with $\xi_3^{\Sigma}$. The join of the fixed point with $A_0$ is the rotated image of the line $A_3D_3$ and its intersection with the $\Xi$-slope line through $D_3$ is the point $D_0$. Using the parallelism, distances, or affinity, we can complete the face $A_0B_0C_0D_0$ and also the face $I_0J_0K_0L_0$. In the affinity, we find the point $I_3$ in the direction of the slope lines and on the original line of the line $A_0I_0$. The $\Xi$-image of the face $I_3J_3K_3L_3$ is parallel to the face $A_3B_3C_3D_3$. Since the edge $AI$ is perpendicular to the face $ABCD$ and lies in the 3-space $\Sigma$, the $\Omega$-image $I_4$ lies on the line $p_4$ and on the line of recall through $I_3$. The remaining points $J_4, K_4$, and $L_4$ form the square parallel to $A_4B_4C_4D_4$,

In the last step, we construct the cube $IJKLMNOP$ parallel to the cube $ABCDEFGH$. The vertex $M$ lies on the edge $IM$ parallel to $AE$, and $|I_3M_3| = |A_3E_3|$ and $|I_4M_4| = |A_4E_4|$. The face $MNOP$ is parallel to the face $IJKL$, and so are their $\Xi$ and $\Omega$-images.

## 3.5 Constructing sections

### 3.5.1 A construction of a section cut of a cubic pyramid

In the following section, we cut a four-dimensional regular pyramid with a cubical base with a 3-space. The chosen polychoron may be considered as a simple analogy to a right square pyramid. In Monge's projection onto a two dimensional drawing plane, if the square pyramid stands with its base on the horizontal ground, the top view of the base square appears in its true shape and the image of the vertex is in its center. The front view of the square appears as a segment, and the distance between the segment and the vertex is the true length of the altitude. This placement is convenient for cutting the square pyramid with a plane perpendicular to the vertical plane of projection. The section in the front view is only a segment, and its image in the horizontal plane is a quadrilateral, which is a collinear image of the base square. In the 4-space, we place the regular cubic pyramid analogically with respect to the pair of perpendicular 3-spaces. The pyramid stands with the base cube in one of the spaces, and its altitude is perpendicular to the cube. Therefore in the first view, the base cube appears in its true shape, and the image of the vertex is its center. In the second perpendicular view, the image of the cube merges into a square, and the image of the whole cubic pyramid seems as a three-dimensional square pyramid. The cubic pyramid is in a good position to be cut with a 3-space. We choose the cutting space similarly as in Monge's projection above. Hence one image of the section is a quadrilateral (instead of the segment), and the second is a hexahedron (instead of the quadrilateral). The hexahedral image of the section is again a collinear image of the base cube.

In the interactive model, we can rotate the cutting 3-space around its intersecting line with the reference plane by manipulating the $\Omega$-trace of the cutting 3-space.

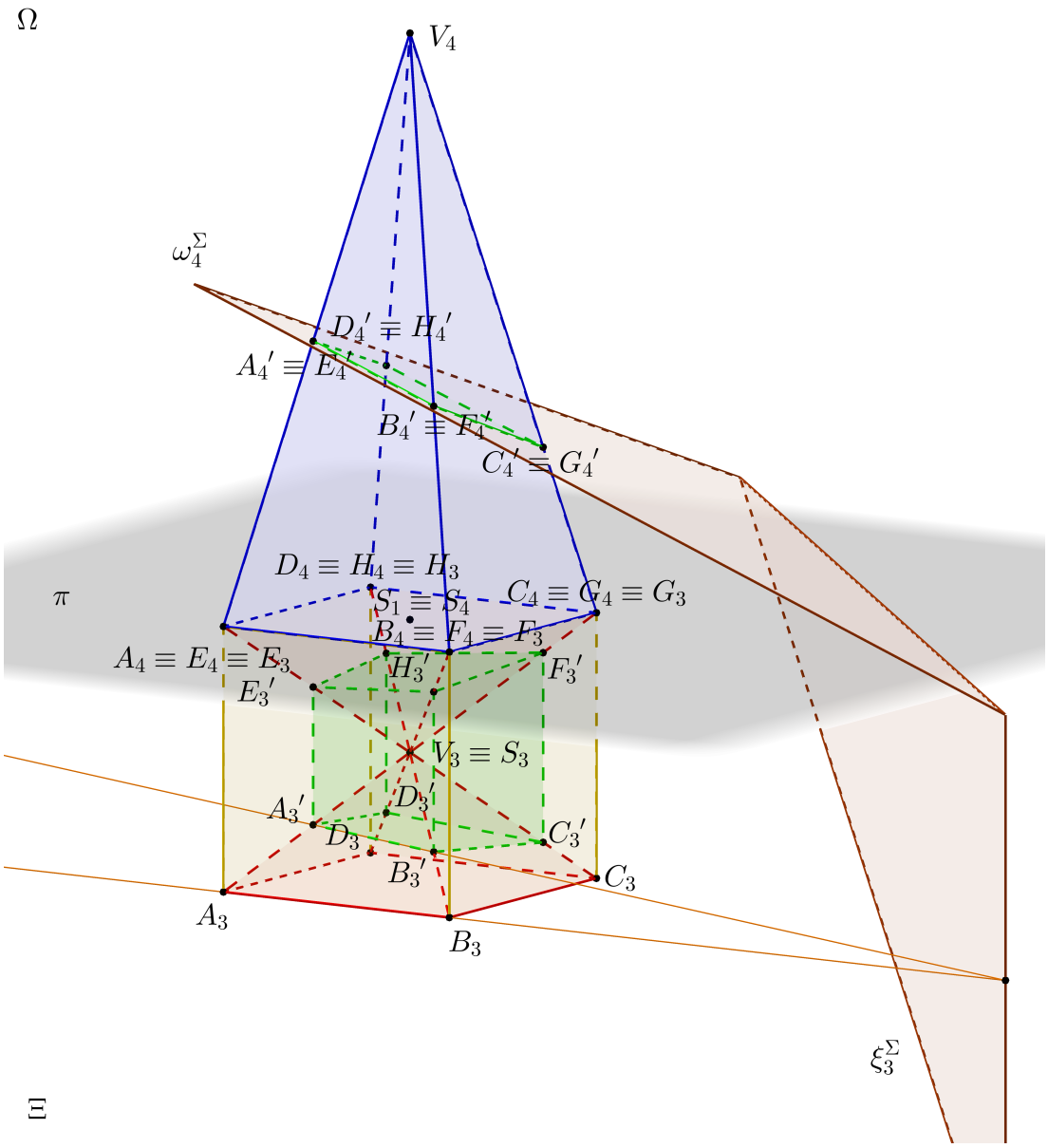



Figure 3.23: A construction of a section cut of a cubic pyramid with a 3-space. Step-by-Step construction: `https://www.geogebra.org/m/txc6h8rz`

Let us construct a cubic pyramid $ABCDEFGHV$ with the base cube $ABCDEFGH$ in the 3-space $\Xi(x, y, z)$ and with the face $ABCD$ in the reference plane $\pi(x, y)$ (Figure 3.23). Let the vertex $V$ lie on the perpendicular line to the cube through its center $S$ ($SV$ is the altitude). We will describe the construction of the section cut, hexahedron $A'B'C'D'E'F'G'H'$, of the cubic pyramid by a 3-space $\Sigma$ perpendicular to $\Omega(x, y, w)$ given by its traces $\xi^\Sigma$, $\omega^\Sigma$.

The given altitude $SV$ is perpendicular to the 3-space $\Xi(x, y, z)$, and hence the Ξ-images $S_3$, $V_3$ merge into one point in $\Xi(x, y, z)$. The Ξ-image $A_3B_3C_3D_3E_3F_3G_3H_3$ of the base cube appears in its true shape. The Ξ-image of the cubic pyramid appears as the union of six pyramids $A_3B_3C_3D_3V_3$, $A_3B_3F_3E_3V_3$, $B_3C_3G_3F_3V_3$, $C_3D_3H_3G_3V_3$, $D_3A_3E_3H_3V_3$, $E_3F_3G_3H_3V_3$. On the other hand, the Ω-image of the cube $ABCDEFGH$ appears as the square $A_4B_4C_4D_4 \equiv E_4F_4G_4H_4$ in $\Omega(x, y, w)$. Thus, the Ω-image of the cubic pyra-

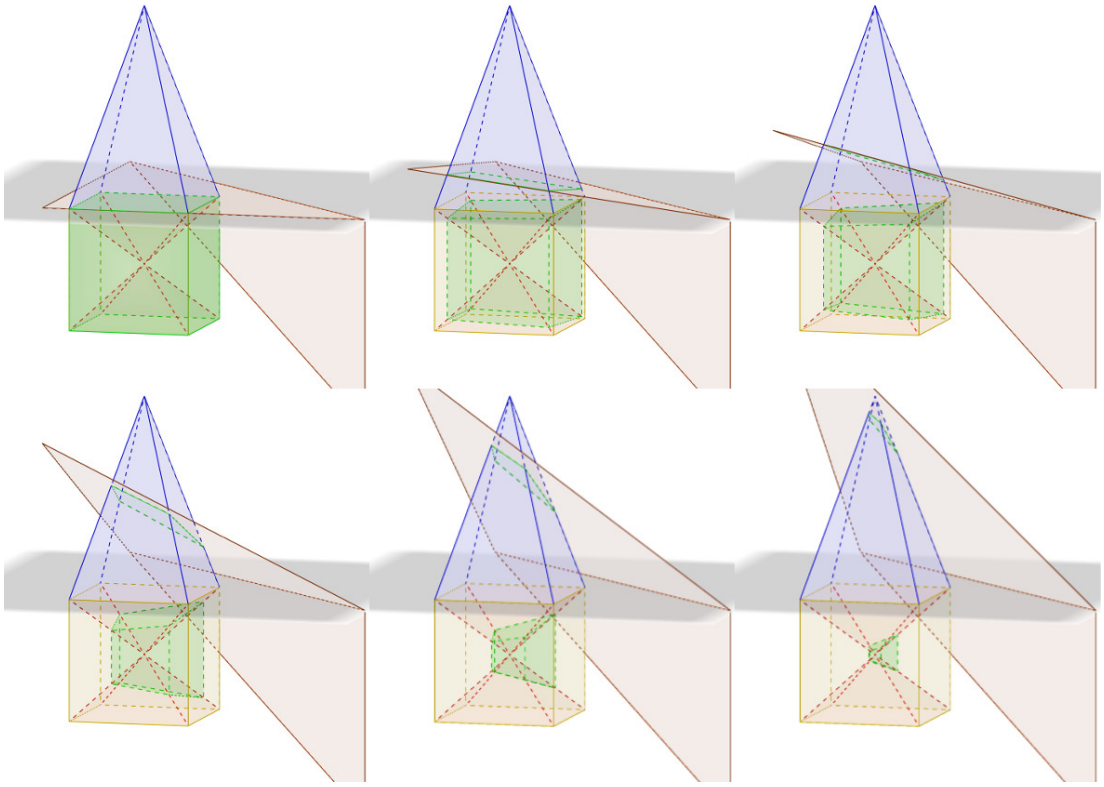


Figure 3.24: Development of section cuts of a cubic pyramid with a rotating 3-space.

mid is the pyramid $A_4B_4C_4D_4V_4$.

The $\Omega$-image of the section is the quadrilateral $A'_4B'_4C'_4D'_4 \equiv E'_4F'_4G'_4H'_4$ lying in the $\Omega$-trace $\omega_4^\Sigma$ of $\Sigma$, where the points $A'_4, B'_4, C'_4, D'_4, E'_4, F'_4, G'_4, H'_4$ are the intersections of the plane $\omega_4^\Sigma$ and the edges $A_4V_4$, $B_4V_4$, $C_4V_4$, $D_4V_4$, $E_4V_4$, $F_4V_4$, $G_4V_4$, $H_4V_4$, respectively. To construct the $\Xi$-image of the section, we can conveniently use lines of recall or the perspective collineation between two 3-spaces: $\Sigma$ - the cutting 3-space and $\Xi(x, y, z)$ - the 3-space of the base cube. The center of the collineation is the point $V$, and the axial plane of the collineation is the $\Xi$-trace $\xi^\Sigma$ of the cutting 3-space $\Sigma$. The construction of the image $B'_3$ of the point $B_3$ in the collineation, with the use of the fixed point of the line $A_3B_3$, is visualized in the figure. The same process is applied to find the remaining vertices of the hexahedron $A'B'C'D'E'F'G'H'$.

Section cuts, which are obtained by rotating the cutting 3-space around its line of intersection with the reference plane, are shown in Figure 3.24.

## 3.6 Constructing shadows

To examine objects in the 4-space, we can construct their three-dimensional shadows. When we return to the idea of the two dimensional creature, we can think about its perception of a shadow. It would probably understand that a two-dimensional object, illuminated from either finite or infinite point source, casts a shadow on a line. The shadow is a mere projection of the object onto the line, and we construct it as the set of intersection points of the light rays with this

line. Similarly, our concept of a shadow is a two-dimensional projection, planar image, of a three-dimensional object. Hence for example, the shadow that the square pyramid casts on the plane of the creature is in its visual possibilities (recall Plato). In Monge's projection, we merge the vertical and the horizontal plane into one drawing plane and construct shadows cast on both of them. If the shadow of the object breaks in their edge — their line of intersection, it appears broken in the drawing plane, too. The geometric constructions are realized in the drawing plane, and therefore, the two-dimensional creature is able to apply them. At last, the four-dimensional story is similar. At this point, we can visualize a four-dimensional polychoron, light rays, and also their intersections with the given perpendicular 3-spaces, which form its three-dimensional shadows. Instead of the edge line, the shadow may break in the edge plane, which is the plane of intersection of these spaces. Certainly, the following methods are another analogy of visualizing the 3-space in a plane and constructing shadows of three-dimensional objects. For methods of constructing shades and shadows in the three-dimensional descriptive geometry on various examples see classical textbooks on descriptive geometry such as [Warren, 1890]. We introduce central and parallel lighting techniques on illumination of a segment, and afterwards, we construct the shadow of the cubic pyramid described in the previous section.

### 3.6.1 Parallel and central lighting

The shadow of a segment in both central and parallel lighting is constructed below as the set of points of intersection of light rays with the given pair of perpendicular 3-spaces of projection. The shadows cast on both 3-spaces are again segments, which intersect in the edge plane in our examples. To construct the segments, we only need to find the shadows of their end points. The final shadow of the segment appears as a broken line in the modeling space.

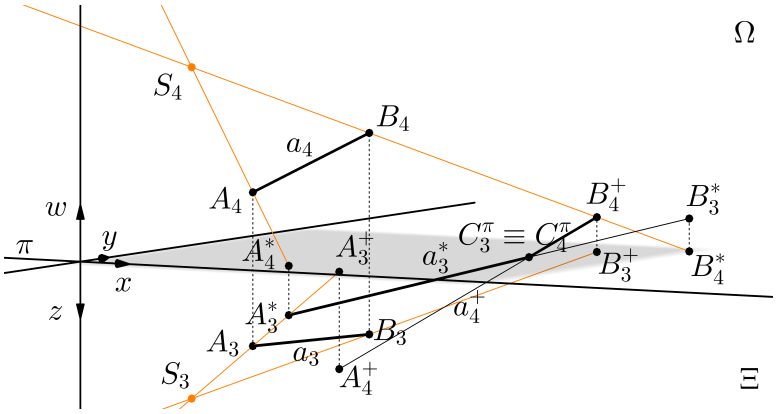



Figure 3.25: The shadow of a segment $AB$ in a central lighting.
Step-by-Step construction: `https://www.geogebra.org/m/qecek82p`

Let us have a segment $AB$ and a point source of light $S$ in the 4-space (Figure 3.25). The segment $AB$ casts shadows $A^*B^*$ and $A^+B^+$ on the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$, respectively.

The shadow $A^*B^*$ is the intersection of light rays from the source $S$ through the points of the segment $AB$ with the 3-space $\Xi(x, y, z)$. Therefore the $\Omega$-image

$A_4^*B_4^*$ lies in the reference plane $\pi(x, y)$. To find the $\Xi$ and $\Omega$-images of the point $A^*$, we construct the ray $S_4A_4$, which intersects the reference plane $\pi(x, y)$ in $A_4^*$. The point $A_3^*$ lies on the ray $S_3A_3$ and on the line of recall through the point $A_4^*$. The $\Xi$ and $\Omega$-images of the point $B^*$ are constructed in the same fashion.

The $\Xi$-image of the shadow $A^+B^+$ of the segment $AB$ cast on the 3-space $\Omega(x, y, w)$ lies in the reference plane $\pi(x, y)$. Thus, the point $A_3^+$ is the intersection of the ray $S_3A_3$ and the reference plane $\pi(x, y)$. Its $\Omega$-image $A_4^+$ lies on the line of recall and on the ray $S_4A_4$. With a similar construction, we finish the images of the point $B^+$ and the shadow $A^+B^+$.

The shadow may break in the edge plane — the reference plane $\pi(x, y)$. If this is the case, we only show the parts of the shadow in the visible half-3-spaces[20] of $\Xi(x, y, z)$ and $\Omega(x, y, w)$. Our shadows $A^*B^*$ and $A^+B^+$ intersect in the point $C^\pi$, in which the shadow of the segment $AB$ is broken.

If the source of light is given by a direction $\overrightarrow{s}$ (Figure 3.26), the lighting is parallel. Instead of the rays through the point $S$, we need rays parallel to the direction $\overrightarrow{s}$. Otherwise, the construction is similar to the central lighting.

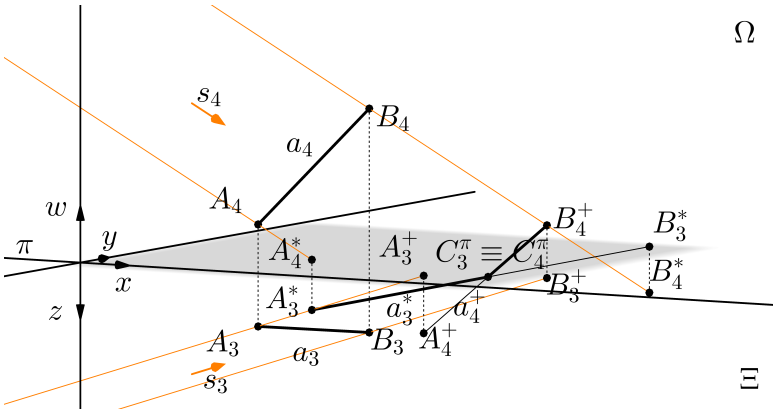



Figure 3.26: The shadow of a segment $AB$ in a parallel lighting.
Step-by-Step construction: `https://www.geogebra.org/m/nxgzgbtx`

### 3.6.2 A construction of a shadow of a cubic pyramid

Previously, we placed the right cubic pyramid to a convenient position for the construction of its sections. We use the same setting for the construction of its shadow. Now, we again skip back to the lower dimension and imagine that the two-dimensional creature is in the same plane as the square base of a right square pyramid. Hence the whole pyramid appears simply as the square. Furthermore, let there be a point light source, which is not in the creature's plane. When the square pyramid is illuminated from the light source, it casts a shadow on the base plane. The shadow of the base square is obviously the base square itself. To construct the rest of the shadow, the creature needs only to find the shadow of the vertex of the pyramid. After connecting the shadow of the vertex and the base square, the construction is finished. This is the main idea of the first step of our four-dimensional construction. The base cube of the cubic pyramid

[20] The half-3-spaces of $\Xi(x, y, z)$ with a nonnegative $z$ coordinate and $\Omega(x, y, w)$ with a nonnegative $w$ coordinate are visible.

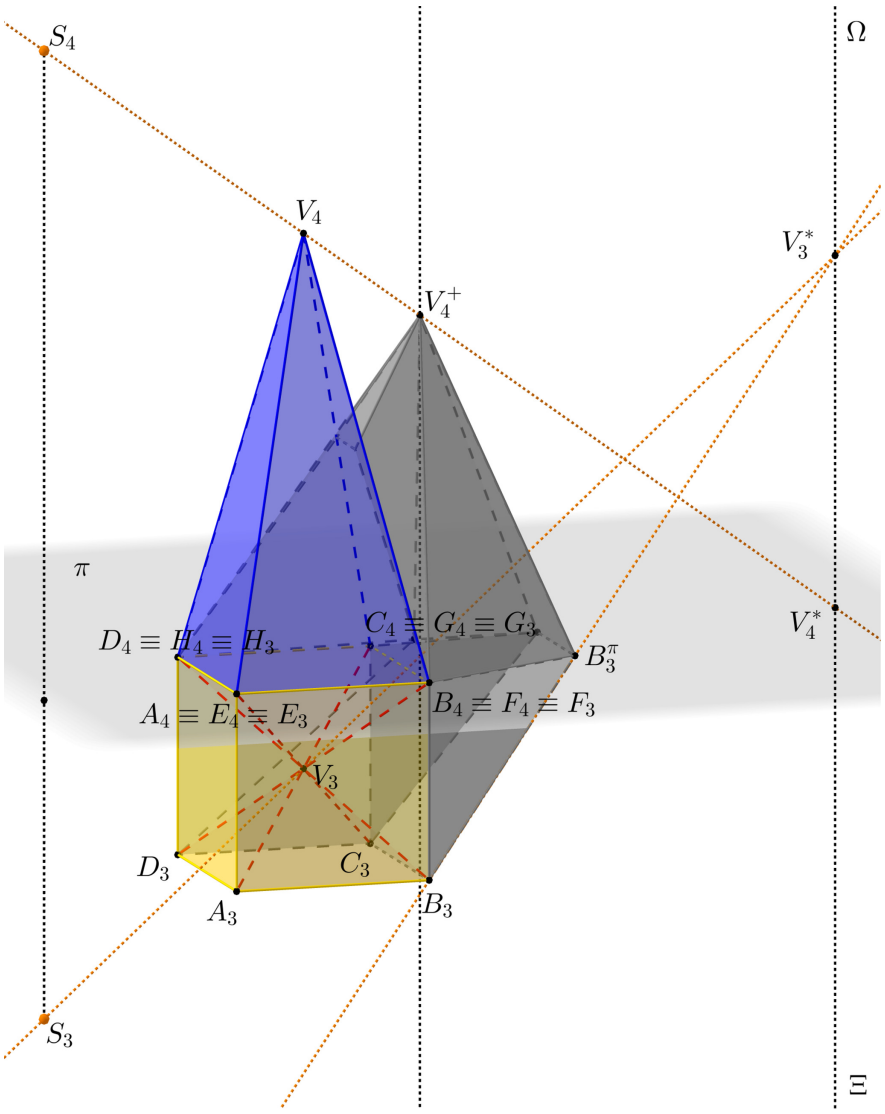



Figure 3.27: A construction of the shadow of a cubic pyramid on the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$. The full construction of the points $V$ and $B$ is shown. Step-by-Step construction: `https://www.geogebra.org/m/kukcfp2q`

lies in one of the 3-spaces of projection, and it overlaps its shadow cast on this 3-space. We only find the shadow of the vertex and join it with the base cube. In our example in the double orthogonal projection, the shadow breaks in the reference plane, thus we construct the shadow of the vertex also in the second perpendicular 3-space, and connect it with the break points of the sides of the cubic pyramid. Finally, the shadow cast on the two 3-spaces looks like a broken polyhedron.

In this last construction (Figure 3.27), we use the same model of the cubic pyramid $ABCDEFGHV$ as in Section 3.5.1. Let us have a point $S$, the point light source, given by its $\Xi$ and $\Omega$-images. The cubic pyramid, illuminated from the source $S$, casts shadows on the 3-spaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$. Firstly, we identify the shade of the cubic pyramid. The surface of shadow is the shadow of its surface of shade.[21] From the $\Xi$-image, we see that the squares $ABCD$,

[21] This is a generalization of the famous descriptive geometry theorem: *The curve of shadow*

$ABFE$, $DAEH$ are illuminated. From the $\Omega$-image, we see the illuminated faces $ABV$, $AEV$, $BFV$, $EFV$, $DAV$, $EHV$, $HDV$. The rest of the cubic pyramid, the union of cells $BCGFV$, $CDHGV$, $EFGHV$, is the shade. The boundary of the shade — faces $BCGF$, $CDHG$, $EFGH$, $BCV$, $CDV$, $DHV$, $HEV$, $EFV$, $FBV$, forms the surface of shade. For the clarity of visualization, only those parts of the shadow, which do not intersect the projections of the body, are constructed.

The shadows of the squares $BCGF$, $CDHG$, $EFGH$ on the 3-space $\Xi(x, y, z)$ remain their $\Xi$-projections $B_3C_3G_3F_3 \equiv B_3^*C_3^*G_3^*F_3^*$, $C_3D_3H_3G_3 \equiv C_3^*D_3^*H_3^*G_3^*$, $E_3F_3G_3H_3 \equiv E_3^*F_3^*G_3^*H_3^*$, since they lie in the 3-space $\Xi(x, y, z)$. To construct the shadow of the vertex $V$, we find the point of intersection $V^*$ of the light ray $SV$ with the 3-space $\Xi(x, y, z)$ according to Figure 3.25. Further on, we construct the shadow of the edge $BV$ by joining $B_3^*$ and $V_3^*$. In our setting, the shadow of the edge $B_3V_3$ intersects the reference plane $\pi$ in the point $B_3^\pi$, where the shadow breaks and continues in the 3-space $\Omega(x, y, w)$. The rest of the shadow in the 3-space $\Xi(x, y, z)$ can be constructed analogically.

The shadow cast on the 3-space $\Omega(x, y, w)$ can be constructed in the same manner, but we can also conveniently use the breaking points of the shadow in the reference plane $\pi(x, y)$. Therefore the last missing point is the shadow $V^+$ of the vertex $V$, which is the intersection of the 3-space $\Omega(x, y, w)$ and the light ray $SV$. Finally, we only need to join all the points of the shadow in the reference plane $\pi(x, y)$ such as $B_3^\pi$ with $V_4^+$. The images of the final shadow are three-dimensional solids in the modeling space.

## 3.7 A pole of a plane with respect to a tetrahedron

The following construction is generalization of Constructions 1 and 5, in which we used a higher dimensional space to construct a harmonic conjugate or the pole of a line with respect to a triangle.

**Construction 6** (Pole of a plane with respect to a tetrahedron, Figure 3.28)**.** *Let us have a tetrahedron $ABCD$ and an arbitrary plane $\sigma$.*

1. *Choose an arbitrary point $E$ not in the 3-space $ABCD$ and join it with the vertices of the tetrahedron.*

2. *Choose an arbitrary distinct 3-space $\Sigma$ through the plane $\sigma$ and find its intersection points $A', B', C'$, and $D'$ with the lines $\overline{AE}, \overline{BE}, \overline{CE}$, and $DE$, respectively.*

3. *Find the intersection point $P'$ of the 3-spaces generated by tetrahedra $ABCD', ABC'D, AB'CD$, and $A'BCD$.*

4. *The pole $P$ of the plane $\sigma$ is the intersection of the line $EP'$ and the 3-space of the tetrahedron $ABCD$.*

Generalizations into higher dimensions naturally follow. Spatial constructions are often used to solve problems in planimetry. This example only shows

*is the shadow of its curve of shade.*

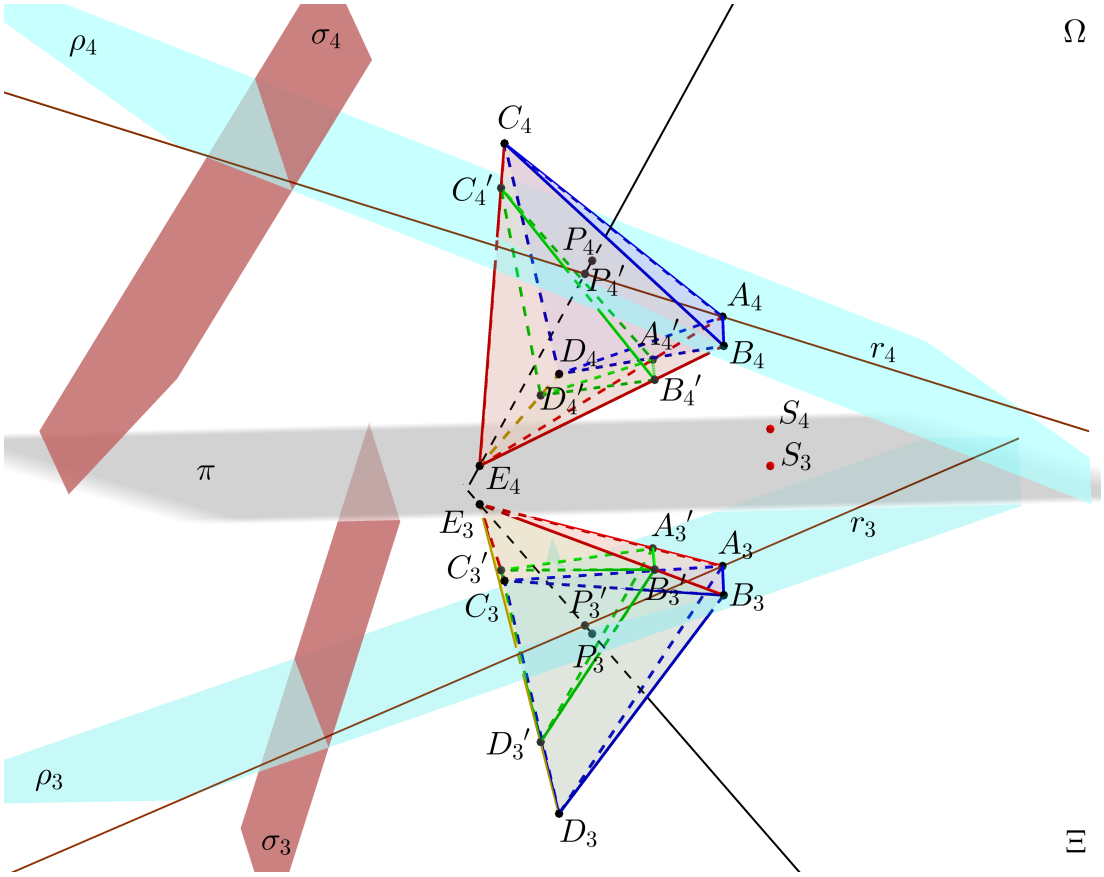



Figure 3.28: The pole $P$ of the polar $\sigma$ with respect to a tetrahedron $ABCD$. The 3-space $\Sigma$ is given by its plane $\sigma$ and a point $S$. The intersection point $P'$ of the four 3-spaces generated by the tetrahedra $ABCD', ABC'D, AB'CD$, and $A'BCD$ is found by consecutive intersections – a plane $\rho$ is the intersection of the 3-spaces $ABCD'$ and $ABC'D$, a line $r$ is the intersection of the plane $\rho$ and the 3-space $AB'CD$, and finally the point $P'$ is the intersection of the line $r$ and the 3-space $A'BCD$.
Interactive model: `https://www.geogebra.org/m/cxe4aszz`

an analogical use of the fourth dimension to solve a three-dimensional problem. Alternatively without the four-dimensional space, one can construct the intersection lines of the faces of the tetrahedron with the plane $\sigma$ and find their poles with respect to the triangular faces. The intersection of the joining lines of the poles with the opposite vertices is the pole of the given plane with respect to the tetrahedron. At last, if we consider the plane $\sigma$ to be the improper plane in $\mathbb{RP}^3$, its pole is the centroid of the tetrahedron (this idea could be used in the construction of the centroid of the tetrahedron in Section 3.4.1, too).

## 3.8 Conclusion

Monge's orthographic projection in three-dimensional geometry is not only a tool for constructions in technical drawing, it is also a method of visualizing the three-dimensional space in the drawing plane and studying its properties. We have used the same concept of two orthogonal projections to study and visualize the

four-dimensional space in a three-dimensional modeling 3-space. It is true that we, three-dimensional people, will probably never naturally see four-dimensional objects as a whole. But none of us has seen a point, a line, a plane, or a 3-space either. The more important question araises: Can we visualize it in our mind, can we imagine it? There were many successful attempts to visualize the objects and their dynamic transformations, mostly by computer graphics. There are also methods of constructing four-dimensional objects in projections into planes. We tried to add a brick into this wall, not only by showing the objects in projections onto two 3-spaces, but also building them constructively and independently of their analytic representations using simple descriptive geometry tools. With the power of interactive graphics, we are able to move the objects and discover their properties intuitively, enhancing our spatial ability in four and multi-dimensional spaces.

Even though, we emphasize the synthetic point of view on the four-dimensional geometric space, we are not opposed to analytic methods of visualization and examination. While analytic representations are valuable for verifications, the synthetic approach leads to intuition and further creations. There is one more question, which often arises in discussions about visualizations of the four-dimensional space. What is their usefulness? Firstly, didactical essence of meditations about the fourth dimension is in abstraction of the ideal geometric space. We mostly use analogies of methods empirically known from the plane and 3-space. But, neither our spatial ability nor geometric space ends where $n$ equals 3. Furthermore, if we can describe the four-dimensional space, we should use all the possible methods to examine it. When the door to the room of the fourth dimension is opened, why should we not walk inside and find all its mysteries. Last, but not least, is the aspect of applications. Last one hundred and fifty years human science has tested many hypothesis based on multidimensionality. Four-dimensional visualizations are now used in physics, computer science and other mathematically related sciences. Knowledge and experience with the four-dimensional space is essential for searching for answers and asking further questions about the world we live in, and where is the limit of the architecture of four-dimensional environments of such applications?

# 4. Selected results

## 4.1 A projective proof of Chasles's theorem for non-developable ruled quadrics

We reconstructed the historical proof of Chasles's theorem 1.2.1 with its non-projective missteps in Section 1.2. Now we combine synthetic and analytic methods to prove it projectively in the framework of definitions derived in Sections 1.4 and 2.1. Firstly, we generalize the concept of a conic into a quadric using homogenous coordinates in $\mathbb{RP}^3$. For sake of simplicity, the set of solutions in synthetic constructions contains also imaginary elements. After definitions, we show a projectivity between a range of points and their polar planes induced by a quadric. The property for tangent planes of a non-developable ruled surface in Chasles's theorem will become a simple corollary.

A quadric $\mathcal{Q}$ is given by the zero set of a real quadratic form

$$\mathcal{Q} : \boldsymbol{F_2}(X) = \sum_{i,j=0}^{3} a_{i,j} x_i x_j = 0, \text{ for any } X(x_1, x_2, x_3, x_0) \in \mathbb{RP}^3 \setminus \{(0,0,0,0)\}$$

with an associated matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,0} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,0} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,0} \\ a_{0,1} & a_{0,2} & a_{0,3} & a_{0,0} \end{pmatrix}.$$

If $\det \mathbf{A} \neq 0$, the quadric is *regular*, otherwise it is called *singular*. Two points $P(p_1, p_2, p_3, p_0)$ and $Q(q_1, q_2, q_3, q_0)$ are polar conjugates with respect to the quadric $\mathcal{Q}$, when

$$\boldsymbol{F}(P, Q) = (p_1, p_2, p_3, p_0) \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,0} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,0} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,0} \\ a_{0,1} & a_{0,2} & a_{0,3} & a_{0,0} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_0 \end{pmatrix} = P^{\mathrm{T}} \mathbf{A} X = 0.$$

A point on a quadric, which is the polar conjugate of all points in $\mathbb{RP}^3$ with respect to the quadric, is its *singular point*. Otherwise, it is a *regular point* of the quadric. Instead of a polar line of a conic, the set of polar conjugates of a non-singular point $P$ of a quadric $\mathcal{Q}$ forms a *polar plane*. The homogenous coordinates $(\pi_1, \pi_2, \pi_3, \pi_0)$ of the polar plane $\pi$ of a pole $P$ with respect to the quadric $\mathcal{Q}$ are obtained as the product $\pi^{\mathrm{T}} = P^{\mathrm{T}} \mathbf{A}$. A *tangent plane* is the polar plane of a regular point of a quadric, and the pole is its *tangent point*.

Although we work with the polarity and tangency on more general surfaces in Section 4.2, to avoid repetitions, we formulate the following definitions now, since they also hold for quadrics. If the tangent planes to a surface are the same for all points on a line of the surface, then this line is said to be a *torsal line*. A line that is not torsal is called *non-torsal*. A *ruled surface* is defined algebraically in [Bydžovský, 1948, p. 510] as a surface with the property that through each of its points passes a line of the surface. A ruled surface is *developable* if each of its lines is torsal. Otherwise, the surface is *non-developable*. At last, we state

the well-known fact that the non-developable ruled quadrics are regular ruled quadrics — a one sheet hyperboloid and hyperbolic paraboloid, which contain no torsal lines.

Synthetically, a (point) quadric is projectively generated, similarly to a conic, by the common points of corresponding elements of a bundle of lines and a bundle of planes with distinct centers (for details see [Baker, 1923, pp. 11–13], [Hestenes and Ziegler, 1991, pp. 29–30]). To construct the polar plane $\pi$ of some pole $P$, we can find three non-collinear harmonic conjugates on three non-coplanar lines of the bundle (the harmonic conjugates form the polar plane) or use the following construction with conics.

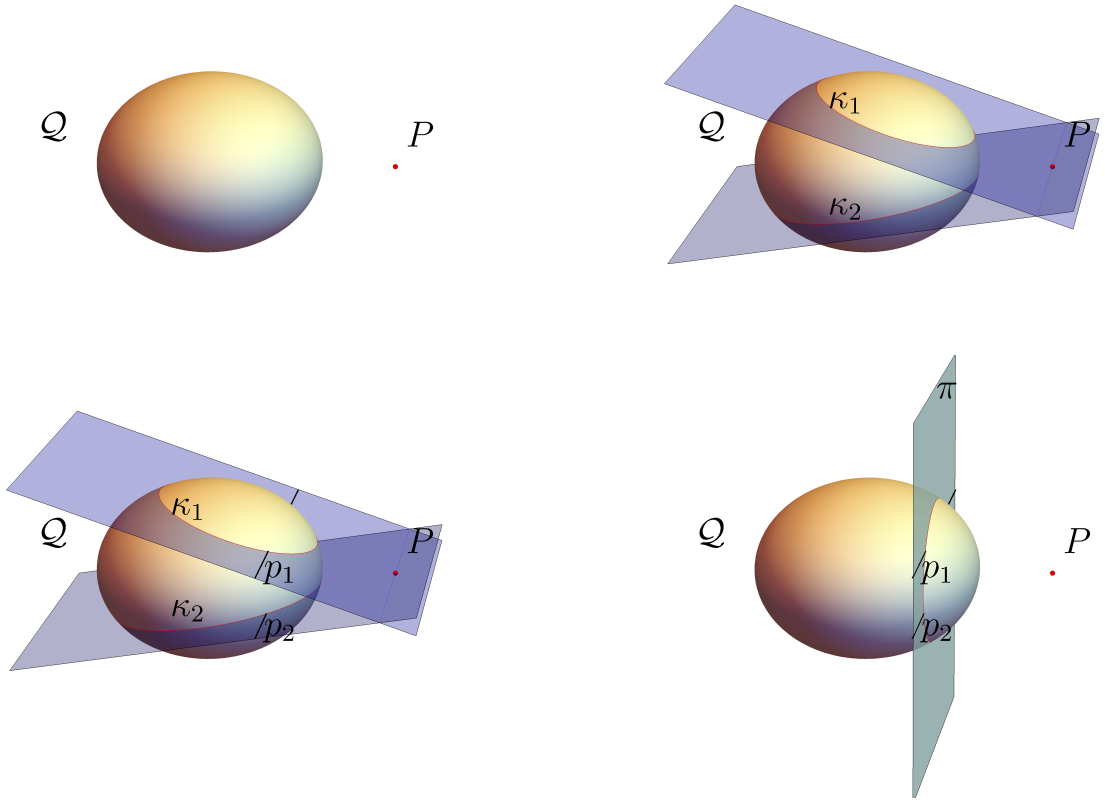



Figure 4.1: A construction of a polar plane.

**Construction 7** (Figure 4.1)**.** *Let $\mathcal{Q}$ be a regular quadric and $P$ a non-singular point of $\mathcal{Q}$, then:*

1) *Choose 2 arbitrary distinct planes through $P$, which cut $\mathcal{Q}$ in conics $\kappa_1, \kappa_2$.*

2) *Find polars $p_1, p_2$ of the pole $P$ with respect to $\kappa_1$ and $\kappa_2$ (Construction 2).*

3) *The plane $\pi$ through the lines $p_1, p_2$ is the polar plane of $P$ with respect to $\mathcal{Q}$.*

**Theorem 4.1.1.** *Let $P(p_1, p_2, p_3, p_0)$ and $R(r_1, r_2, r_3, r_0)$ be two distinct points and $\mathcal{Q}$ a regular quadric. There is a projectivity between the range of points on $\overline{PR}$ and the pencil of polar planes of these points with respect to $\mathcal{Q}$.*

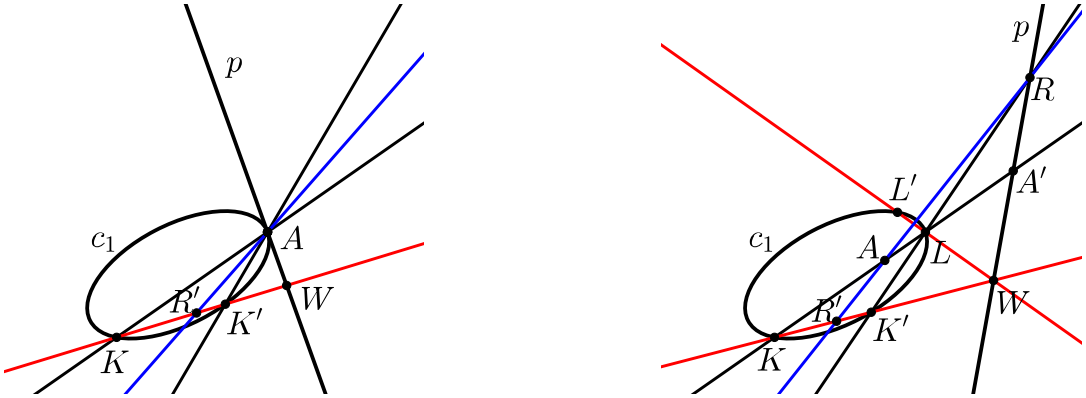



Figure 4.2: The proof of Lemma 4.1.1.

*Proof.* Let $P$ and $R$ be two points on a line $p$, and $\alpha$ and $\beta$ be two planes through $p$, which cut the given quadric $\mathcal{Q}$ in regular conics $c_1, c_2$. The line $p$ has a pole $A$ with respect to the conic $c_1$ in $\alpha$. The polar of an arbitrary point $W$ on $p$ with respect to $c_1$ passes through $A$. Because of the duality of the polar line and its pole, for a range of points on $p$ there exists a pencil of lines through $A$ in $\alpha$. The same assumption can be made in the plane $\beta$ for the conic $c_2$, and we denote $B$ the pole of the line $p$. The quadric $\mathcal{Q}$ induces the polarity of $W$ and its polar plane $\rho$. Thus, the polars of $W$ in planes $\alpha$ and $\beta$ and also $\overline{AB}$ are coplanar. Therefore for each pole $W$ on $p$, the polar plane will pass through the axis $\overline{AB}$. The pencil of polar planes is clearly in a projectivity with the pencils of polar lines in conics. To finish the proof, we use the following lemma:

**Lemma 4.1.1.** *A range of points is in a projectivity with the pencil of their polars with respect to a regular conic in the same plane.*

*Proof.* The proof is divided into two separate cases.

(Figure 4.2, left) At first, let $A$ be a point of a conic $c_1$. The polar line $p$ of the pole $A$ is the tangent to $c_1$ through $A$. Let $K$ be a fixed point on $c_1$ and $W \neq A$ is an arbitrary point on $p$. The point $K'$ is the intersection point of the line $\overline{WK}$ and $c_1$. The range of all points $W$ on $p$ and the respective pencil of lines $\overline{KK'}$ through the center $K$ are in a projectivity. According to Theorem 2.1.5, the pencil of lines $\overline{AK'}$ with the center $A$ is in a projectivity with the pencil of lines $\overline{KK'}$ with the center $K$ (assuming different positions of $W$). Let $R'$ be a point such that $(W, R'; K', K) = -1$, then $\overline{AR'}$ is the polar line of $W$. By Theorem 2.2.1, it holds that $(W, R'; K', K) = (R', W; K, K')$, and hence there exists a projectivity between the ranges of points $W, R', K', K$ and $R', W, K, K'$, in which $W$ is conjugated with $R'$. The pencil of lines $\overline{AR'}, \overline{AW}, \overline{AK}, \overline{AK'}$ with the center $A$ is in a perspectivity with the range of points $R', W, K, K'$ on $\overline{WK}$, and therefore the polar line $\overline{AR'}$ is conjugated with the pole $W$ on $p$.

(Figure 4.2, right) Let $A$ be a point not on $c_1$, and $K$ and $L$ be two distinct fixed points on $c_1$ such that $A$ lies on $\overline{KL}$. Let $W$ be an arbitrary point on $p$, which is the polar line of the pole $A$ with respect to $c_1$. The lines $\overline{WK}$ and $\overline{WL}$ intersect $c_1$ in $K'$ and $L'$. For various positions of $W$ on $p$, we obtain the pencil of lines $\overline{KK'}$ with the center $K$ and the pencil of lines $\overline{LK'}$ with the center $L$. As a consequence of Theorem 2.1.5 these 2 pencils are in a projectivity. The pencil of lines $\overline{KK'}$ and the range of points $W$ on $p$ are in a perspectivity, because the

line $p$ intersects the pencil in points $W$. Let $R$ be the point of intersection of $\overline{K'L}$ and $p$. The pencil of lines $\overline{LK'}$ is in a perspectivity with the range of points $R$ on $p$. Therefore there exists a projectivity between the range of points $R$ and the range of points $W$.

Further on, let $A', R'$ be the points of intersection of $p$ with $\overline{LK}$ and $\overline{RA}$ with $\overline{KK'}$, respectively. The point $A'$ lies on the polar of the pole $A$, so the pair $K, L$ separates the pair $A, A'$ harmonically, and $(A, A'; K, L) = -1$. Projecting from the center $R$ to the line $\overline{WK}$

$$AA'KL \overset{(R)}{\overline{\wedge}} R'WKK',$$

and we have $(R', W, K, K') = -1$, which implies that $R'$ lies on the polar of the point $W$. The point $A$ lies on the same polar, and therefore so does the point $R$. Since the pencil of lines $\overline{AR}$ with the center $A$ is in a projectivity with the range of points $R$ on the axis $p$, then there is also a projectivity between the range of points $W$ on the axis $p$ and the pencil of its polar lines $\overline{AR'}$ with the center $A$. □

Using Lemma 4.1.1, a range of points is in a projectivity with the pencil of their polar planes induced by the quadric $\mathcal{Q}$, which proves Theorem 4.1.1 synthetically. □

To compare, see a brief but non-visual, analytic proof of the previous theorem.

*Proof.* Let $W$ be a point of the range $W = k_1R + k_2P$ for $(k_1, k_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. The polar plane of a variable point $W$ with respect to a quadric $\mathcal{Q}$ is

$$\boldsymbol{F}(W, X) = 0.$$

After substitution we have

$$\boldsymbol{F}(W, X) = \boldsymbol{F}(k_1R + k_2P, X) = k_1\boldsymbol{F}(R, X) + k_2\boldsymbol{F}(P, X) = 0.$$

However $\boldsymbol{F}(P, X)$ and $\boldsymbol{F}(R, X)$ are the polar forms of the points $P$ and $R$, and $k_1\boldsymbol{F}(R, X) + k_2\boldsymbol{F}(P, X)$ describes the pencil of polar planes of $W$. The pole $W$ and its polar plane have the same projective coordinates in their one-dimensional forms. Therefore there is a projectivity between them. □

**Corollary 4.1.1** (Chasles's theorem for quadrics, Figure 4.3)**.** *A pencil of tangent planes through a ruling of a non-developable ruled quadric is in a projectivity with the range of their contact points.*

*Proof.* Let $\mathcal{Q}$ be a non-developable ruled quadric and points $P$ and $Q$ be distinct points on a ruling of $\mathcal{Q}$ with non-identical tangent planes. Each point $W$ of a range with the axis $\overline{PQ}$ on $\mathcal{Q}$ is also a point on $\mathcal{Q}$. By virtue of Theorem 4.1.1, the polar plane of a point on a regular quadric is its tangent plane and the line $\overline{PQ}$ is the axis of both — the range of points and the pencil of planes in a projectivity. □

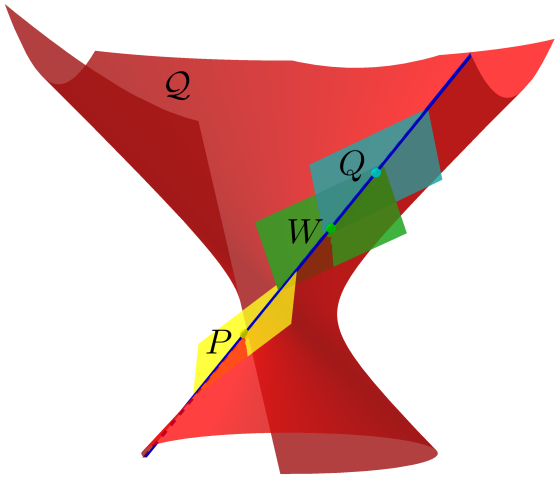



Figure 4.3: A pencil of tangent planes on a ruling of a one-sheet hyperboloid. Interactive model: `https://www.geogebra.org/m/zvwbfjqa`

## 4.2 A projective proof and generalization of Chasles's theorem along non-torsal lines

We generalize Chasles's theorem in Corollary 4.1.1 to algebraic surfaces and prove it strictly projectively in this section. A proof of the original Chasles's theorem based on the differential approach can be found in the article by Önder and Uğurlu [2013, pp. 509–510] and with more insight in projective differential geometry by Hlavatý [1941, p. 48] and in [Pottmann and Wallner, 2009, pp. 228–229].

The algebraic language also enables a straightforward generalization to higher dimensions. We use the standard axiomatic system of the projective extension of the real space with elements at infinity mentioned in Section 1.4.1. An extension to higher-dimensional spaces is described in [Veblen and Young, 1910, pp. 2, 3, 29–33].

### 4.2.1 Chasles's theorem along non-torsal lines

We give an algebraic proof in $\mathbb{RP}^m$ for $m \geq 3$. Consecutive steps of the proof are visualized on the Clebsch surface in $\mathbb{RP}^3$, which is an algebraic surface of 3rd degree (Figure 4.4). We first define necessary terms and show that all polars of a pole on a surface with respect to this surface are tangent surfaces at this point. Then we prove existence of a projectivity between the pencil of first polars and the range of its poles on a non-torsal line to a surface. At the end, we also give a projective proof of Chasles's theorem for non-developable ruled surfaces.

Let $\mathcal{S}$ be a surface of degree $n$ described by the algebraic form $\boldsymbol{F}_n(X)$ of degree $n$ induced by the multilinear form $\boldsymbol{F}(\underbrace{X, X, \ldots, X}_{\times n})$. A point $P$ lies on $\mathcal{S}$ if $\boldsymbol{F}(P) = 0$. This point is said to be *singular* if

$$\frac{\partial \boldsymbol{F}_n(P)}{\partial p_i} = 0 \text{ for all } i = 0, \ldots, m.$$

Otherwise, we call $P$ a *regular* point of the surface $\mathcal{S}$. The polar hypersurface of a regular point $P(p_1, p_2, \ldots, p_m, p_0)$

$$\pi_1(\mathcal{S}, P) : \boldsymbol{F}(P, \underbrace{X, \ldots, X}_{\times(n-1)}) = \sum_{i_1=0,\ldots,m} p_{i_1} \frac{\partial \boldsymbol{F}_n(X)}{\partial x_{i_1}} = 0$$

is said to be its *first polar*. In the formula, we use only the formal derivation of a polynomial.

Applying the same process on $\pi_1$ will give us the *2nd polar* of $P$ with respect to $\mathcal{S}$

$$\pi_2(\mathcal{S}, P) : \boldsymbol{F}(P, P, \underbrace{X, \ldots, X}_{\times(n-2)}) = \sum_{i_2} p_{i_2} \frac{\sum\limits_{i_1} p_{i_1} \frac{\partial \boldsymbol{F}_n(X)}{\partial x_{i_1}}}{\partial x_{i_2}} = 0 \text{ for all } i_1, i_2 = 0, \ldots, m.$$

The *r–th polar* of $P$ with respect to $\mathcal{S}$ can be, after some arithmetic modifications, written as

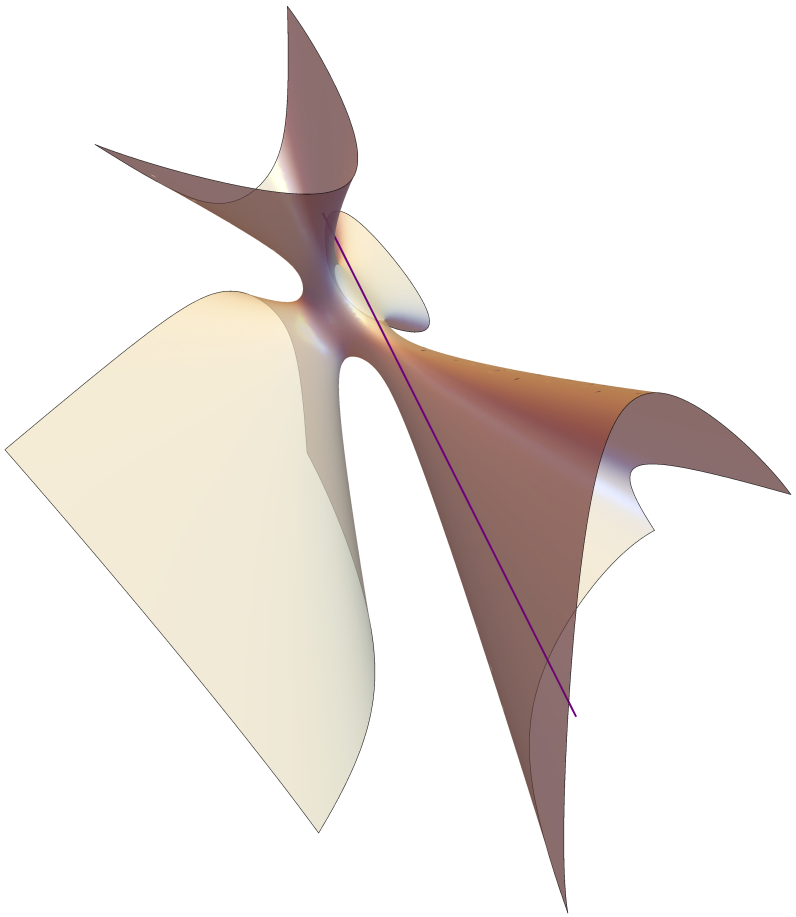


Figure 4.4: Clebsch surface and its non-torsal line.

$$\pi_r(\mathcal{S},P):\boldsymbol{F}(\underbrace{P,\ldots,P}_{\times r}\underbrace{X,\ldots,X}_{\times(n-r)})=$$
$$\sum\frac{r!}{r_1!r_2!\ldots r_m!r_0!}p_1^{r_1}\ldots p_m^{r_m}p_0^{r_0}\frac{\partial^r\boldsymbol{F}_n(X)}{\partial x_1^{r_1}\ldots\partial x_m^{r_m}\partial x_0^{r_0}}=0,$$

where we sum over all $(m+1)$-tuples $(r_1,\ldots,r_m,r_0)$ such that $r_j\geq 0$ and $\sum\limits_{j=0}^{m}r_j=r$. For our purposes the *last polar* ($r=n-1$) will be of major interest. As we are operating on symmetric multilinear forms, we can dually formulate[22] the last polar hereby as

$$\pi_{n-1}(\mathcal{S},P):\boldsymbol{F}(\underbrace{P,\ldots,P}_{\times(n-1)},X)=\sum_{i=0,\ldots,m}x_i\frac{\partial\boldsymbol{F}_n(P)}{\partial p_i}=0.$$

The last polar is a linear form; so it is a polar hyperplane.

**Example 4.2.1.** For visualizations (see Figures 4.4,4.5,4.6) the following equations of the Clebsch surface $\mathcal{S}_{CL}$ are used:

$$\text{in }\mathbb{E}^3:(x^3+y^3+z^3+1)-(x+y+z+1)^3=0,$$
$$\text{in }\mathbb{RP}^3:(x_1^3+x_2^3+x_3^3+x_0^3)-(x_1+x_2+x_3+x_0)^3=0.$$

The points $P=(1,-1,0,1),Q=(4,-4,0,1),W=(3,-3,0,1),R=(7,-7,0,1)$ on the line $p$ have the first polars:
$\pi_1(\mathcal{S}_{CL},P):\boldsymbol{F}(P,X,X)=-6x_0x_1-6x_0x_2-6x_1x_2-6x_2^2-6x_0x_3-6x_1x_3-6x_2x_3-3x_3^2=0$
$\pi_1(\mathcal{S}_{CL},Q):\boldsymbol{F}(Q,X,X)=-6x_0x_1+9x_1^2-6x_0x_2-6x_1x_2-15x_2^2-6x_0x_3-6x_1x_3-6x_2x_3-3x_3^2=0$
$\pi_1(\mathcal{S}_{CL},W):\boldsymbol{F}(W,X,X)=-6x_0x_1+6x_1^2-6x_0x_2-6x_1x_2-12x_2^2-6x_0x_3-6x_1x_3-6x_2x_3-3x_3^2=0$
$\pi_1(\mathcal{S}_{CL},R):\boldsymbol{F}(R,X,X)=-6x_0x_1+18x_1^2-6x_0x_2-6x_1x_2-24x_2^2-6x_0x_3-6x_1x_3-6x_2x_3-3x_3^2=0.$

Let $t$ be a line through points $P$ and $Q$. Each point on $t$ has homogenous coordinates $k_1Q+k_2P$ for $(k_1,k_2)\in\mathbb{R}^2\setminus\{(0,0)\}$. The intersection points of $t$ and $\mathcal{S}$ are the roots of the polynomial after the Taylor expansion

$$\begin{aligned}\boldsymbol{F}_n(k_1Q+k_2P)=k_2^n\boldsymbol{F}_n(P)+\frac{k_2^{n-1}k_1}{1!}\boldsymbol{F}(\underbrace{P,\ldots,P}_{\times(n-1)},Q)+\ldots\\+\frac{k_2k_1^{n-1}}{(n-1)!}\boldsymbol{F}(P,\underbrace{Q,\ldots,Q}_{\times(n-1)})+\frac{k_1^n}{n!}\boldsymbol{F}_n(Q)=0.\end{aligned}\tag{4.1}$$

If the projective coordinates $(k_1',k_2')$ of a point $T$ on $t$ are a double root of the polynomial (4.1), we call $T$ a double point of intersection of $t$ and $\mathcal{S}$. A line, for which a regular point of a surface is at least a double point of intersection, is called a *tangent line*, and the point is its *tangent point*. If each line of a plane through the same point is a tangent line of a surface, the plane is said to be a *tangent plane* at this point.

[22] A combinatorial proof can be found in [Bydžovský, 1948], Section 31, p. 70-74.

**Lemma 4.2.1.** *The last polar of a regular point with respect to a surface is the tangent plane of the surface at this point.*

*Proof.* Let $T$ be a regular point on the surface $\mathcal{S} : \boldsymbol{F}_n(T) = 0$, and $Q$ be a point not on $\mathcal{S}$, but on $\pi_{n-1}(\mathcal{S}, T)$, i.e.,

$$\boldsymbol{F}(\underbrace{T, \ldots, T}_{\times(n-1)}, Q) = 0.$$

Then the substitution of the line $\overline{TQ}$ into (4.1) implies $k_2 = 0$ as a root with a multiplicity of at least 2. Hence the point $T$ is at least a double point of this intersection. □

**Lemma 4.2.2.** *Each polar of a regular point $P$ on a surface with respect to this surface contains $P$, and $P$ is the tangent point of the common tangent plane of all polars.*

*Proof.* The equation of a surface $\mathcal{S}$ can be rearranged with respect to the last coordinate

$$\boldsymbol{F}_n(X) = x_0^n u_0 + x_0^{n-1} u_1 + \cdots + x_0 u_{n-1} + u_n = 0,$$

where the $u_i$ are forms of degree $i$ in $x_1, \ldots, x_m$. Assume $O_0 = (0, \ldots, 0, 1)$ to be a regular point of $\mathcal{S}$.[23] Substituting $O_0$ into $X$ will vanish all forms $u_1, \ldots, u_n$. It holds that

$$\boldsymbol{F}_n(O_0) = u_0 = 0.$$

Therefore the equation of $\mathcal{S}$ (through $O_0$) simplifies to

$$\boldsymbol{F}_n(X) = x_0^{n-1} u_1 + \cdots + x_0^{r+1} u_{n-r-1} + x_0^r u_{n-r} + \cdots + x_0 u_{n-1} + u_n = 0,$$

and

$$\pi_r(\mathcal{S}, O_0) : (n-1)\ldots(n-r) x_0^{n-1-r} u_1 + (n-2)\ldots(n-r-1) x_0^{n-r-2} u_2 + \\ \cdots + (r+1)\ldots 2 x_0 u_{n-r-1} + r(r-1)\ldots 1 u_{n-r} = 0.$$

The point $O_0$ certainly satisfies the equation of this polar. The last polar of $O_0$ on $\mathcal{S}$ is the polar hyperplane

$$\pi_{n-1}(\mathcal{S}, O_0) : (n-1)! u_1 = 0,$$

and it is obviously the last polar for each of the previous polars. □

**Theorem 4.2.1.** *The pencil of first polars through a non-torsal line of an algebraic surface in $\mathbb{RP}^m$ for $m \geq 3$ is in projectivity with the range of their contact points.*

*Proof.* Let $\mathcal{S}$ be an algebraic surface of degree $n \geq 2$ and $p$ be a non-torsal line of this surface. Let $P$ and $Q$ be distinct regular points on $p$ with different tangent planes. The tangent plane to $\mathcal{S}$ at $P$ contains all tangent lines through $P$, and therefore it also contains the line $p$ and the point $Q$. Applying Lemmas 4.2.1 and 4.2.2, each polar of the points $P$ and $Q$ with respect to $\mathcal{S}$ must also contain $p$ (Figure 4.5). Furthermore, the line $p$ is the axis of the pencil of $r$-th polars of the points in the range $k_1 Q + k_2 P$.

[23] There is always a projective transformation, which maps the considered point to $O_0$.

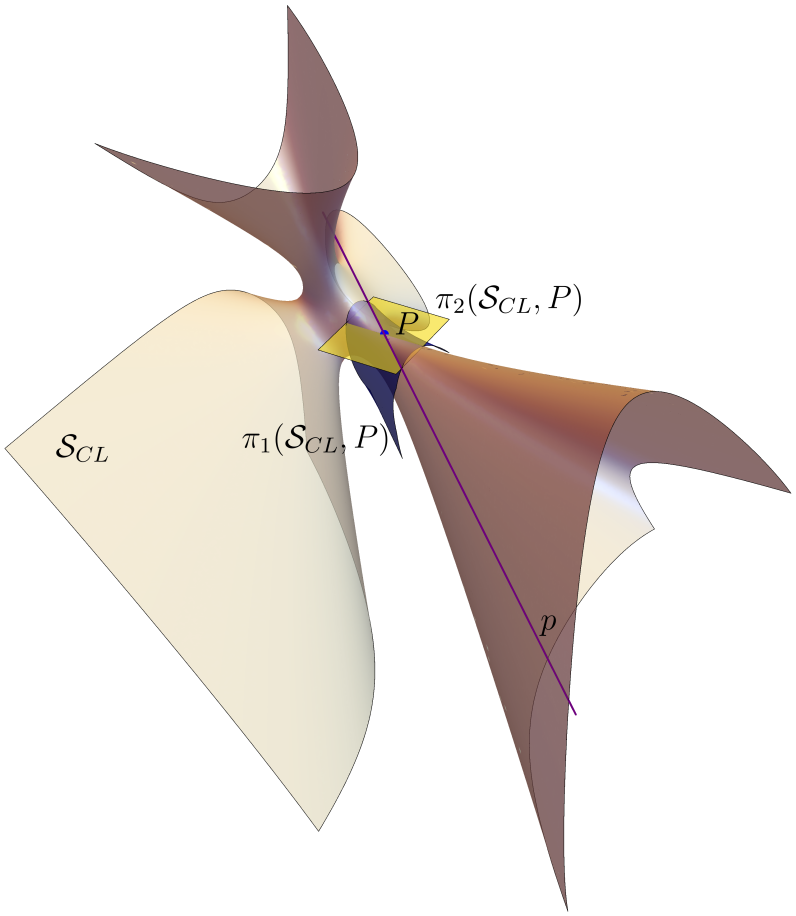



Figure 4.5: First and last polar at the point $P$.

The last step is to show that there is a projectivity between the range of points on $p$ and their first polars. Let $W$ be a point in the range $k_1Q + k_2P$ with the projective coordinates $(k_1, k_2)$. The first polar of $W$ with respect to $\mathcal{S}$ is

$$\begin{aligned}\pi_1(\mathcal{S}, W) : \boldsymbol{F}(W, \underbrace{X, \ldots, X}_{\times(n-1)}) = \boldsymbol{F}(k_1Q + k_2P, \underbrace{X, \ldots, X}_{\times(n-1)}) = \\ = k_1\boldsymbol{F}(Q, \underbrace{X, \ldots, X}_{\times(n-1)}) + k_2\boldsymbol{F}(P, \underbrace{X, \ldots, X}_{\times(n-1)}) = 0.\end{aligned} \tag{4.2}$$

It is a surface of degree $(n-1)$ in the one-dimensional form (pencil of surfaces) with the same projective coordinates $(k_1, k_2)$ as the point $W$ in its range. Therefore there is a projectivity between the range of points on a non-torsal line and the pencil of first polars of these points with respect to $\mathcal{S}$ (Figure 4.6). □

Let us verify the validity of the previous theorem on Example 4.2.1:
The point $W = (3, -3, 0, 1)$ in the range $k_1P + k_2Q$ for $P = (1, -1, 0, 1)$ and $Q = (4, -4, 0, 1)$ has the projective coordinates $(k_1, k_2) = (\frac{1}{3}, \frac{2}{3})$. The same holds for the first polars:

$k_1\boldsymbol{F}(P, X, X) + k_2\boldsymbol{F}(Q, X, X) = \frac{1}{3}(-6x_0x_1 - 6x_0x_2 - 6x_1x_2 - 6x_2^2 - 6x_0x_3 - 6x_1x_3 - 6x_2x_3 - 3x_3^2) + \frac{2}{3}(-6x_0x_1 + 9x_1^2 - 6x_0x_2 - 6x_1x_2 - 15x_2^2 - 6x_0x_3 - 6x_1x_3 - 6x_2x_3 - 3x_3^2) = -6x_0x_1 + 6x_1^2 - 6x_0x_2 - 6x_1x_2 - 12x_2^2 - 6x_0x_3 - 6x_1x_3 - 6x_2x_3 - 3x_3^2 = \boldsymbol{F}(W, X, X).$

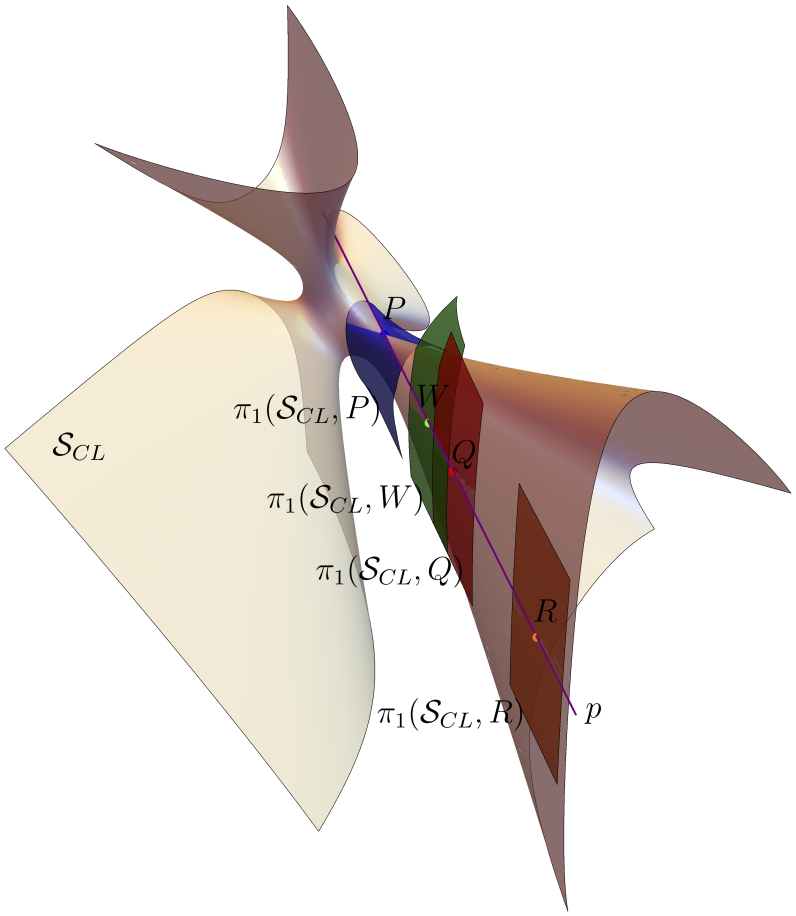



Figure 4.6: First polars of four distinct points on a non-torsal line.

The point $R = l_1P + l_2Q$ and its first polar have projective coordinates $(l_1, l_2) = (-1, 2)$ in their one-dimensional forms.

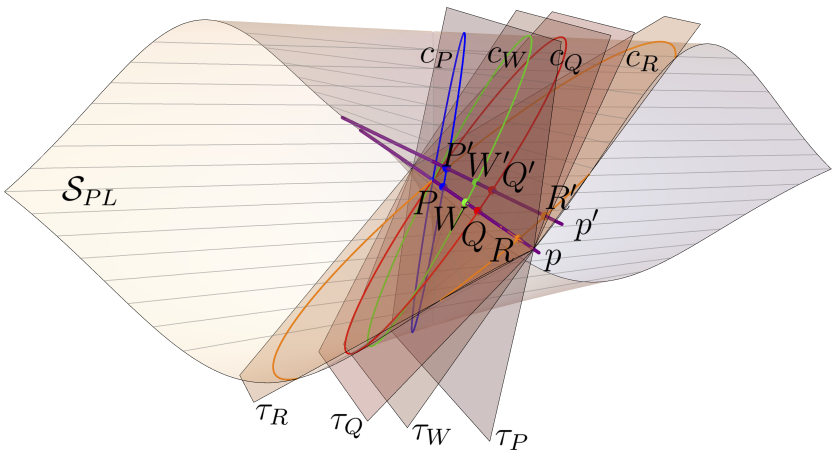



Figure 4.7: A pencil of tangent planes at a range of points on a ruling.

**Example 4.2.2.** For the visualization (Figure 4.7) the following equations of

the Plücker conoid $\mathcal{S}_{PL}$ are used:

$$\text{in } \mathbb{E}^3 : z(x^2+y^2) - 4(x^2-y^2) = 0,$$
$$\text{in } \mathbb{RP}^3 : x_1^2x_3 - 4x_1^2x_0 + x_2^2x_3 + 4x_2^2x_0 = 0.$$

In the upcoming Chasles's theorem, we need the following property: through each regular point of a non-developable ruled surface of degree $n > 2$ passes exactly one ruling[24].

**Theorem 4.2.2** (Chasles's theorem)**.** *The pencil of tangent planes at the points of a non-torsal line of a ruled surface in $\mathbb{RP}^m$ for $m \geq 3$ is in projectivity with the range of their contact points.*

*Proof.* For surfaces of 2nd degree the theorem is a straightforward consequence of Theorem 4.2.1. Let $\mathcal{S}$ be a surface of degree $n > 2$ and $P$ and $Q$ be regular points which lie on a ruling $p$. At each regular point on $p$ there is only one tangent plane of $\mathcal{S}$ — its last polar. Dually, each tangent plane through $p$ touches $\mathcal{S}$ in one contact point. Let $\tau_P$ and $\tau_Q$ be tangent planes of $\mathcal{S}$ at $P$ and $Q$, respectively. Furthermore let $\tau_W$ be a plane in the pencil $k_1\tau_P + k_2\tau_Q$ with the projective coordinates $(k_1, k_2)$ and axis $p$. We prove that $W = k_1P + k_2Q$.

Let $p'$ be another ruling of $\mathcal{S}$ such that $p'$ intersects pencil of planes $\tau_W = k_1\tau_P + k_2\tau_Q$ in regular points $W' = k_1P' + k_2Q'$. The planes $\tau_P, \tau_Q$, and $\tau_W$ cut $\mathcal{S}$ in $p$ and in plane curves $c_P, c_Q$, and $c_W$ of degree $n-1$, respectively. $P', Q'$, and $W'$ are the intersections of $c_P, c_Q$, and $c_W$ with $p'$. Since $n > 2$, no other ruling passes through the points $P', Q'$, and $W'$. This holds for any choice of $p' \neq p$. Each algebraic curve of degree $n$ is determined by $\frac{n(n+3)}{2}$ points. If we construct $\frac{n(n+3)}{2}$ rulings, then for each point on $c_W$ we obtain $c_W = k_1c_P + k_2c_Q$. The points $P, Q$ and $W$ are intersections of $c_P, c_Q$, and $c_W$ with $p$, and therefore so $W = k_1P + k_2Q$. □

## 4.2.2 Conclusion

Chasles's theorem is, by its nature, a great result in projective geometry. However, proofs of the theorem have always slipped into the use of more powerful methods of infinite closeness in the 19th century, or projective differential and differential geometry nowadays. In this section the theorem was proven purely projectively and generalized to non-torsal lines of algebraic surfaces in the projective extension of the real space. The use of the language of algebraic projective geometry has brought Chasles's theorem back to its place in projective geometry. This also leads to a question: "Is there a general method to avoid the use of non-projective methods for theorems of projective geometry originally proven with the use of infinite closeness?". The importance of purification is given not only in algebraic but especially in synthetic geometry, as the computer 3D graphics has opened the gate to visualizing and manipulating objects, which are almost impossible to draw by hand. Projective geometry is therefore a convenient tool for computers, since one obtains precise results, not based on numerical methods.

[24]The statement can be found in general literature on geometry of ruled surfaces and we will not prove it here (see [Bydžovský, 1948, Pottmann and Wallner, 2009]).

## 4.3 The synthetic classification of regular quadrics

### 4.3.1 Introduction

Quadrics are often considered to be a generalization of conics. However, when we define conics, probably the first intuitive picture is a conic section of a circular cone. We use a similar method to derive the regular (or proper, non-degenerate) real quadrics as quadric sections. Our focus lies on the synthetic constructions given in Section 4.3.3, where models of quadrics are depicted. We visualize a four-dimensional cone, its cutting 3-space, and the section simultaneously in the double orthogonal projection onto two perpendicular three-dimensional spaces. The method of projection and constructions of sections of four-dimensional polytopes are discussed in Chapter 3. For understanding visualizations, the reader should have knowledge of Monge's projection introduced in [Monge, 1799] and following textbooks on descriptive geometry. Since synthetic properties of conic sections are often used, we suggest [Glaeser et al., 2016] for further details. To follow the examination of quadrics in homogenous coordinates in Section 4.3.2 see textbooks [Semple and Kneebone, 1952, Part II, Chapter XI] and [Casse, 2006, Chapter 8]. For even more general and detailed approach see the recent book [Casas-Alvero, 2014, Chapters 6 and 7].

Apart from the maybe difficult to digest synthetic-visual four-dimensional part of the section, we support our reasoning with analytic derivation in Section 4.3.2.[25] The synthetic part is realised in the four-dimensional Euclidean space with coordinates $x, y, z$, and $w$. The analytic approach is held in the framework of projective extension of the real (Euclidean) space $\mathbb{RP}^4$, where we use homogenous coordinates $x_1, x_2, x_3, x_4$, and $x_0$ (we assume an infinite point to have the last coordinate $x_0 = 0$). To describe orthogonal projections, we need in many cases only the three-dimensional subspaces $\Xi(x, y, z)$ and $\Omega(x, y, w)$ in their Euclidean and also extended versions $\Xi(x_1, x_2, x_3, x_0)$ and $\Omega(x_1, x_2, x_4, x_0)$.

Throughout this section a quadratic cone is frequently abbreviated to cone, conical surfaces and hypersurfaces are also simplified as cones with respect to the current context.

### 4.3.2 The analytic approach

In the four-dimensional projective extension of the real space $\mathbb{RP}^4$ with homogenous coordinates $x_1, x_2, x_3, x_4$, and $x_0$, the reduced homogenous equation of a quadric $\Gamma$ with a unique singular point - four-dimensional ordinary quadratic cone, has one of these forms:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \tag{4.3}$$

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0, \tag{4.4}$$

[25] It is not absolutely necessary for a reader well-experienced in the double orthogonal projection to follow the analytic part. However, the analytic part is recommended for verifications of intuitive false conclusions.

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0. \tag{4.5}$$

If the quadric is represented by Equation 4.3, it is a point (or an imaginary cone), and it will not play any role in this thesis. Equations 4.4 and 4.5 represent an ellipsoidal cone and one-sheet hyperboloidal cone, respectively. The vertex – singular point of the cones has homogenous coordinates (0,0,0,0,1). For simplicity of further manipulations, we translate the vertex into the point $V = (0, 0, 1, 1, 1)$.

**Quadric sections of an ellipsoidal cone**

The ellipsoidal cone $\Gamma$ with the vertex $V$ has the homogenous equation:

$$x_1^2 + x_2^2 + (x_3 - x_0)^2 - (x_4 - x_0)^2 = 0. \tag{4.6}$$

Sections of this ellipsoidal cone with 3-spaces $x_4 = \alpha x_1$, for $\alpha \in \mathbb{R}$, which are perpendicular to the 3-space $\Omega(x_1, x_2, x_4, x_0)$, are represented by the equation dependent on the parameter $\alpha$:

$$\begin{aligned} x_1^2 + x_2^2 + (x_3 - x_0)^2 - (\alpha x_1 - x_0)^2 = 0 \\ x_1^2(1 - \alpha^2) + 2\alpha x_1 x_0 + x_2^2 + x_3^2 - 2x_3 x_0 = 0. \end{aligned} \tag{4.7}$$

We can interpret the last equation as the equation of a quadric in the 3-space $\Xi(x_1, x_2, x_3, x_0)$. The determinant of the matrix of this quadric equals -1 and is independent on $\alpha$. In the affine classification, the quadric is an ellipsoid $\varepsilon$, elliptic paraboloid $\phi$, or two-sheet hyperboloid $\eta$. To identify the affine type, we consider the set of all its improper points with the homogenous element $x_0 = 0$. Equation 4.7 reduces to

$$x_1^2(1 - \alpha^2) + x_2^2 + x_3^2 = 0. \tag{4.8}$$

For $\alpha \in (-1, 1)$, there is only the trivial solution $(0, 0, 0, 0)$, which is not a point, and the quadric is an ellipsoid $\varepsilon$. For $\alpha = \pm 1$, the set of solutions is the point $(1, 0, 0, 0)$ and the quadric is an elliptic paraboloid $\psi$. In the other cases $\alpha \in (-\infty, -1) \cup (1, \infty)$, the set of improper points is an ellipse, and the type of the quadratic is a two-sheet hyperboloid $\eta$.

In Section 4.3.3, we visualize the ellipsoidal cone $\Gamma$ in the double orthogonal projection onto the perpendicular 3-spaces $\Xi(x_1, x_2, x_3, x_0)$ and $\Omega(x_1, x_2, x_4, x_0)$. In the orthogonal projections, apparent contours $\Gamma_3$ and $\Gamma_4$ of the ellipsoidal cone $\Gamma$ with Equation 4.6 in the 3-spaces $\Xi(x_1, x_2, x_3, x_0)$ and $\Omega(x_1, x_2, x_4, x_0)$ can be in the projective extension determined from its contour generators, which are the intersections of the ellipsoidal cone and polar space of the direction of the projection – improper pole, with respect to the ellipsoidal cone $\Gamma$. The homogenous coordinates of the polar space of the point $Z_\infty = (0, 0, 1, 0, 0)$ with respect to the ellipsoidal cone $\Gamma$, represented by the matrix $\mathbf{A}$, is derived as follows:

$$Z_\infty^T \mathbf{A} = (0, 0, 1, 0, 0) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix} = (0, 0, 1, 0, -1). \tag{4.9}$$

The polar 3-space of $Z_\infty$ with respect to the ellipsoidal cone $\Gamma$ has the homogenous equation $x_3 - x_0 = 0$. The intersection of this polar 3-space with the ellipsoidal cone reduces after the substitution $x_3 = x_0$ into the equation of the contour generator:

$$x_1^2 + x_2^2 - (x_4 - x_0)^2 = 0. \tag{4.10}$$

Independently of the $x_3$ coordinate, Equation 4.10 of a real quadratic cone in the 3-space $\Omega(x_1, x_2, x_4, x_0)$ is also the apparent contour $\Gamma_3$ of the ellipsoidal cone in the 3-space $\Omega(x_1, x_2, x_4, x_0)$ (its $\Omega$-image).

The apparent contour $\Gamma_4$ in the 3-space $\Xi(x_1, x_2, x_3, x_0)$ is the whole 3-space, and therefore we visualize only the images of intersections of the ellipsoidal cone with the 3-spaces given by equations $x_4 = 0$ and $x_4 = 2x_0$. After substitutions into Equation 4.4, we obtain two overlapping ellipsoids in the 3-space $\Xi(x_1, x_2, x_3, x_0)$ with the equation

$$x_1^2 + x_2^2 + (x_3 - x_0)^2 - x_0^2 = 0. \tag{4.11}$$

The last remark also explains the name ellipsoidal cone.

### Quadric sections of a one-sheet hyperboloidal cone

Equation 4.5 of the one-sheet hyperboloidal cone $\Gamma$ after the translation of the vertex $V$ becomes:

$$x_1^2 + x_2^2 - (x_3 - x_0)^2 - (x_4 - x_0)^2 = 0. \tag{4.12}$$

Sections of this cone with the 3-spaces given by the parametric equation $x_4 = \alpha x_1$, for $\alpha \in \mathbb{R}$, reduce Equation 4.12 into:

$$\begin{aligned} x_1^2 + x_2^2 - (x_3 - x_0)^2 - (\alpha x_1 - x_0)^2 = 0 \\ x_1^2(1 - \alpha^2) + 2\alpha x_1 x_0 + x_2^2 - x_3^2 + 2x_3 x_0 - 2x_0^2 = 0. \end{aligned} \tag{4.13}$$

In the 3-space $\Xi(x_1, x_2, x_3, x_0)$, Equation 4.13 represents a quadric with the determinant of its matrix equal to 1. The quadric is therefore in the affine classification a one-sheet hyperboloid $\eta$ or hyperbolic paraboloid $\psi$. Examination of the set of improper points of this quadric leads to the following parametric equation:

$$x_1^2(1 - \alpha^2) + x_2^2 - x_3^2 = 0. \tag{4.14}$$

For $\alpha \neq \pm 1$ the set of improper points is an ellipse and the quadric is a one-sheet hyperboloid $\eta$. Otherwise, for $\alpha = \pm 1$, we obtain two intersecting lines of some hyperbolic paraboloid $\psi$.

If we take a better look at Equation 4.5, the symmetry in the $x_3$ and $x_4$ coordinates implies the following procedure to find the apparent contours of $\Gamma$. The apparent contour $\Gamma_4$ in the 3-space $\Omega(x_1, x_2, x_4, x_0)$ is derived from the polar plane of $Z_\infty$, with the homogenous coordinates

$$Z_\infty^T \mathbf{A} = (0, 0, 1, 0, 0) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} = (0, 0, -1, 0, 1). \tag{4.15}$$

Substituting $x_3 = x_0$ into Equation 4.12, we obtain the following equation

$$x_1^2 + x_2^2 - (x_4 - x_0)^2 = 0 \tag{4.16}$$

of a cone $\delta'$ in the 3-space $\Omega(x_1, x_2, x_4, x_0)$. Opposed to the ellipsoidal cone in Section 4.3.2, we also restrict the visualization in the 3-space $\Omega(x_1, x_2, x_4, x_0)$ between the 3-spaces parallel to $\Xi(x_1, x_2, x_3, x_0)$ with the equations $x_3 = 0$ and $x_3 = 2x_0$. Both substitutions reduce Equation 4.12 into

$$x_1^2 + x_2^2 - (x_4 - x_0)^2 - x_0^2 = 0 \tag{4.17}$$

of overlapping one-sheet hyperboloids $\delta$ in the 3-space $\Omega(x_1, x_2, x_4, x_0)$. The points of the apparent contour $\Gamma_4$ of the one-sheet hyperboloidal cone $\Gamma$ lie between the cone $\delta'$ with Equation 4.16 and one-sheet hyperboloid $\delta$ with Equation 4.17. The substitution $x_0 = 0$ into Equations 4.16 and 4.17 leads to a conic, which is the set of improper points of the cone $\delta'$ and the one-sheet hyperboloid $\delta$. Therefore $\delta'$ is the asymptotic cone of the one-sheet hyperboloid $\delta$.

The apparent contour $\Gamma_3$ in the 3-space $\Xi(x_1, x_2, x_3, x_0)$ is infinite and we restrict our visualization between the parallel 3-spaces, $x_4 = 0$ and $x_4 = 2x_0$, as in the ellipsoidal case in Section 4.3.2. After substitutions of the last formulas into Equation 4.12, we obtain two overlapping one-sheet hyperboloids $\beta_3$ with the equation:

$$x_1^2 + x_2^2 - (x_3 - x_0)^2 - x_0^2 = 0. \tag{4.18}$$

In this case, the 3-space with the equation $x_4 = x_0$ also plays a role. Its intersection with the one-sheet hyperboloidal cone $\Gamma$ is the elliptic cone $\beta_3'$ in the 3-space $\Xi(x_1, x_2, x_3, x_0)$ (in case of an ellipsoidal cone, we would obtain only the vertex $V$) with its equation:

$$x_1^2 + x_2^2 - (x_3 - x_0)^2 = 0. \tag{4.19}$$

One more useful remark is seen if we describe the sets of infinite points of the hyperboloid $\beta_3$ and cone $\beta_3'$ with Equations 4.18 and 4.19. Since both sets of improper points are equal, we deduce that the cone $\beta_3'$ is the asymptotic cone of the one-sheet hyperboloid $\beta_3$. The $\Xi$-image $\Gamma_3$ of the one-sheet hyperboloidal cone $\Gamma$ is therefore a set of points between the elliptic cone $\beta_3'$ given by Equation 4.19 and the one-sheet hyperboloid $\beta_3$ with Equation 4.18.

### 4.3.3 Synthetic approach

We derive the above results with the use of descriptive geometry tools described in Chapter 3. In the upcoming GeoGebra models, the user has a possibility to move some input elements, and therefore, the analytic section with the equations reduced to simple formulas should be assumed as illustrative. The constructions are divided into two cases. First, we construct a spherical cone as a special case of the ellipsoidal cone and its quadric sections — an ellipsoid, two-sheet hyperboloid and their limiting elliptic paraboloid. Then, a one-sheet hyperboloid and hyperbolic paraboloid are derived from sections of a one-sheet hyperboloidal cone. The clarity of necessity of two kinds of cones is also more obvious, if we think about a quadric as a collinear image of another quadric. Sections of four-dimensional cone with a 3-space, not passing through the vertex, are collinear

images of its directrix. Collineation preserves collinearity, thus, if the directrix is a sphere, its possible collinear images are unruled regular quadrics – an ellipsoid, two-sheet hyperboloid and elliptic paraboloid. If the directrix is a one-sheet hyperboloid, the quadric sections are ruled quadrics – a one-sheet hyperboloid or hyperbolic paraboloid.

### A spherical cone

A spherical cone may be created as a set of joining lines of all the points of a spherical directrix with a vertex point noncospatial with the points of the sphere. Let us visualize a right spherical cone $\Gamma$ and its quadric section in the special position. Let the directing sphere be placed in the 3-space $\Xi(x, y, z)$. The vertex $V$ lies on the axis $o$ perpendicular to the 3-space $\Xi(x, y, z)$ of the directing sphere through its center $S$. Further on, the spherical cone is restricted between the 3-space $\Omega(x, y, w)$ and its parallel 3-space at the same distance from the vertex $V$. The apparent contour of the spherical cone in the 3-space $\Omega(x, y, w)$, its $\Omega$-image, is the double-cone $\Gamma_4$ of revolution with the axis $o_4$ perpendicular to the reference plane $\pi(x, y)$. The angle $\kappa$ between a generatrix and the axis $o$ of the spherical cone appears in its true size in the $\Omega$-image in the modeling space. The $\Xi$-image is the sphere $\Gamma_3$ with the $\Xi$-image $o_3$ of the whole axis located in the center $S$ of the sphere. The apparent contours were described analytically with the expressions 4.10 and 4.11.

The cutting 3-space $\Sigma$ not incident with the vertex $V$ is given by its $\Xi$ and $\Omega$-traces, $\xi_3^\Sigma$ and $\omega_4^\Sigma$, respectively. We choose the 3-space $\Sigma$ perpendicular to the 3-space $\Omega$. Therefore the $\Xi$-trace $\xi_3^\Sigma$ is perpendicular to the reference plane $\pi(x, y)$, and the angle $\lambda$ between the $\Omega$-trace $\omega_4^\Sigma$ and the axis $o_4$ of the cone is the true size of the angle between the axis $o$ and the 3-space $\Sigma$. The relation between the angles $\kappa$ and $\lambda$ is equivalent to the choice of the parameter $\alpha$ in Equation 4.8 in the analytic approach.

**An ellipsoid** In case of $\lambda > \kappa$ ($\alpha \in (-1, 1)$) the quadric section of the spherical cone $\Gamma$ with the 3-space $\Sigma$ is an ellipsoid of revolution $\varepsilon$ (Figure 4.8). Synthetically we construct its center, vertices, foci, and symmetry axes. Since the 3-space $\Sigma$ is perpendicular to the 3-space $\Omega(x, y, w)$, the $\Omega$-image of each point in the 3-space $\Sigma$ lies in the $\Omega$-trace $\omega_4^\Sigma$. Hence the $\Omega$-image $\varepsilon_4$ of the section is the ellipse $e_4$ of intersection of $\Gamma_4$ and $\omega_4^\Sigma$ with the inner points of the ellipse.

The vertices $A_4$ and $B_4$ of the ellipse $e_4$ lie on the $\Xi$-slope line $s^\Sigma$ of the 3-space $\Sigma$, which intersects the $\Omega$-image $o_4$ of the axis of $\Gamma$ in the point $O_4$. The line $s_4^\Sigma$ is the major axis of the ellipse $e_4$. The center of the segment $A_4B_4$ is the $\Omega$-image $Z_4$ of the center $Z$ of the ellipsoid $\varepsilon$. The $\Xi$-image $e_3$ lies on the lines of recall through $e_4$ and in the $\Omega$-plane $\varphi^\Sigma$ of $\Sigma$ through the point $O$ on the axis $o$. The $\Xi$-image $\varphi_3^\Sigma$ of the $\Omega$-plane is parallel to the reference plane $\pi(x, y)$ and includes the center $S_3$ of the sphere $\Gamma_3$. The $\Xi$-image $s_3^\Sigma$ of the $\Xi$-slope line is the intersection line of its plane of recall and the plane $\varphi_3^\Sigma$. The parallel projection implies that the line $s_3^\Sigma$ is the principal axis of the $\Xi$-image $\varepsilon_3$ of the quadric section $\varepsilon$. Moreover the $\Xi$-images of the vertices $A, B$, and the center $Z$ are constructed as the intersections of their lines of recall and the $\Xi$-image $\varphi_3^\Sigma$ of the $\Omega$-plane. The minor axis of the ellipse $e_4$ lies in the $\Omega$-trace $\omega_4^\Sigma$, passes

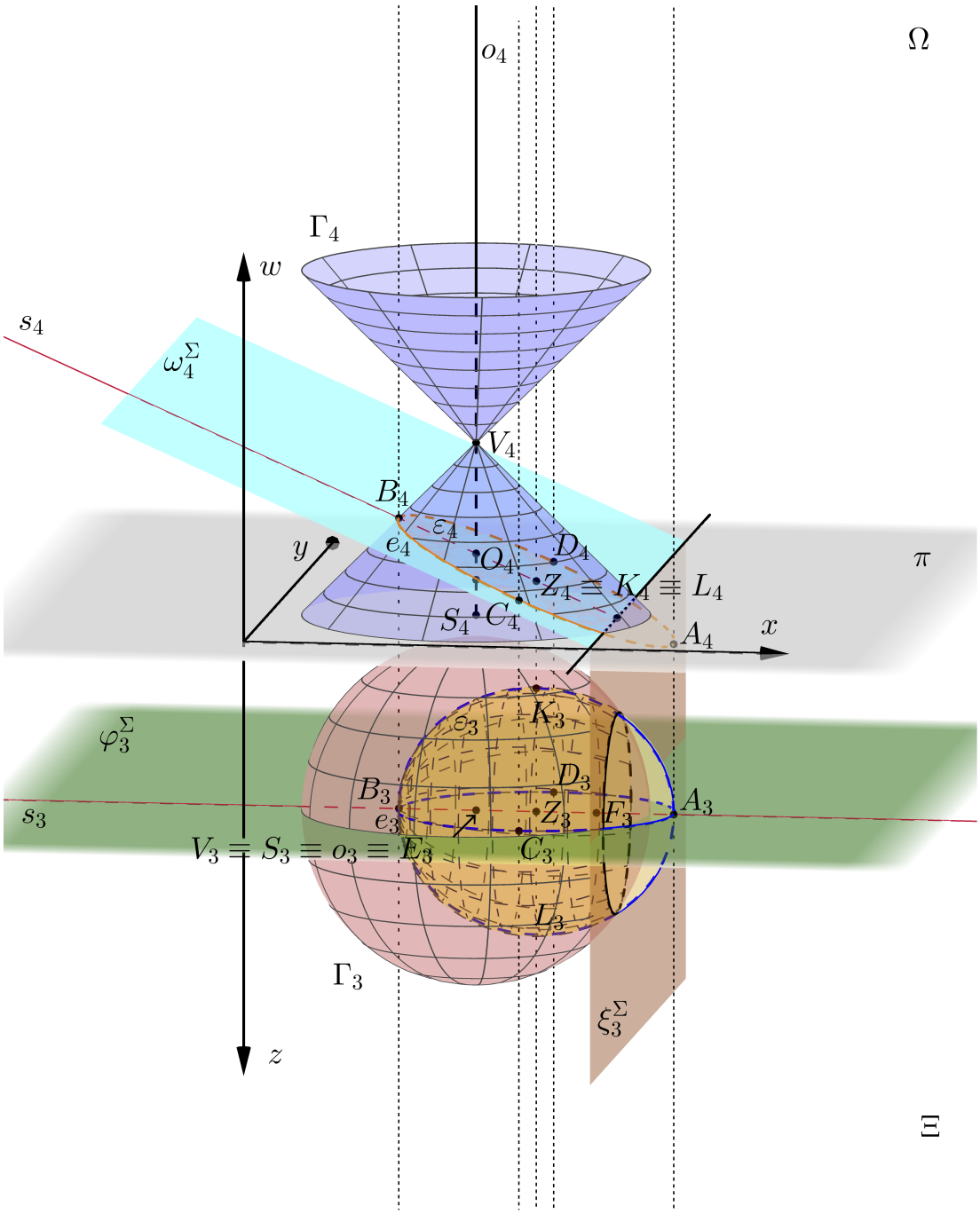



Figure 4.8: A projection of an elipsoid $\varepsilon$ as a quadric section of a spherical cone $\Gamma$ with a 3-space $\Sigma$.
Interactive model: `https://www.geogebra.org/m/dfgqzv3b`

through the center $Z_4$ of the ellipse and is perpendicular to the major axis $s_4^\Sigma$. The co-vertices $C_4$ and $D_4$ are the intersections of the cone $\Gamma_4$ and the minor axis of the ellipse $e_4$. Their $\Xi$-images $C_3$ and $D_3$ are the intersecting points of their lines of recall and the $\Omega$ plane $\varphi_3^\Sigma$. In this point, we have all the vertices, center and axes of the ellipse $e_3$, and its construction is elementary.

Let us consider a chord of the ellipse $e_4$ parallel to its minor axis $C_4D_4$. In the $\Omega$-image, the chord lies in the plane parallel to the reference plane $\pi(x, y)$. The section of this parallel plane and the spherical cone is a sphere in the 4-space,

which appears in its true size in the 3-space $\Xi(x, y, z)$. From these thoughts, we conclude that the $\Xi$-image of the chord is a circle. Because of the symmetry of the ellipse $e_4$ in the 3-space $\Omega(x, y, w)$, the $\Xi$-image $\varepsilon_3$ is the ellipsoid of revolution with the principal axis $s_3^\Sigma$. The third principal axis $K_3L_3$ of the ellipsoid is perpendicular to the lines $\overline{A_3B_3}$ and $\overline{C_3D_3}$ through the center $Z_3$ with the semi-diameter equal to $C_3D_3$.

The construction of the foci $E_3$ and $F_3$ of the ellipsoid of revolution $\varepsilon$ is with the images of the vertices simple. However, one more remark can be added. The focus $E_3$ merges into the $\Xi$-image of the vertex $V$. In Monge's projection, the three-dimensional analogy — one focus of a conic section lies in the horizontal projection of the vertex, is a consequence of Dandelin's theorem. In our spherical cone, if we rotate the ellipse $e$ around its major axis $AB$, the resulting $\Xi$-image is the ellipse $e_3$ rotated around $s_3$, while the $\Omega$-image shrinks between the ellipse $e_4$ and the segment $A_4B_4$. The plane of rotated ellipse intersects the base sphere in a circle. Assuming this circle to be the base of a three-dimensional right circular cone with the vertex $V$, we can use Dandelin's theorem for all the positions in the rotation and the focus $E_3$ is fixed.

We have constructed the projections $\varepsilon_3$ and $\varepsilon_4$ of the whole ellipsoid $\varepsilon$. Apparently, only the parts of the ellipsoid with positive $w$ coordinate are visible. In the $\Xi$-image the visible points on $\varepsilon_3$ are those inside the sphere $\Gamma_3$, in the $\Omega$-image, the points "above" the reference plane $\pi(x, y)$ are visible.

**A two-sheet hyperboloid** The construction of a two-sheet hyperboloid of revolution $\eta$ (Figure 4.9) as a quadric section of a right spherical cone $\Gamma$ and a 3-space $\Sigma$ with an angle $\lambda < \kappa$ ($\alpha \in (-\infty, -1) \cup (1, \infty)$, see Equation 4.8 in Section 4.3.2) is carried out in the same fashion, and we leave the derivation to the kind reader. We add only a short note on a construction of the asymptotic cone of the $\Xi$-image $\eta_3$, the two-sheet hyperboloid of revolution in the 3-space $\Xi(x, y, z)$. To find the asymptotic cone, we construct the quadric section of the spherical cone $\Gamma$ with the 3-space parallel to $\Sigma$ through the vertex $V$, or differently we find the asymptotes of the hyperbola $h_3$ in the $\Xi$-image of the $\Omega$-plane $\varphi_3^\Sigma$ and rotate them around $s_3$.

**An elliptic paraboloid** In case of cutting the spherical cone with a 3-space $\Sigma$ for $\lambda = \kappa$ ($\alpha = \pm 1$ in Equation 4.8), the quadric section is a paraboloid of revolution (Figure 4.10). Let the spherical cone $\Gamma$ be in the same position as in the construction of the ellipsoidal quadric section. The 3-space $\Sigma$ is again perpendicular to the 3-space $\Omega(x, y, w)$. Therefore its $\Xi$-trace $\xi_3^\Sigma$ is perpendicular to the reference plane $\pi(x, y)$, and we may choose it arbitrarily. The condition of the equality of the angles $\kappa$ and $\lambda$ is satisfied if the $\Xi$-slope line $s$ is parallel to a generator of the spherical cone. We choose the generator $s'$ such that its $\Omega$-image $s'_4$ is the line through the point $V_4$ and perpendicular to the line of intersection of the $\Xi$-trace $\xi_3^\Sigma$ and the reference plane $\pi(x, y)$. The $\Omega$-trace $\omega_4^\Sigma$ is parallel to the $\Omega$-image $s'_4$ of the generator. The $\Omega$-image $s_4$ of the $\Xi$-slope line $s$, which intersects the axis $o$ in the point $O$, is the line in the plane $\omega_4^\Sigma$ parallel to the line $s'_4$.

The intersection of the $\Omega$-trace $\omega_4^\Sigma$ with the $\Omega$-image $\Gamma_4$ of the spherical cone is a parabola $p_4$. The $\Omega$-image of each point of the 3-space $\Sigma$ lies in $\omega_4^\Sigma$, and therefore

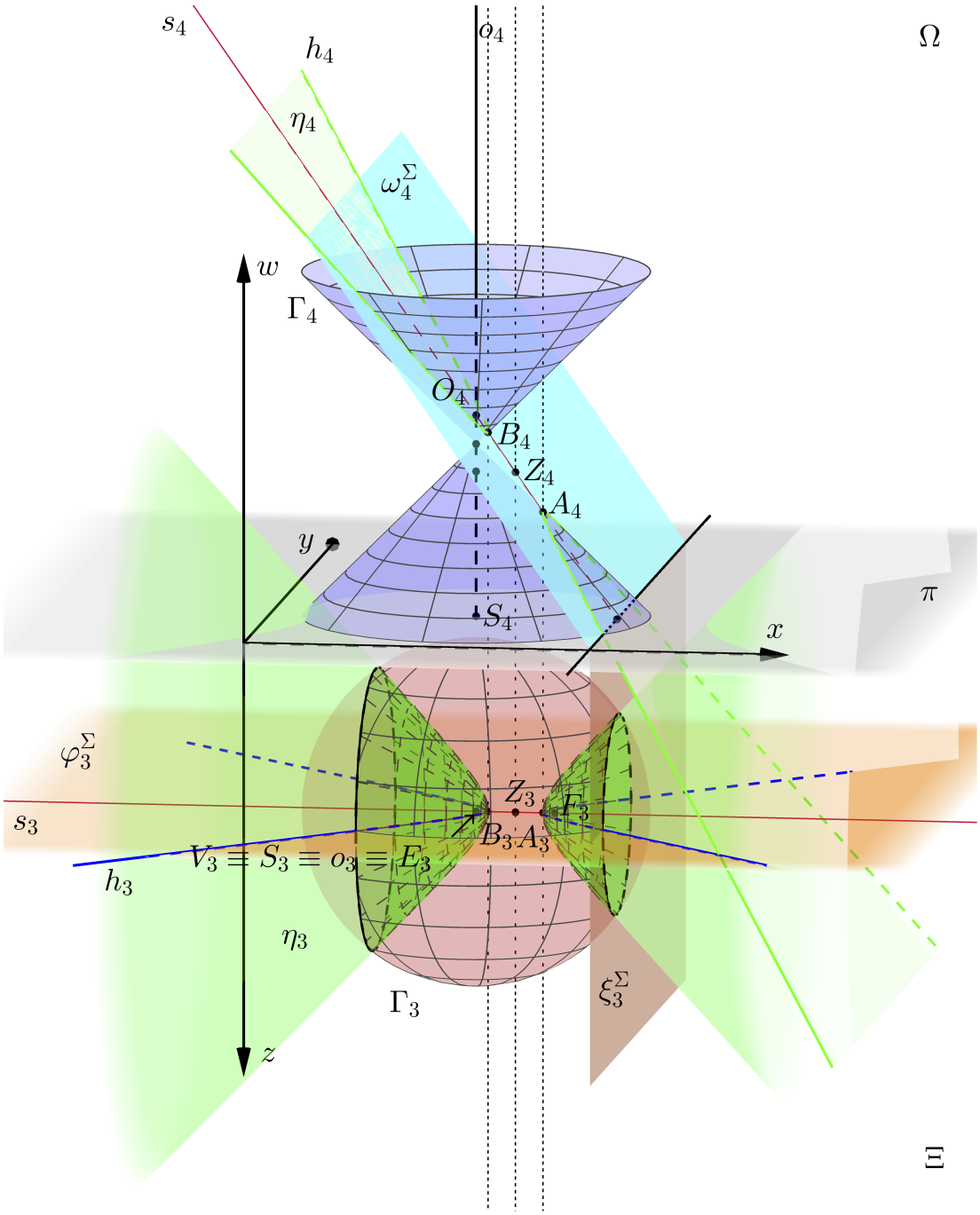



Figure 4.9: A projection of a two-sheet hyperboloid $\eta$ as a quadric section of a spherical cone $\Gamma$ with a 3-space $\Sigma$.
Interactive model: `https://www.geogebra.org/m/m6fyz3dq`

the parabola $p_4$ with its inner points forms the $\Omega$-image $\phi_4$ of the paraboloid $\phi$. The axis of the parabola $p_4$ is the line $s_4$ and its vertex $W_4$ is the intersection of $s_4$ and the cone $\Gamma_4$, and hence the axis of the paraboloid $\phi$ is the original line $s$, and its vertex is the point $W$.

From the $\Omega$-images we construct the $\Xi$-images $s_3$ and $W_3$ of the axis and vertex of the paraboloid $\phi$, which lie in the $\Omega$-plane $\varphi_3^\Sigma$ constructed analogically to the ellipsoidal case. As same as the previous cases, the focus $F_3$ of the paraboloid is the $\Xi$-projection $V_3$ of the vertex of the spherical cone $\Gamma$.

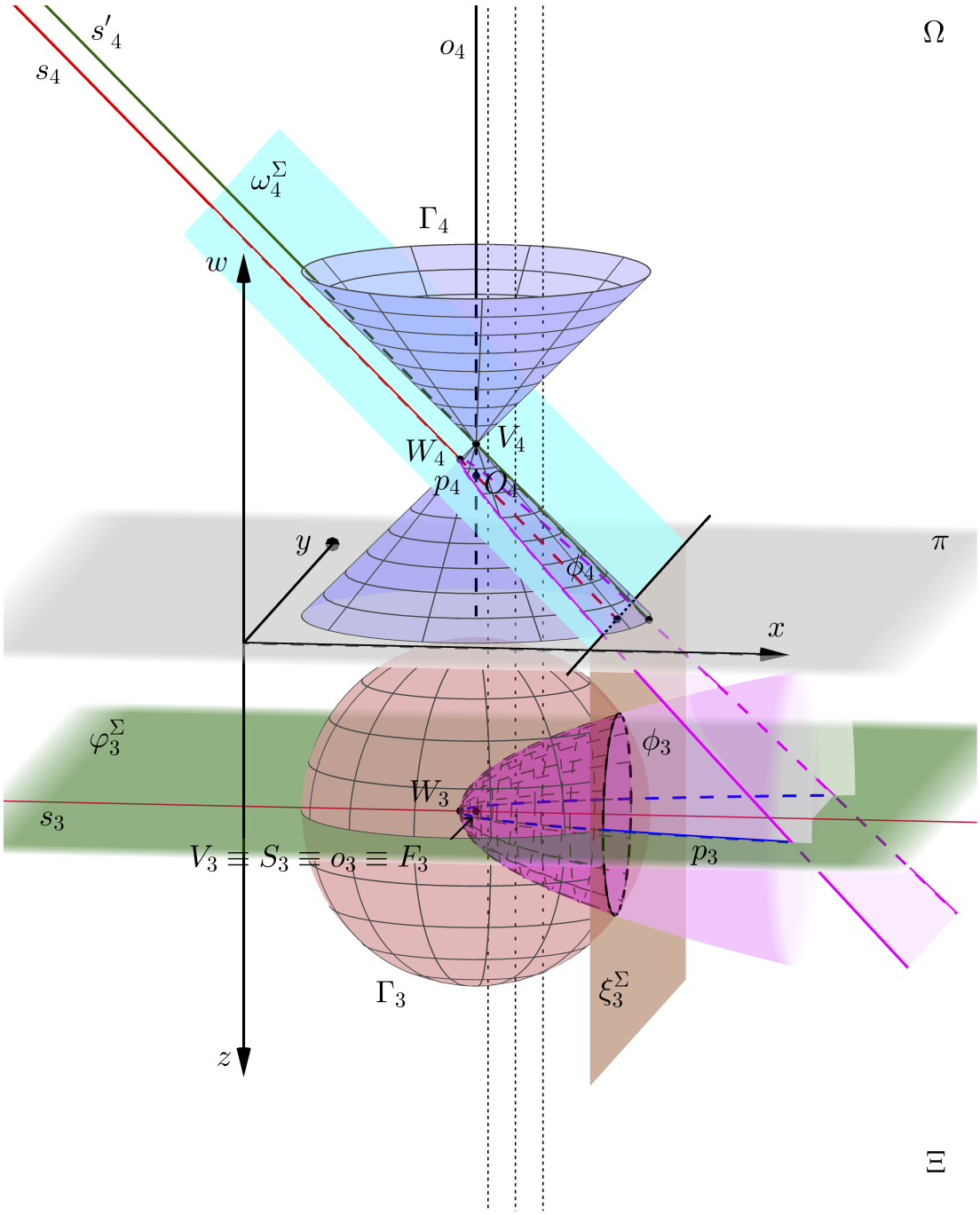



Figure 4.10: A projection of a paraboloid of revolution $\psi$ as a quadric section of a spherical cone $\Gamma$ with a 3-space $\Sigma$.
Interactive model: `https://www.geogebra.org/m/suyfamv6`

### A one-sheet hyperboloidal cone

All the lines connecting a point and a one-sheet hyperboloid, which lies in another 3-space, create a one-sheet hyperboloidal cone in the 4-space. Let a one-sheet hyperboloid of revolution $\beta$ lie in the 3-space $\Xi(x, y, z)$ with the axis of revolution perpendicular to the reference plane $\pi(x, y)$ (Figure 4.11). Let a point $S$ be the center of the hyperboloid $\beta$. The vertex $V$ lies in the 3-space $\Omega(x, y, w)$ on the line perpendicular to the 3-space $\Xi(x, y, z)$ through the center $S$ of the hyperboloid $\beta$.

The hyperboloid $\beta$ is the directrix of a one-sheet hyperboloidal cone $\Gamma$ with the vertex $V$. Unlike the spherical cone above, where the directing sphere is finite, we restrict not only the $\Omega$-image $\Gamma_3$, but also the $\Xi$-image $\Gamma_4$ between two 3-spaces parallel to $\Xi(x, y, z)$ in equal distances (the $z$ coordinate of $S$) from the center $S$.

The inner boundary of the apparent contour of $\Gamma$ in the 3-space $\Xi(x, y, z)$ is the asymptotic cone $\beta_3'$ of the one-sheet hyperboloid of revolution $\beta_3$ (see Section 4.3.2). The $\Omega$-image $\Gamma_4$ of the hyperboloidal cone is derived from the $\Xi$-image. The section of the hyperboloidal cone $\Gamma$ with a 3-space through the center $S$ of the hyperboloid $\beta$ perpendicular to its axis appears as a circle $b_3 \equiv b_3'$ in the 3-space $\Xi(x, y, z)$. The $\Omega$-image $b_4$ lies in the reference plane $\pi(x, y)$ and is the base circle of the double-cone of revolution $\delta_4'$ with the vertex $V_4$. The circle $b_4'$ is the top base of the cone $\delta_4'$ — the inner boundary of the apparent contour in the 3-space $\Omega(x, y, w)$. Let us denote $\kappa$ the angle between a generatrix of the cone $\delta_4'$ and its axis $o_4$. The outer boundary is the (overlapping) one-sheet hyperboloid of revolution $\delta_4$ in the 3-space $\Omega(x, y, w)$ created by cutting the hyperboloidal cone $\Gamma$ with the restricting 3-spaces. The circle section $c_3$ of the hyperboloid $\beta_3$ with the reference plane $\pi(x, y)$ is also the circular boundary $c_4$ and $c_4'$ (bottom and top base) of the hyperboloid $\delta_4$. The neck circle $d_4 \equiv d_4'$ of the one-sheet hyperboloid of revolution $\delta_4$ lies in the plane parallel to the reference plane $\pi(x, y)$ through the point $V_4$ and is the $\Omega$-image of the circle section $d_3$ of the asymptotic cone $\beta_3'$ and the reference plane $\pi(x, y)$, and also the $\Omega$ image of the bottom circular base $d_3'$ of the asymptotic cone $\beta_3'$. To construct a generating hyperbola of the hyperboloid of revolution $\delta_4$, we can choose its axial plane through $o_4$ and find the vertices on the circle $d_4$ and four other intersecting points on the base circles $c_4'$ and $c_4''$.

The cutting 3-space $\Sigma$ is perpendicular to the 3-space $\Omega(x, y, w)$, and hence its $\Xi$-trace $\xi_3^\Sigma$ is perpendicular to the reference plane $\pi(x, y)$. The angle $\lambda$ between the $\Omega$-trace $\omega_4^\Sigma$ and the axis $o_4$ of $\delta_4'$ appears in its true size.

**A one-sheet hyperboloid** In case of $\lambda \neq \kappa$ (analytically $\alpha \neq \pm 1$ in Equation 4.14), the quadric section is a one-sheet elliptic hyperboloid $\eta$.

The $\Omega$-image $\eta_4$ of the hyperboloid is the planar area in the $\Omega$-trace $\omega_4^\Sigma$ of the 3-space $\Sigma$ between its elliptic or hyperbolic[26] cuts $e_4$ and $f_4 \equiv f_4'$ with the cone $\delta_4'$ and the one-sheet hyperboloid $\delta_4$, respectively. The construction of the $\Xi$-image $e_3$ of the ellipse $e$ in the $\Omega$ plane $\varphi^\Sigma$ is clear from the previous cases. Its vertices are denoted $A, B, C$, and $D$, center $Z$, and foci $E$ and $F$. Further on, the one-sheet hyperboloid $\delta_4$ is the $\Omega$-image of two overlapping one-sheet hyperboloids in the 4-space. Therefore the $\Xi$-image $f_3$ of the ellipse $f$ lies in the reference plane, and the $\Xi$-image $f_3'$ of the ellipse $f'$ is in the second boundary plane of the $\Xi$-image $\Gamma_3$. The vertices of the ellipses $f$ and $f'$ are $K, L, M, N$ and $K', L', M', N'$, respectively. The points $A_3, B_3$ are vertices of the principal hyperbola through the points $K_3, K_3', L_3, L_3'$ of the one-sheet hyperboloid $\eta_3$. The second principal hyperbola has the vertices $C_3, D_3$ and points $M_3, M_3', N_3, N_3'$. The principal ellipse $e_3$ mentioned above is in the last plane of symmetry. To construct the asymptotic cone of the hyperboloid $\eta_3$, we can conveniently use the $\Xi$-image of the cone, which is the quadric section of the one-sheet hyperboloidal cone $\Gamma$ with the 3-space parallel to $\Sigma$ through the vertex $V$, and translate the

[26] We visualize and continue the construction for the elliptic section with $\lambda > \kappa$.

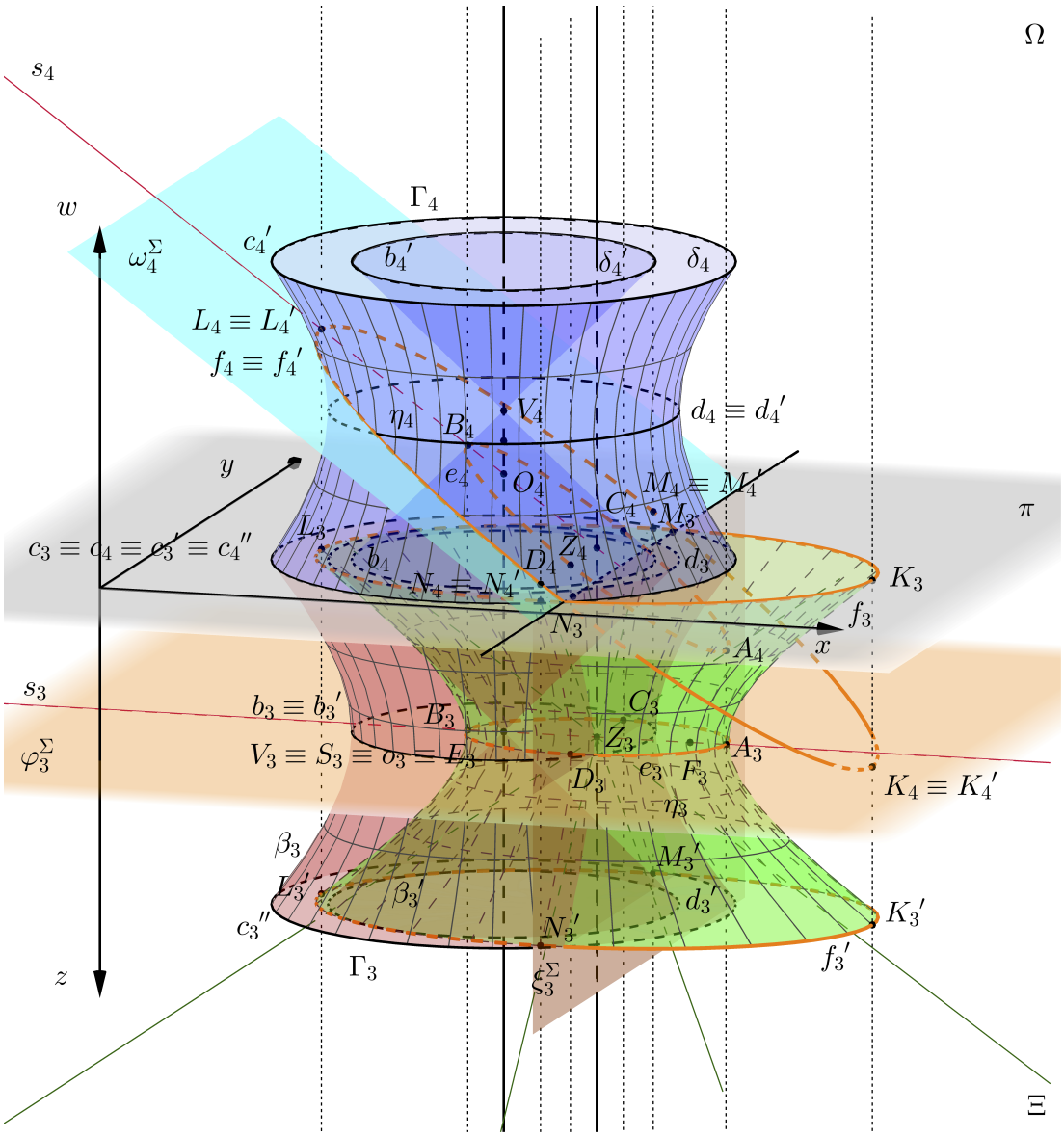



Figure 4.11: A projection of a one-sheet elliptic hyperboloid $\eta$ as a quadric section of a one-sheet hyperboloidal cone $\Gamma$ with a 3-space $\Sigma$.
Interactive model: `https://www.geogebra.org/m/guykkre3`

vertex of this cone to the center $Z_3$ (for clarity of visualization only the asymptotes of the generating hyperbolas are depicted).

**A hyperbolic paraboloid** The last quadric section — hyperbolic paraboloid $\psi$ lies in a 3-space $\Sigma$ parallel to a generatrix of the one-sheet hyperboloidal cone $\Gamma$, and so $\lambda = \kappa$ (or $\alpha \neq \pm 1$ in Equation 4.14) (Figure 4.12).

In the 3-space $\Omega(x, y, w)$, the condition is satisfied if $\Omega$-trace $\omega_4^\Sigma$ is parallel to a generatrix of the cone $\delta_4'$. The cone $\delta_4'$ is the asymptotic cone of the hyperboloid $\delta$, and their cuts with the $\Omega$-trace $\omega_4^\Sigma$ are parabolas $p_4$ and $p_4' \equiv p_4''$. The axis of the parabola $p_4$ is the $\Xi$-slope line $s_4$ of the 3-space $\Sigma$, and the vertex $W_4$ is the point of intersection of $s_4$ and $\delta_4'$. The $\Xi$-image $W_3$ of the vertex $W$ is in the $\Omega$-plane $\varphi_3^\Sigma$ and on its line of recall. The $\Xi$-image $p_3$ is the parabola with the vertex $W_3$ and focus $F_3 \equiv V_3$. The axis of the parabola $p_4' \equiv p_4''$ is also $s_4$, and its vertex $W_4' \equiv W_4''$

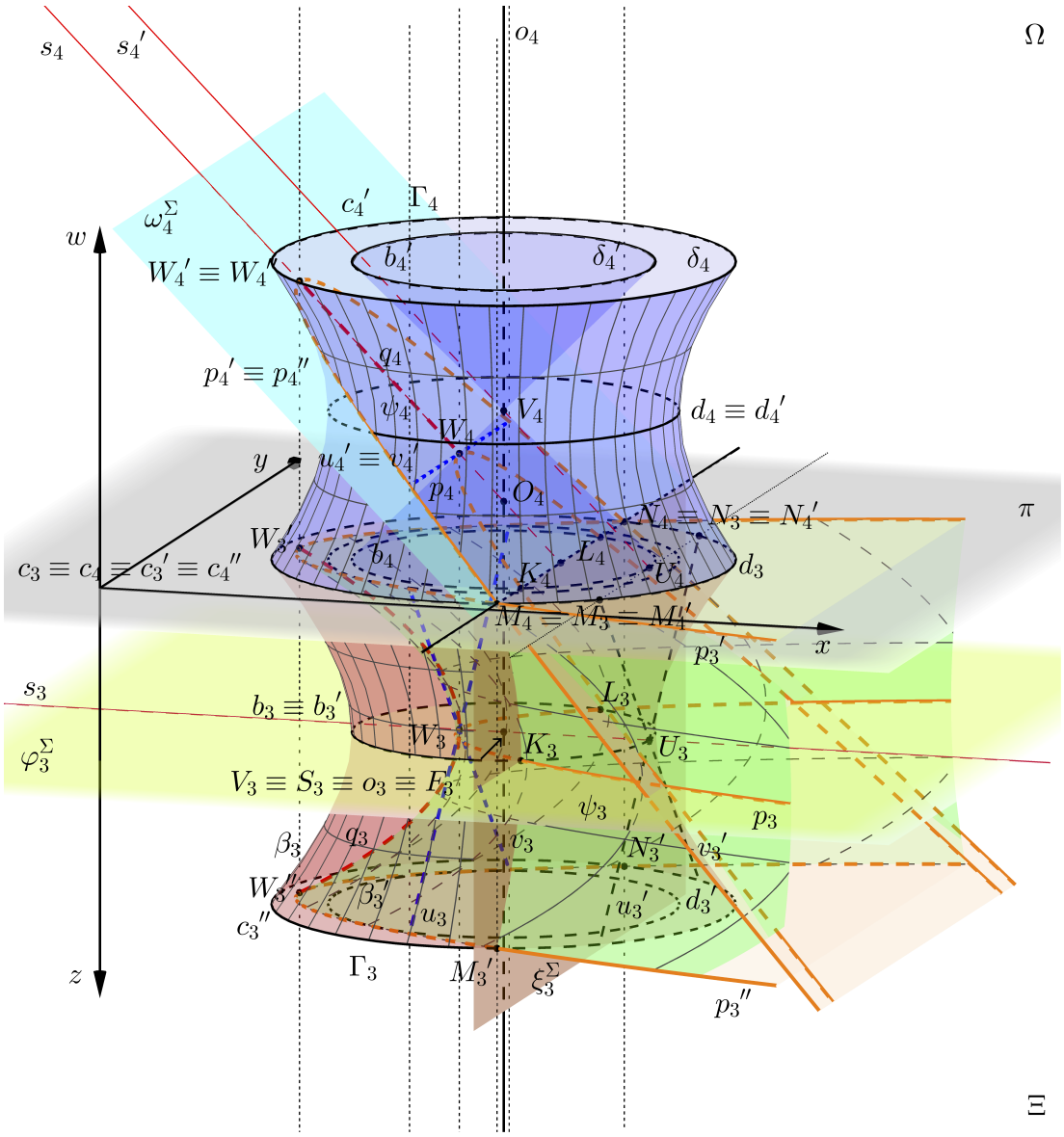



Figure 4.12: A projection of a hyperbolic paraboloid $\psi$ as a quadric section of a one-sheet hyperboloidal cone $\Gamma$ with a 3-space $\Sigma$.
Interactive model: `https://www.geogebra.org/m/jt6ktfq2`

is the point of intersection of $s_4$ and $\delta$. The $\Xi$-images $p_3'$ and $p_3''$ are parabolas in the base (restricting) planes of the $\Xi$-image $\Gamma_3$ of the hyperboloidal cone $\Gamma$, with the vertices $W_3'$ and $W_3''$. For construction of $p_3'$ and $p_3''$, we can either translate the principal parabola $p_3$ into the vertices $W_3'$ and $W_3''$ or construct them with the use of their subtangents. Few more elements are constructed in the figure. The ray $q_4 = W_4W_4'$ is the $\Omega$-image of a parabola $q$ which appears as the parabola $q_3$ with the vertex $W_3$ and two points $W_3'$ and $W_3''$ in the 3-space $\Xi(x, y, z)$. The parabola $q_3$ is the principal parabola of the hyperbolic paraboloid $\psi_3$, and it can be easily constructed with the use of subtangents. The last piece is the construction of the vertex generating lines $u$ and $v$. First, we construct the singular quadric section $\psi'$ — two intersecting planes (only their intersecting line $s'$ is visualized in the figure), of the one-sheet hyperboloidal cone $\Gamma$ with the 3-space $\Sigma'$ through the vertex $V$ parallel to the 3-space $\Sigma$. The $\Xi$-slope line $s_4'$ is parallel to $s_4$. The

construction of the asymptotic lines $u'_3$ and $v'_3$ in the $\Xi$-trace of the 3-space $\Sigma'$ is indicated. The vertex generators $u$ and $v$ are parallel to $u'$ and $v'$, respectively, and so are their $\Xi$ and $\Omega$-images.

### 4.3.4 Conclusion

We have described regular quadric sections as cuts of four-dimensional cones with 3-spaces both — analytically and synthetically. The visual aspect of synthetic constructions was of our major interest, while the analytic approach has enlightened our reasoning. Admittedly, the method of cutting four-dimensional cones has significant drawbacks — we need more kinds of cones, but after all, we would not even be able to obtain all the real singular cases of quadrics. Algebraically speaking, from a four-dimensional cone with the rank of its matrix 4 in $\mathbb{RP}^4$, we can derive at best only those three-dimensional quadrics of the rank 2 in $\mathbb{RP}^3$. However, the graphic element, constructive creation and ability to grasp the four-dimensional spatiality have overcome these issues. Of course, nowadays, we mostly use computers to deal with graphics, and we can easily make projections of four-dimensional objects from their analytic representations. Nonetheless, it seems that classical techniques of descriptive geometry may be invaluable for visualization of the four-dimensional space.

**Further issues** There are many further directions to be taken from the presented method. First, we have touched singular quadric sections only slightly throughout our constructions. In the constructions of ruled quadrics, we have not really used their "ruledness" and there may be significant simplifications for their constructions. Also, other interesting properties may arise from such examination. The models of quadrics may be used for more detailed synthetic derivation of their focal properties and a synthetic proof of generalized Dandelin's theorem. Apart from four-dimensional cones, graphic models of other four-dimensional quadrics are ready to be explored.

# Conclusion

Projective geometry was created by synthetic reasoning and synthetic methods are still an essential part of the theory. We have shown how such reasoning with the support of visualizations or interactive models can penetrate into different areas. Particular topics were already concluded with related further thoughts, and so, we only give a short survey.

To understand some theory, it is always useful to explore its roots. The pioneers of projective geometry were dealing with measurements, infiniteness, continuity, imaginarity, etc. We commented the original proof of Chasles's theorem for non-developable ruled surfaces and highlighted the problematic parts, which are or are not possible to correct within the (currently assumed pure) theory. Especially the infinite closeness bothered synthetic projective geometry for a long time. Thus, we reproved the theorem in pure projective nature at first for quadrics, and then, we generalized it for algebraic surfaces. The underlaying problems of the naive era of projective geometry were in a great extent resolved by von Staudt's contribution, to which we gave a detailed description of the main ideas.

The subject of projective geometry is nowadays usually approached either axiomatically or analytically (algebraically). In contrast, we tried to refer to interrelations of the classical synthetic, axiomatic and analytic constructions of projective space, its objects and transformations, in the spirit of Richter-Gebert [2011], Glaeser et al. [2016]. A brief theoretical introduction into projective geometry was given with accent to demonstrations and visualizations.

Synthetic methods of solving problems in projective geometry have been occasionally noted, and if, then mostly in texts on history of geometry. With further details and examples, we explained the method of projection and the method of selection of the improper hyperplane. Particularly the latter is connected with our experience and intuitiveness in affine geometry, from which fundamental constructions of projective geometry were derived. Apart from other examples, we gave an alternative proof of Pappus's theorem and projective generalizations and proofs of the Menelaus's and Ceva's theorem, and a construction of the pole of a triangle. Applications of the methods were demonstrated on problems of elementary geometry — for example, on a novel synthetic proof of the characteristic property of a point with respect to a parabola. The generality of von Staudt's system of elements was pointed out on the synthetic construction of the real field with the use of an analytic representation of a model of a cow in computer graphics.

A descriptive geometry method of four-dimensional visualization was introduced in details. A whole chapter was devoted to methodically develop elementary constructions. With 3D modeling software, we were able to synthetically construct images of complex four-dimensional objects and examine their sections and shadows. Connections with synthetic projective geometry were described in a construction of the pole of plane with respect to a tetrahedron with respect to a plane, in which we used the fourth dimension to derive the three-dimensional result. The same idea was applied to the classification of three-dimensional quadrics as sections of four-dimensional cones and their respective constructions in the

double orthogonal projection of the four-dimensional space onto two mutually perpendicular three-dimensional spaces.

Throughout the thesis, we touched few kinds of geometric subjects, on most of which whole books were or may be written. There are still many further issues. At first, continuous work with the purification and revision of the golden era of geometry in the 19th century. Further on, recent advances in computer visualization allow us to widely incorporate synthetic reasoning, and so, classical synthetic methods, translated into modern language, can be well incorporated. On the top of that, we are able to automatize and involve laborious synthetic methods and generalize them. With such a generalization, we have created a synthetic framework — playground, in which we can build and examine objects of the four-dimensional space without their analytic representation. Applications of the method are possible not only in geometry, but in any field of science that works with the four-dimensional imaging.

# Bibliography


URL `http://eusebeia.dyndns.org/4d/`.

URL `https://www.geogebra.org/`.

URL `http://www.wolfram.com/mathematica/`.

Edwin Abbott Abbott and Rosemary Jann (ed.). *Flatland, a Romance of Many Dimensions*. Oxford University Press, 2006. ISBN 978-0-19-953750-1.

Arseny V. Akopyan. *Geometry in figures*. IST Austria, 2nd edition, 2017. ISBN 978-1548710781.

Henry Frederick Baker. *Principles of geometry, Cambridge, vol. III (1923)*. Cambridge University Press, 1923.

Alexander Bogomolny. Interactive mathematics miscellany and puzzles. URL `https://www.cut-the-knot.org/`.

Charles Julien Brianchon. Mémoire sur les surfaces courbes du second degré. *Journal de l'Ecole Polytechnique*, VI:297–311, 1806.

Charles Julien Brianchon. *Mémoires sur les lignes du second ordre*. Bachelier, Paris, 1817.

Charles Julien Brianchon. *Application de la Théorie des Transversales*. Paris, 1818.

Bohumil Bydžovský. *Úvod do algebraické geometrie*. Prometheus, Praha, 1948.

Oliver Byrne and Werner Oechslin (ed.). *The First Six Books of the Elements of Euclid*. Taschen, multilingual edition, 2017. ISBN 978-3-8365-5938-6.

Florian Cajori. *A History of Mathematics*. Macmillan, 1919.

Lazare Nicolas Marguérite Carnot. *De la corrélation des figures de géométrie*. Crapelet, Paris, 1801.

Lazare Nicolas Marguérite Carnot. *Géométrie de position*. Crapelet, Paris, 1803.

Lazare Nicolas Marguérite Carnot. *Mémoire sur la relation qui existe entre les distances respectives de cinq points quelconques pris dans l'espace: suivi d'un Essai sur la théorie des transversales*. Courcier, Paris, 1806.

Fernando R. Casas. Polar perspective: A graphical system for creating two-dimensional images representing a world of four dimensions. *Leonardo*, 17(3): 188, 1984. doi: 10.2307/1575189.

Eduardo Casas-Alvero. *Analytic Projective Geometry*. European Mathematical Society Publishing House, Zürich, 2014. ISBN 978-3-03719-138-5.

Rey Casse. *Projective geometry: an introduction*. Oxford University Press, 2006. ISBN 9780199298860.

Augustin-Louis Cauchy. *Leçons sur les applications du calcul infinitésimal à la géométrie*, volume 1. Imprimerie Royale, Paris, 1826.

Michel Chasles. *Aperçu historique sur l'origine et le développement des méthodes en géométrie*. M. Hayez, Bruxelles, 1837.

Michel Chasles. Sur les surfaces engendrées par une ligne droite; particulièrement sur l'hyperboloïde, le paraboloïde et le cône du second degré. *Correspondance mathématique et physique*, 11:49–113, 1839.

Michel Chasles. *Traité des sections coniques: faisant suite au Traité de géométrie supérieure*, volume 1. Gauthier-Villars, Paris, 1865.

Michel Chasles. *Traité de géométrie supérieure*. Gauthier-Villars, Paris, 2nd edition, 1880.

Bruce L. Chilton. Shadows of four-dimensional polytopes. *Mathematics Magazine*, 44(5):269–273, 1971. doi: 10.1080/0025570X.1971.11976168.

Bruce L. Chilton. Principal shadows of the 12 pentagonal regular 4-dimensional objects (polytopes). *Leonardo*, 13(4):288–294, 1980. doi: 10.2307/2688637.

Julian Lowell Coolidge. The rise and fall of projective geometry. *The American Mathematical Monthly*, 41(4):217–228, 1934. doi: 10.2307/2302023.

Harold Scott Macdonald Coxeter. *Regular polytopes*. Dover Publications, Inc, 2nd edition, 1973. ISBN 9780486614809.

Harold Scott Macdonald Coxeter. *Projective geometry*. Springer, New York, 2nd edition, 2003. ISBN 9780387406237.

Girard Desargues. *Brouillon Project d'une atteinte aux événemens des recontres du Cône avec un plan*. 1639.

François Pierre Charles Dupin. *Développements de géométrie*. Courcier, Paris, 1813.

Albrecht Dürer. *Underweysung der Messung, mit dem Zirckel und Richtscheyt, in Linien, Ebenen unnd gantzen corporen*. Nürnberg, 1525.

Giovanni Falcone and Marco Pavone. Kirkman's tetrahedron and the fifteen schoolgirl problem. *The American Mathematical Monthly*, 118(10):887–900, 2011. ISSN 19300972. doi: 10.4169/amer.math.monthly.118.10.887.

Andrew Russell Forsyth. *Geometry of Four Dimensions*. The University Press, Cambridge, 1930.

Joseph-Diez Gergonne. Géométrie de situation. recherches sur quelques lois générales qui régissent les lignes et surfaces algébriques de tous les ordres. In *Annales de mathématiques pures et appliquées*, volume 17, pages 214–229, 1826.

Joseph-Diez Gergonne. Géométrie de situation. rectification de quelques théorèmes énoncés dans les annales. *Annales de mathématiques pures et appliquées*, 18:149–154, 1827.

Georg Glaeser, Hellmuth Stachel, and Boris Odehnal. *The Universe of Conics: From the ancient Greeks to 21st century developments*. Springer, Berlin Heidelberg, 2016. ISBN 978-3-662-45449-7. doi: 10.1007/978-3-662-45450-3.

George Bruce Halsted. *Synthetic projective geometry*. Number 2. J. Wiley & Sons, New York, 4th edition, 1906.

Andrew J. Hanson and Pheng Ann Heng. Visualizing the fourth dimension using geometry and light. In *Proceedings of the 2nd conference on Visualization '91*, pages 321–328, 1991. doi: 10.1109/VISUAL.1991.175821.

John Leigh Smeathman Hatton. *The Theory of the Imaginary in Geometry: Together with the Trigonometry of the Imaginary*. Cambridge University Press, 1920.

Karel Havlíček. *Cesty moderní matematiky*. Horizont, Praha, 2nd edition, 1976.

David Hestenes and Renatus Ziegler. Projective geometry with clifford algebra. *Acta Applicandae Mathematica*, 23(1):25–63, 1991. doi: 10.1007/BF00046919.

Charles Howard Hinton. *The fourth dimension*. S. Sonnenschein & Company, Ltd. London, John Lane, New York, 1904.

Charles Howard Hinton and Rudolf von Bitter Rucker (ed.). *Speculations on the Fourth Dimension; Selected Writings of Charles H. Hinton*. Dover Publications, Inc, New York, 1980. ISBN 978-0486239163.

Václav Hlavatý. *Differenciální přímková geometrie I.-II.* Česká akademie věd a umění, Praha, 1941.

Václav Hlavatý. *Projektivní geometrie I.* Melantrich, Praha, 1944.

Václav Hlavatý. *Projektivní geometrie II.* Melantrich, Praha, 1945.

Christoph M. Hoffmann and Jianhua Zhou. Some techniques for visualizing surfaces in four-dimensional space. *Computer-Aided Design*, 23(1):83–91, 1991. doi: 10.1016/0010-4485(91)90083-9.

Fritz Hohenberg and Josef Tschupik. Grundzüge der darstellenden geometrie. *H. Behnke et al.: Grundzüge der Mathematik*, 2, 1960.

František Kadeřávek, Josef Klíma, and Josef Kounovský. *Deskriptivní geometrie II.* Jednota československých matematiků a fysiků, Praha, 1932.

Akira Kageyama. A visualization method of four dimensional polytopes by oval display of parallel hyperplane slices. *Journal of Visualization*, 19(3):417–422, 2016. doi: 10.1007/s12650-015-0319-5.

Martin Kemp. Dali's dimensions. *Nature*, 391(6662):27–27, Jan 1998. doi: 10.1038/34063.

Felix Klein. A comparative review of recent researches in geometry. *Bulletin of the New York Mathematical Society*, 2(10):215–249, 1893.

Andrei N. Kolmogorov and Adolf-Andrei P. Yushkevich. *Mathematics of the 19th Century: Geometry, Analytic Function Theory*. Birkhäuser, Basel, 1996. ISBN 978-3-7643-5048-2. doi: 10.1007/978-3-0348-9173-8.

Erwin Kruppa. Ein beitrag zur darstellenden geometrie des $R_4$. *Monatshefte für Mathematik und Physik*, 43(1):157–160, Dec 1936. doi: 10.1007/bf01707596.

Ladislav Kvasz. *Patterns of change: linguistic innovations in the development of classical mathematics*, volume 36. Birkhäuser, Basel, 2008. ISBN 978-3-7643-8839-3. doi: 10.1007/978-3-7643-8840-9.

Ladislav Kvasz. Jazyk matematiky, jeho zmeny a didaktika matematiky. *Pokroky matematiky, fyziky a astronomie*, 58(4):315–325, 2013.

Robert Lachlan. *An elementary treatise on modern pure geometry*. Macmillan and Company, London, 1893.

Miroslav Lávička. *Geometrie 2*. Supplementary teaching material KMA/G2, University of West Bohemia, Pilsen, February 2006.

Carlos Ernesto S. Lindgren and Steve M. Slaby. *Four-Dimensional Descriptive Geometry*. McGraw-Hill, New York, 1968.

Henry Parker Manning. *Geometry of Four Dimensions*. Dover, New York, 1956.

Yuri Matijasevich. Первое знакомство с номограммами. Квант, 2(5):25, 1971.

Koji Miyazaki. Visualization of a die and other objects in four-dimensional space. *Leonardo*, 21(1):56–60, 1988. doi: 10.2307/1578417.

August Ferdinand Möbius. *Der barycentrische Calcul*. J..A Barth, Leipzig, 1827.

August Ferdinand Möbius. *Lehrbuch der Statik*, volume 1. G. J. Göschen, Leipzig, 1837.

Gaspard Monge. *Géométrie descriptive*. Baudouin, Paris, 1799.

Mehmet Önder and H. Hüseyin Uğurlu. Frenet frames and invariants of timelike ruled surfaces. *Ain Shams Engineering Journal*, 4(3):507–513, 2013. doi: 10.1016/j.asej.2012.10.003.

Alexander Ostermann and Gerhard Wanner. *Geometry by Its History*. Springer, Berlin Heidelberg, 2012. ISBN 978-3-642-29162-3. doi: 10.1007/978-3-642-29163-0.

Blaise Pascal. *Essay pour les coniques*. 1640.

Jean Pélerin. *De artificiali perspectiva*. Toul, 1505.

Plato, William Lowe Bryan (ed.), and Charlotte Lowe Bryan (ed). *Plato the teacher: Being selections from the Apology, Euthydemus, Protagoras, Symposium, Phædrus, Republic, and Phædo of Plato.*, pages 187–409. Charles Scribner's Sons, New York, 1897.

Peter Plaumann and Karl Strambach, editors. *Geometry—von Staudt's Point of View: Proceedings of the NATO Advanced Study Institute*, volume 70, Bad Windsheim, West Germany, 1980. Dordrecht : D. Reidel Pub. Co.

Julius Plücker. *Analytisch-geometrische entwicklungen*, volume 1. G.D. Baedeker, Essen, 1828.

Julius Plücker. *System der analytischen Geometrie, auf neue Betrachtungsweisen gegründet, und insbesondere eine ausführliche Theorie der Curven dritter Ordnung enthaltend.* Duncker und Humblot, Berlin, 1835.

Julius Plücker. *Theorie der algebraischen Curven, gegründet auf eine neue Behandlungsweise der analytischen Geometrie.* A Marcus, Bonn, 1839.

Julius Plücker. *System der Geometrie des Raumes in neuer analytischer Behandlungsweise: insbesondere die Theorie der Flächen 2. Ordnung und Classe enthaltend.* W. H. Scheller, Düsseldorf, 1846.

Julius Plücker. *Neue Geometrie des Raumes: gegründet auf die Betrachtung der geraden Linie als Raumelement*, volume 1. B. G. Teubner, Leipzig, 1868.

Henri Poincaré. *Science and method.* Translated by F. Maitland with preface by B. Russell, Dover Publications, Inc, Mineola, New York, 2nd edition, 2003.

Jean-Victor Poncelet. *Traité des propriétés projectives des figures: ouvrage utile à ceux qui s' occupent des applications de la géométrie descriptive et d'opérations géométriques sur le terrain.* Gauthier-Villars, Paris, 1865.

Helmut Pottmann and Johannes Wallner. *Computational Line Geometry.* Springer, Berlin Heidelberg, 2009. ISBN 978-3-540-42058-3. doi: 10.1007/978-3-642-04018-4.

Theodor Reye. *Die Geometrie der Lage.* Baumgärtner's Buchhandlung, Leipzig, 4th edition, 1899.

Jürgen Richter-Gebert. *Perspectives on Projective Geometry: A Guided Tour Through Real and Complex Geometry.* Springer, Berlin Heidelberg, 2011. ISBN 978-3-642-17285-4. doi: 10.1007/978-3-642-17286-1.

Christoph J. Scriba and Peter Schreiber. *5000 Years of Geometry: Mathematics in History and Culture.* Birkhäuser, Basel, 2015. ISBN 978-3-0348-0897-2. doi: 10.1007/978-3-0348-0898-9.

Henry Segerman. *Visualizing Mathematics with 3D Printing.* JHU Press, 2016. ISBN 978-1421420356.

John Greenlees Semple and Geoffrey Thomas Kneebone. *Algebraic projective geometry.* Oxford University Press, 1952.

Carlo Heinrich Séquin. 3d visualization models of the regular polytopes in four and higher dimensions. In Reza Sarhangi, editor, *Proceedings of Bridges: Mathematical Connections in Art, Music, and Science*, pages 37—48, Southwestern College, Winfield, Kansas, 2002.

Bogdan V. Şerbănoiu and Adrian A. Şerbănoiu. Multidimensional descriptive geometry. In Sorin Nistor and Gabriela A. Popovicu, editors, *16th edition National Technical-Scientific Conference, Modern Technologies for the 3rd Millennium*, pages 257–266, Oradea, Romania, 2017. Editographica, Bologna.

Zbyněk Šír. *Les sections coniques chez Philippe de La Hire*. PhD thesis, Université Pierre at Marie Curie (Paris VI), 2002.

Hellmuth Stachel. The right-angle-theorem in four dimensions. *Journal for Theoretical Graphics and Computing*, 3(1):4–13, 1990.

Jakob Steiner and Arthur Oettingen (ed.). *Systematische Entwicklung der Abhangigkeit geometrischer Gestalten von einander: mit Berucksichtigung der Arbeiten alter and neuer Geometer uber Porismen, Projections-Methoden, Geometrie der Lage, Transversalen, Dualitat, und Reciprocitat, etc.* W. Engelmann, Leipzig, 1896.

Oswald Veblen and John Wesley Young. *Projective geometry*. Ginn and company, Boston, 1910.

Klaus Volkert. On models for visualizing four-dimensional figures. *Mathematical Intelligencer*, 39(2):27–35, 2017. doi: 10.1007/s00283-016-9699-1.

Karl Georg Christian von Staudt. *Geometrie der Lage*. Fr. Korn'schen Buchhandlung, Nürnberg, 1847.

Karl Georg Christian von Staudt. *Beiträge zur Geometrie der Lage I, II & III*. Fr. Korn'schen Buchhandlung, Nürnberg, 1856.

Samuel Edward Warren. *The Elements of Descriptive Geometry, Shadows and Perspective*. John Wiley and Sons, New York, 1890.

Gunther Weiss. (n,2)-axonometries and the contour of hyperspaces. *Journal for Geometry and Graphics*, 1(2):157–167, 1997.

Jonathan Woodring, Chaoli Wang, and Han-Wei Shen. High dimensional direct rendering of time-varying volumetric data. In *Proceedings of the 14th IEEE Visualization 2003 (VIS'03)*, page 55, Seattle, WA, USA, USA, 2003. IEEE Computer Society. doi: 10.1109/VISUAL.2003.1250402.

Wesley S. B. Woolhouse, editor. *Lady's and gentleman's diary.* Company of Stationers, London, 1850.

Svätopluk Zachariáš and Daniela Velichová. Projection from 4d to 3d. *Journal for Geometry and Graphics*, 4(1):55–69, 2000.

Michal Zamboj. Projective Proof and Generalization of Chasles's Theorem Along Non-Torsal Lines. *Journal for Geometry and Graphics*, 20(2):243–251, 2016a.

Michal Zamboj. Non-projective problems of the Chasles theorem. In Jindřich Bečvář and Martina Bečvářová, editors, *37. Mezinárodní konference historie matematiky*, pages 189–194, Poděbrady, 2016b. Matfyzpress, Praha.

Michal Zamboj. On Methods of Synthetic Projective Geometry. In Daniela Velichová and Světlana Tomiczková Miroslav Lávička, editors, *Proceedings of Slovak-Czech Conference on Geometry and Graphics 2017*, pages 179–184, Vršatské podhradie, Slovakia, 2017. Vydavatelský servis, Plzeň.

Michal Zamboj. Double Orthogonal Projection of Four-Dimensional Objects onto Two Perpendicular Three-Dimensional Spaces. *Nexus Network Journal*, 20(1): 1–15, 2018a. doi: 10.1007/s00004-017-0368-2.

Michal Zamboj. Sections and Shadows of Four-Dimensional Objects. *Nexus Network Journal (online)*, pages 1–9, 2018b. doi: 0.1007/s00004-018-0384-x.

Michal Zamboj. Quadric Sections of Four-Dimensional Cones. In Luigi Cocchiarella, editor, *Proceedings of the 18th International Conference on Geometry and Graphics*, volume 809, pages 500–513, Milan, Italy, 2018c. Springer International Publishing. doi: 10.1007/978-3-319-95588-9.

Michal Zamboj. Parabola a jej podobenstvá. In Naďa Vondrová, editor, *Dva dny s didaktikou matematiky 2018, sborník příspěvků (in press)*, Katedra matematiky a didaktiky matematiky Univerzita Karlova v Praze, Pedagogická fakulta, Praha, 2018d. Katedra matematiky a didaktiky matematiky, Univerzita Karlova v Praze, Pedagogická fakulta, Společnost učitelů matematiky JČMF.

Jianhua Zhou. Visualization of Four Dimensional Space and Its Applications (Ph.D. Thesis). *Computer Science Technical Reports*, Paper 922, 1991.

Karel Žitný and Igor Zolotarev. Tíží projektivní geometrii nesnesitelné břemeno dějin? In Jindřich Bečvář and Eduard Fuchs, editors, *Matematika v 19. století. Sborník přednášek z 15. letní školy Historie matematiky*, pages 88–110, Vyškov, 1996. Prometheus, Praha.

# List of publications

Michal Zamboj. Projective Proof and Generalization of Chasles's Theorem Along Non-Torsal Lines. *Journal for Geometry and Graphics*, 20(2):243–251, 2016a.

Michal Zamboj. Non-projective problems of the Chasles theorem. In Jindřich Bečvář and Martina Bečvářová, editors, *37. Mezinárodní konference historie matematiky*, pages 189–194, Poděbrady, 2016b. Matfyzpress, Praha.

Michal Zamboj. On Methods of Synthetic Projective Geometry. In Daniela Velichová and Světlana Tomiczková Miroslav Lávička, editors, *Proceedings of Slovak-Czech Conference on Geometry and Graphics 2017*, pages 179–184, Vršatské podhradie, Slovakia, 2017. Vydavatelský servis, Plzeň.

Michal Zamboj. Double Orthogonal Projection of Four-Dimensional Objects onto Two Perpendicular Three-Dimensional Spaces. *Nexus Network Journal*, 20(1): 1–15, 2018a. doi: 10.1007/s00004-017-0368-2.

Michal Zamboj. Sections and Shadows of Four-Dimensional Objects. *Nexus Network Journal (online)*, pages 1–9, 2018b. doi: 0.1007/s00004-018-0384-x.

Michal Zamboj. Quadric Sections of Four-Dimensional Cones. In Luigi Cocchiarella, editor, *Proceedings of the 18th International Conference on Geometry and Graphics*, volume 809, pages 500–513, Milan, Italy, 2018c. Springer International Publishing. doi: 10.1007/978-3-319-95588-9.

Michal Zamboj. Parabola a jej podobenstvá. In Naďa Vondrová, editor, *Dva dny s didaktikou matematiky 2018, sborník příspěvků (in press)*, Katedra matematiky a didaktiky matematiky Univerzita Karlova v Praze, Pedagogická fakulta, Praha, 2018d. Katedra matematiky a didaktiky matematiky, Univerzita Karlova v Praze, Pedagogická fakulta, Společnost učitelů matematiky JČMF.

# A. Attachments

## A.1 Interactive constructions

Supplementary GeoGebra constructions with the labels of corresponding figures are on the attached USB flash drive in the folder `/attachments/geogebraconstructions/` and in the online book `https://www.geogebra.org/m/k7hpk2zn`. Instructions how to install GeoGebra and run the files from the folder are on the official website [ggb]. The online interactive book is accessible in common internet browsers, and no further installation is needed. All the figures with GeoGebra constructions are referred to the online book in captions.

The files are divided into two groups

*Interactive model* - the user can move the camera or free elements (usually points, lines, sliders) of objects or mark check boxes and observe changes of dependent elements.

*Step-by-Step construction* - the user is supposed to open *View* → *Construction Protocol* and play the construction of *Breakpoints* in the menu. The final construction may be used as an interactive model, too.

Visibility of elements in constructions is for demonstration purposes (mostly in 3D constructions) adjusted to input settings. Thus, after manipulations, some elements may disappear, and the user is asked to modify the visibility of elements in the menu *View* → *Algebra*.

# A.2 The original proof of Chasles's theorem

An excerpt from [Chasles, 1839].



| | LATITUDE. | LONGITUDE. | INCLINAIS. | INTENSI. |
|---|---|---|---|---|
| Océan Atlantique . . . . | 40° 9′ | 325° 20′ | 69° 32,9′ | 1,504 |
| | 41 27 | 327 25 | 70 3,6 | 1,466 |
| — . . . . | 42 29 | 328 34 | 69 47,6 | 1,512 |
| — . . . . | 44 22 | 330 55 | 71 7,1 | 1,515 |
| — . . . . | 46 46 | 335 42 | 70 18,5 | 1,463 |
| — . . . . | 47 47 | 343 58 | 69 46,0 | 1,421 |
| | 47 46 | 344 25 | 70 14,9 | 1,419 |
| — . . . . | 48 13 | 347 7 | 69 27,8 | 1,422 |
| — . . . . | 49 16 | 351 58 | 69 10,5 | 1,416 |
| Canal britannique. . . . | 50 48 | 358 54 | 68 45,0 | 1,380 |

## MÉMOIRE

***Sur les surfaces engendrées par une ligne droite; particulièrement sur l'hyperboloïde, le paraboloïde et le cône du second degré,*** par M. CHASLES.

La Géométrie, qui doit embrasser toutes les propriétés de l'étendue figurée, se compose d'une multitude de propositions dont le nombre et la diversité peuvent, au premier abord, rebuter les jeunes mathématiciens et leur faire craindre de ne pénétrer bien avant dans cette science qu'au prix de laborieux et de pénibles efforts. Mais une étude réfléchie ne tarde point à dissiper cette première impression et à faire reconnaître dans la géométrie un caractère propre, qui la distingue même parmi les autres sciences mathématiques; c'est que toutes les propositions géométriques





sont soumises essentiellement à une harmonie et à un enchaînement continu qui établissent entre elles des relations certaines : relations qui permettent de réunir un très-grand nombre de vérités diverses comme simples corollaires et conséquences immédiates d'une même proposition générale. On conçoit alors combien la science se simplifie et devient facile, quand on a découvert quelques-unes de ces propositions vraiment fécondes qui dominent une vaste théorie et sont l'expression complète d'une grande variété de vérités qu'elles peuvent remplacer toutes. La recherche de ces propositions mères doit être regardée comme un moyen certain de hâter les progrès et de faciliter l'étude de la géométrie.

Le désir d'atteindre ce but m'a inspiré déjà quelques écrits et me porte à publier le présent mémoire, dans lequel j'ai réuni un certain nombre de propositions relatives aux surfaces gauches, particulièrement à l'hyperboloïde et au paraboloïde, et qui, à peu près toutes du même genre, découlent toutes, naturellement, d'une même proposition. Cette proposition se rapporte à une expression très-simple formée des segmens que quatre points situés en ligne droite comprennent entre eux. Cette expression est celle que j'ai appelée *rapport anharmonique*, dans mon *aperçu historique sur l'origine et le développement des méthodes en géométrie* (1. J'ai déjà insisté, dans plusieurs endroits de cet ouvrage, sur les avantages que doit apporter dans l'étude de la géométrie cette théorie du rapport anharmonique : le mémoire actuel pourra en présenter une nouvelle preuve, par la facilité et la brièveté des démonstrations, presque toujours intuitives, des nombreux théorèmes qu'il contient.

J'aurai, du reste, d'autres occasions de revenir sur les applications de cette théorie.

### § I. *Propriétés générales des surfaces gauches.*

(1). Quand quatre droites A, B, C, D, situées dans un même

1) Formant le tome XI des *Mémoires couronnés de l'académie royale de Bruxelles*, 1 vol. in-4°, 1837, chez Hayez.

plan, sont issues d'un même point, si l'on tire une transversale quelconque qui les rencontre en quatre points $a$, $b$, $c$, $d$, le rapport

$$\frac{ca}{cb} : \frac{da}{db},$$

qui a lieu entre les segmens compris entre ces points deux à deux, a une valeur constante, quelle que soit la transversale.

Cette valeur est

$$\frac{\sin.\ C,A}{\sin.\ C,B} : \frac{\sin.\ D,A}{\sin.\ D,B}.$$

de sorte qu'on a l'équation

$$(1). \quad . \quad . \quad . \quad \frac{ca}{cb} : \frac{da}{db} = \frac{\sin.\ C,A}{\sin.\ C,B} : \frac{\sin.\ D,A}{\sin.\ D,B},$$

quelle que soit la transversale.

Cette proposition se démontre très-facilement; et on voit qu'elle a lieu aussi quand, au lieu des quatre droites A, B, C, D, on prend quatre plans A, B, C, D, passant par une même droite, et qu'on tire arbitrairement une transversale qui les rencontre aux points $a$, $b$, $c$, $d$.

L'équation (1) exprime une propriété du rapport $\frac{ca}{cb} : \frac{da}{db}$, qui le distingue de toutes autres expressions qu'on peut former avec les segmens compris entre les quatre points $a$, $b$, $c$, $d$; c'est-à-dire que pour ces autres expressions une pareille équation, entre les segmens et les sinus des angles qui les comprennent, n'a pas lieu.

(2). J'ai eu à faire un grand usage du rapport $\frac{ca}{cb} : \frac{da}{db}$, dans mon *Traité historique des méthodes géométriques*, et particulièrement dans le mémoire qui lui fait suite, où j'ai pris ce rapport pour le type unique de toutes les relations de longueur transformables par les principes de *dualité* et d'*homographie*. Par cette raison, j'ai dû lui donner un nom particulier, et je l'ai appelé *rapport anharmonique*, parce que dans le cas où il est égal à l'unité,



ce qui se présente souvent dans les recherches géométriques, on dit que les quatre points $a$, $b$, $c$, $d$, forment une proportion *harmonique*.

Ainsi l'expression $\frac{ca}{cb} : \frac{da}{db}$ est le *rapport anharmonique* des quatre points $a$, $b$, $c$, $d$, situés en ligne droite, et l'expression

$$\frac{\sin.\ C,A}{\sin.\ C,B} : \frac{\sin.\ D,A}{\sin.\ D,B}$$

est le *rapport anharmonique* des quatre droites A, B, C, D, issues d'un même point, ou des quatre plans A, B, C, D, menés par une même droite. Et l'équation (1) exprime que :

*Quand quatre plans passent par une même droite, une transversale quelconque les rencontre en quatre points dont le rapport anharmonique est égal à celui des quatre plans.*

J'aurai occasion de me servir souvent de cette proposition et de la dénomination de *rapport anharmonique*. C'est pourquoi je suis entré ici dans cette explication préliminaire, sur laquelle je n'aurai plus besoin de revenir.

(3). *Étant données trois droites fixes, situées d'une manière quelconque dans l'espace, si l'on mène quatre autres droites dont chacune s'appuie sur les trois premières, les rapports anharmoniques faits sur chacune de celles-ci respectivement, entre ses points de rencontre avec les quatre autres droites seront égaux entre eux.*

Soient L, L', L'', les trois droites fixes; $a$, $b$, $c$, $d$, les points où les quatre autres droites rencontrent la première L, et $a'$, $b'$, $c'$, $d'$; $a''$, $b''$, $c''$, $d''$, les points où elles rencontrent les deux autres L', L''; je dis qu'on aura

$$\frac{ca}{cb} : \frac{da}{db} = \frac{c'a'}{c'b'} : \frac{d'a'}{d'b'} = \frac{c''a''}{c''b''} : \frac{d''a''}{d''b''}.$$

En effet, les quatre droites $aa'$, $bb'$, $cc'$, $dd'$, déterminent quatre plans passant par la droite L. Les quatre points $a'$, $b'$, $c'$, $d'$, étant les intersections de ces plans par la droite L', leur rapport anharmonique est égal à celui des quatre plans. Pareillement le

rapport anharmonique des quatre points $a''$, $b''$, $c''$, $d''$, est égal à celui des quatre plans. Donc le rapport anharmonique des quatre points $a'$, $b'$, $c'$, $d'$, est égal à celui des quatre points $a''$, $b''$, $c''$, $d''$. On prouve de même que ce rapport est égal à celui des quatre points $a$, $b$, $c$, $d$, en considérant les quatre plans qui ont pour intersection commune la droite L'' et qui passent respectivement par les quatre droites $aa'$, $bb'$, $cc'$, $dd'$.

Le théorème est donc démontré.

Ce théorème exprime, comme on voit, une propriété générale de trois droites situées d'une manière quelconque dans l'espace. C'est cette propriété que je prends pour base du présent mémoire, et dont je vais développer les conséquences.

(4). Concevons que les trois droites L, L', L'', soient trois génératrices consécutives et infiniment voisines d'une surface gauche. Les quatre plans menés par la droite L et passant respectivement par les quatre droites $aa'$, $bb'$, $cc'$, $dd'$, seront les plans tangens à la surface aux points $a$, $b$, $c$, $d$. Désignons par A, B, C, D, ces quatre plans. Leur rapport anharmonique est égal à celui des quatre points $a'$, $b'$, $c'$, $d'$, où la droite L' les rencontre. Donc, d'après la proposition ci-dessus, ce rapport est égal à celui des quatre points de contact $a$, $b$, $c$, $d$; c'est-à-dire qu'on a

$$(2). \quad \ldots \quad \frac{\sin.\,C,A}{\sin.\,C,B} : \frac{\sin.\,D,A}{\sin.\,D,B} = \frac{ca}{cb} : \frac{da}{db}.$$

Donc :

*Quatre plans tangens à une surface gauche, menés par une même génératrice, ont leur rapport anharmonique égal à celui de leurs quatre points de contact.*

(5). Cette relation entre les plans tangens à une surface gauche et leurs points de contact, est susceptible de plusieurs conséquences.

Supposons que les deux plans C, D soient à angle droit; on aura

$$\sin.\,D,A = \cos.\,C,A\,;\ \sin.\,D,B = \cos.\,C,B\,;$$



l'équation devient donc

$$\frac{ca}{cb} : \frac{da}{db} = \frac{\sin.\,C,A}{\sin.\,C,B} : \frac{\cos.\,C,A}{\cos.\,C,B} = \frac{\text{tang.}\,C,A}{\text{tang.}\,C,B}.$$

Le plan D étant perpendiculaire au plan C, son point de contact $d$ est précisément le point où ce plan C est normal à la surface. Supposons que le point de contact $c$ de ce plan soit situé à l'infini, ce qui aura lieu si ce plan est mené parallèlement à la génératrice infiniment voisine de celle qui est la commune intersection des plans A, B, C; l'équation se réduira à

$$\frac{da}{db} = \frac{\text{tang.}\,C,B}{\text{tang.}\,C,A}.$$

Le point $d$ est celui où le plan C est normal à la surface : comme c'est un point unique sur chaque génératrice L, appelons-le le point *central* relatif à cette génératrice, et désignons-le par $o$. Appelons O le plan tangent en ce point, lequel sera perpendiculaire au plan C, on aura

$$\text{tang.}\,C,A = \frac{1}{\text{tang.}\,A,O}\,,\ \text{tang.}\,C,B = \frac{1}{\text{tang.}\,B,O}\,;$$

et l'équation deviendra

$$\frac{ao}{bo} = \frac{\text{tang.}\,A,O}{\text{tang.}\,B,O}.$$

Conséquemment, on a

$$ao = \lambda\ \text{tang.}\,A,O\,,$$
$$bo = \lambda\ \text{tang.}\,B,O\,,$$

$\lambda$ étant une constante, quels que soient les plans A, B. Donc

*Un plan quelconque étant mené par une génératrice d'une surface gauche, la distance du point où il est tangent à la surface, au*

# A.3 Chasles's text on polarity

An excerpt from [Chasles, 1837].



§ II. *Méthode analytique. — Propositions préliminaires.*

(1) Théorème I. *Si l'on conçoit dans l'espace un plan mobile, déterminé dans chacune de ses positions par une équation rapportée à trois axes coordonnés quelconques des* x, y *et* z; *et que les paramètres de cette équation contiennnent, au premier degré, les coordonnées d'un point que nous appellerons* directeur;

1° *Quand ce point parcourra un plan, le plan mobile tournera autour d'un point fixe;*

2° *Quand le point parcourra une droite, le plan mobile tournera autour d'une seconde droite;*

3° *Quand le point parcourra une surface courbe, le plan mobile roulera sur une autre surface courbe;*

*Si la première surface est du second degré, la seconde sera aussi du second degré;*

*Et si la première surface est géométrique et du degré* m, *la seconde surface sera aussi géométrique, et telle que par une droite quelconque, on pourra lui mener* m *plans tangens.*

En effet soient $x'$, $y'$, $z'$, les coordonnées du point directeur, l'équation du plan mobile sera de la forme

(1). . . . . $Xx' + Yy' + Zz' = U;$

X, Y, Z et U étant des polynomes de la forme $ax + by + cz - d$, où les coordonnées courantes $x$, $y$, $z$ n'entrent qu'au premier degré.

Soit

(2) . . . . . $Lx + My + Nz = 1$

l'équation du plan que parcourt le point directeur; L, M, N étant des constantes. Les coordonnées $x'$, $y'$, $z'$ de ce point auront entre elles la relation

(3). . . . . $Lx' + My' + Nz' = 1.$

On voit aisément que les valeurs des trois coordonnées $x$, $y$, $z$, tirées des trois équations suivantes:

(4). . . . . $X = LU,\ Y = MU,\ X = NU,$



déterminent un point qui se trouve sur le plan mobile dans toutes ses positions. Car si on substitue ces coordonnées dans l'équation (1) du plan mobile, il en résultera l'équation $Lx' + My' + Nz' = 1$, qui est identique, puisqu'elle n'est autre que l'équation de condition (3).

Ainsi le plan mobile, dans toutes ses positions, passe par un point fixe, dont les coordonnées sont déterminées par les trois équations (4); ce qui démontre la première partie du théorème.

Appelons ce point fixe le *pôle* du plan que le point directeur a parcouru.

Quand le point directeur parcourt une droite, qu'on imagine deux plans passant par cette droite; le point directeur se mouvra dans ces deux plans en même temps; le plan mobile passera donc par deux points fixes qui seront les *pôles* de ces plans; ce plan mobile tournera donc autour de la droite qui joint ces points; ce qui est la seconde partie du théorème.

Si le point directeur parcourt une surface courbe A, le plan mobile enveloppera une seconde surface courbe A'.

Supposons la surface A géométrique et du degré $m$ : une transversale menée arbitrairement la rencontrera en $m$ points; à chacun de ces points, considéré comme point directeur, correspondra un plan tangent à la surface A'; ces plans tangens, qui seront en nombre $m$, passeront tous par une même droite ( d'après la seconde partie du théorème); la surface A' admettra donc $m$ plans tangens passant par une même droite quelconque.

Si la surface A est du second degré, la surface A' sera donc aussi du second degré; ce qui démontre les deux dernières parties du théorème.

(2) La surface A', qui est l'enveloppe du plan mobile, quand le point directeur parcourt la surface A, peut être déterminée d'une seconde manière, par points, au moyen du théorème suivant:

Théorème II. *Quand le point directeur parcourt une surface courbe A, la surface enveloppe du plan mobile est le lieu géométrique des* pôles *de tous les plans tangens à la surface A;*

*Et le point où le plan mobile, dans une de ses positions, touche*

Digitized by G

*cette surface enveloppe, est le* pôle *du plan tangent à la surface A, mené par le point directeur auquel correspond cette position du plan mobile.*

En effet, pour obtenir le point où le plan mobile, dans une de ses positions, touche sa surface enveloppe A′, on regarde ce point comme l'intersection de ce plan tangent et de deux autres plans tangens infiniment peu différens; or ces trois plans correspondent à trois positions du point directeur infiniment voisines, c'est-à-dire à trois points d'un élément de la surface A, lesquels sont sur son plan tangent. Le point de contact du plan mobile et de la surface A′ est donc le *pôle* de ce plan tangent à la surface A; ce qui démontre, en même temps, les deux parties du théorème énoncé.

(3) *Remarque.* Ce théorème sera très-utile pour construire, par points, la surface enveloppe du plan mobile, quand le point directeur parcourra une surface donnée A.

Car il suffira de mener les plans tangens de la surface A, et de chercher leurs *pôles;* ce seront les points de la nouvelle surface. Or ces pôles se déterminent aisément, car soit $Lx+My+Nz=1$ l'équation d'un plan tangent à la surface A, les coordonnées de son pôle seront données par les trois équations linéaires

$$X=LU,\quad Y=MU,\quad Z=NU.$$

Ce moyen de construire par points la surface enveloppe du plan mobile, sert aussi pour trouver immédiatement l'équation de cette surface, sans recourir au calcul différentiel, ordinairement indispensable dans les questions de surfaces enveloppes.

Car le calcul se réduira à éliminer entre les trois équations que nous venons d'écrire, et l'équation de la surface proposée A, les coordonnées $x'$, $y'$, $z'$ appartenant à cette surface; l'équation résultante en X, Y, Z, qui sont des fonctions des coordonnées $x$, $y$, $z$ de la surface enveloppe du plan mobile, sera précisément l'équation de cette surface.

Supposons, par exemple, que le point directeur parcourre la surface du second degré représentée par l'équation



$$Ax^2+By^2+Cz^2+2(Dx+Ey+Fz)=K;$$

l'équation du plan tangent au point $(x', y', z')$ de cette surface sera

$$x(Ax'+D)+y(By'+E)+z(Cz'+F)=K-(Dx'+Ey'+Fz').$$

Les trois équations de condition ci-dessus seront donc

$$\begin{aligned}X[(K-(Dx'+Ey'+Fz')]&=(Ax'+D).U.\\ Y[(K-(Dx'+Ey'+Fz')]&=(By'+E)\ U.\\ Z[(K-(Dx'+Ey'+Fz')]&=(Cz'+F)\ U.\end{aligned}$$

Il suffit d'éliminer $x'$, $y'$, $z'$ entre ces trois équations et la suivante

$$Ax'^2+By'^2+Cz'^2+2(Dx'+Ey'+Fz')=K.$$

Le résultat

$$\left(\frac{X^2}{A^2}+\frac{Y^2}{B^2}+\frac{Z^2}{C^2}\right)\left(K+\frac{D^2}{A}+\frac{E^2}{B}+\frac{F^2}{C}\right)=\left(1+\frac{LD}{A}+\frac{ME}{B}+\frac{NF}{C}\right)^2,$$

est l'équation de la surface enveloppe. Cette équation est du second degré, puisque X, Y, Z sont des fonctions linéaires des coordonnées courantes $x$, $y$, $z$.

(4) Théorème III. *Quand le point directeur se meut à l'infini, le plan mobile tourne autour d'un point fixe, comme si le point directeur parcourait un plan.*

En effet, si le point directeur est situé à l'infini sur une droite ayant pour équations

$$x=az,\quad y=bz,$$

on aura

$$x'=az',\quad y'=bz';$$

et l'équation (1) du plan mobile, correspondante à cette position du point directeur, deviendra

$$(aX+bY+Z)z'=U.$$

Or $z'$ est infini; cette équation se réduit donc à

$$aX + bY + Z = o.$$

Telle est l'équation du plan mobile, dans sa position correspondante au point directeur situé à l'infini sur la droite dont nous avons posé les équations.

Ce plan passe évidemment par le point dont les coordonnées $x$, $y$, $z$ sont déterminées par les trois équations

$$X = o, \quad Y = o, \quad Z = o.$$

Donc, dans quelque position que soit situé, à l'infini, le point directeur, le plan mobile passera par le point fixe que déterminent ces trois équations; comme si le point directeur parcourait un plan. C. Q. F. D.

(5) Si l'on cherchait à déterminer ce plan, il faudrait mettre dans les trois équations (4), à la place de $x$, $y$, $z$, leurs valeurs tirées des trois équations ci-dessus; on aurait ainsi les valeurs des coefficiens L, M, N correspondantes à la position de ce plan. Or ces équations deviennnent

$$LU = o, \quad MU = o, \quad NU = o;$$

U n'est pas nulle, puisque c'est une fonction de $x$, $y$, $z$, qui est devenue un nombre par la substitution des valeurs de ces coordonnées; il faut donc qu'on ait

$$L = o, \quad M = o, \quad N = o;$$

et dès lors l'équation $Lx + My + Nz = 1$ représente un plan situé à l'infini, et indéterminé de direction.

Nous pouvons donc énoncer, avec M. Poncelet, cette idée paradoxale, mais d'une justesse mathématique:

*L'espace indéfini a pour enveloppe une surface plane.* (Traité des propriétés projectives, p. 373).





(6) Théorème IV. *Quand le point directeur prend quatre positions* a, b, c, d, *en ligne droite, le plan mobile prend quatre positions correspondantes, que nous désignons par* A, B, C, D;

*Ces quatre plans passent par une même droite, et leur rapport anharmonique*[1] *est égal à celui des quatre points* a, b, c, d; *c'est-à-dire que l'on a*

$$\frac{\sin.\ C,A}{\sin.\ C,B} : \frac{\sin.\ D,A}{\sin.\ D,B} = \frac{ca}{cb} : \frac{da}{db}.$$

En effet, les quatre plans A, B, C, D passent par une même droite, puisque les quatre points $a$, $b$, $c$, $d$ sont en ligne droite; par conséquent si on tire une transversale quelconque qui rencontre ces quatre plans aux points $\alpha$, $\epsilon$, $\gamma$, $\delta$, on aura

$$\frac{\gamma\alpha}{\gamma\epsilon} : \frac{\delta\alpha}{\delta\epsilon} = \frac{\sin.\ C,A}{\sin.\ C,B} : \frac{\sin.\ D,A}{\sin.\ D,B}.$$

Il suffit donc de prouver que le premier membre de cette équation est égal au second membre de l'équation ci-dessus.

Pour cela, soient $\xi$, $\nu$, $\zeta$, les coordonnées d'un point fixe K, pris sur la droite des quatre points $a$, $b$, $c$, $d$; et soient $r'$, $r''$, $r'''$, $r^{IV}$, les distances de ces quatre points au point K; leurs coordonnées seront

$$\left.\begin{aligned} x' &= \xi + lr', \\ y' &= \nu + mr', \\ z' &= \zeta + nr'; \end{aligned}\right\} \text{ pour le point } a;$$

$$\left.\begin{aligned} x'' &= \xi + lr'', \\ y'' &= \nu + mr'', \\ z'' &= \zeta + nr''; \end{aligned}\right\} \text{ pour le point } b;$$

$$\left.\begin{aligned} x''' &= \xi + lr''', \\ y''' &= \nu + mr''', \\ z''' &= \zeta + nr'''; \end{aligned}\right\} \text{ pour le point } c;$$

et

$$\left.\begin{aligned} x^{IV} &= \xi + lr^{IV}, \\ y^{IV} &= \nu + mr^{IV}, \\ z^{IV} &= \zeta + nr^{IV}; \end{aligned}\right\} \text{ pour le point } d;$$

[1] Nous avons donné dans la Note IX la définition du *rapport anharmonique* de quatre points, ou de quatre plans.

$l$, $m$, $n$ étant les quantités angulaires qui déterminent la direction de la droite $ab$.

L'équation du plan mobile, dans sa position A, correspondante au point directeur $a$, est

$$X\xi + Y\nu + Z\zeta - U + (lX + mY + nZ)\,r' = o.$$

soit $\rho'$ la distance du point K au point où ce plan rencontre le droite $ab$ ; les coordonnées de ce dernier point seront de la forme

$$\begin{aligned} x &= \xi + l\rho', \\ y &= \nu + m\rho', \\ z &= \zeta + n\rho' ; \end{aligned}$$

elles doivent satisfaire à l'équation du plan; les y mettant, on aura une équation qui donnera la valeur de $\rho'$. Cette équation sera évidemment de la forme

$$P + Q\rho' + (R + S\rho')\,r' = o ;$$

P, Q, R, S étant des fonctions de $\xi$, $\nu$, $\zeta$, $l$, $m$, $n$.

Les distances $\rho''$, $\rho'''$, $\rho^{IV}$, du point K aux points où les trois autres plans B, C, D rencontrent la droite $ab$, seront données par les équations semblables

$$\begin{aligned} P + Q\rho'' + (R + S\rho'')\,r'' &= o, \\ P + Q\rho''' + (R + S\rho''')\,r''' &= o, \\ P + Q\rho^{IV} + (R + S\rho^{IV})\,r^{IV} &= o. \end{aligned}$$

De ces équations, on tire

$$\begin{aligned} \frac{\rho''' - \rho'}{\rho''' - \rho''} &= \frac{r''' - r'}{r''' - r''} \cdot \frac{Q + Sr''}{Q + Sr'}, \\ \frac{\rho^{IV} - \rho'}{\rho^{IV} - \rho''} &= \frac{r^{IV} - r'}{r^{IV} - r''} \cdot \frac{Q + Sr''}{Q + Sr'}. \end{aligned}$$

D'où

$$\frac{\rho'' - \rho'}{\rho''' - \rho''} : \frac{\rho^{IV} - \rho'}{\rho^{IV} - \rho''} = \frac{r''' - r'}{r''' - r''} : \frac{r^{IV} - r'}{r^{IV} - r''} ;$$

équation qui démontre le théorème énoncé.

# A.4 Von Staudt's algebra of throws

An excerpt from [von Staudt, 1856].



einander gemein haben, in der die Regelflächen sich berühren, haben auch entweder einen Leitstrahl p, in welchem die Regelflächen sich berühren, oder zwei Leitstrahlen p, q mit einander gemein, je nachdem nämlich irgend eine von a verschiedene Gerade der einen Regelschaar die andere Regelfläche in einem Punkte P berührt oder in zwei Punkten P, Q schneidet. Die Gerade p geht durch den Punkt P und den Pol der Ebene aP und eben so im letztern der erwähnten Fälle die Gerade q durch den Punkt Q und den Pol der Ebene aQ.

Haben zwei Regelflächen die Gerade u mit einander gemein, ohne in dieser Geraden sich zu berühren, so haben die zu einander projektivischen geraden Gebilde, von welchen das eine in Hinsicht auf die eine Fläche und das andere in Hinsicht auf die andere Fläche dem Ebenenbüschel u zugeordnet ist, entweder nur einen Punkt oder zwei Punkte entsprechend gemein. Im erstern Falle geht durch die Gerade u nur eine Ebene, welche beiden Flächen in einem und demselben Punkte sich anschmiegt, während im letztern Falle zwei solche Ebenen in der Geraden u sich schneiden.

255. Wenn von den vier Geraden a, $a_1$, b, $b_1$ keine zwei sich schneiden, aber auch nicht alle vier einer und derselben Regelschaar angehören, so gibt es entweder nur ein involutorisches System, in welchem der Geraden a die Gerade $a_1$ und der Geraden b die Gerade $b_1$ zugeordnet ist, oder es giebt zwei solche Systeme, je nachdem nämlich die Regelschaaren $aa_1b$, $aa_1b_1$, $abb_1$, $a_1bb_1$ nur einen Leitstrahl p oder zwei Leitstrahlen p, q mit einander gemein haben.

Es seien M, N die Punkte, welche sowohl durch die Punkte pa, $pa_1$ als auch durch die Punkte pb, $pb_1$ harmonisch getrennt sind. Wenn nun die durch den Punkt M gehenden Geraden h, $h_1$ der Regelschaaren $aa_1b$, $aa_1b_1$ mit der Geraden p in einerlei Ebene liegen also durch diese Gerade und eine Gerade m harmonisch getrennt sind, so ist in dem involutorischen Systeme (m, n), dessen andere Ordnungslinie n von der erstern m durch die Geraden a, $a_1$ harmonisch getrennt ist, der Geraden h die Gerade $h_1$, der Geraden a die Gerade $a_1$ und also, da $haa_1b \barwedge p\,(haa_1b) \barwedge p\,(h_1a_1ab_1) \barwedge h_1a_1ab_1$ ist, der Geraden b die Gerade $b_1$ zugeord-





net. Wenn es aber noch eine Gerade q gibt, welche die vier Geraden a, $a_1$, b, $b_1$ schneidet und S, T die Punkte sind, welche durch die Punkte qa, $qa_1$ und zugleich durch die Punkte qb, $qb_1$ harmonisch getrennt sind, so ist sowohl in dem involutorischen Systeme (MS, NT) als auch in dem involutorischen Systeme (MT, NS) der Geraden a die Gerade $a_1$ und der Geraden b die Gerade $b_1$ zugeordnet. Auch gibt es in diesem Falle ein involutorisches System (p, q), in welchem jede der Geraden a, $a_1$, b, $b_1$ sich selbst zugeordnet ist.

Anm. Man wird nun auch leicht entscheiden, was in §. 5 und §. 6 zu berichtigen ist, wenn der Begriff Element im weitern Sinne genommen wird.

---

## §. 19.

### Summen von Würfen.

256. Von nun an werden Würfe, welche zu einander projektivisch sind, als gleich betrachtet, daher auch von einem Wurfe gesagt wird, dass er bestimmt sei, wenn irgend ein zu ihm projektivischer Wurf gegeben ist. Ist von einem eigentlichen Wurfe abcd die Rede, so versteht es sich von selbst, dass a, b, c, d vier verschiedene Elemente eines und desselben Elementargebildes sind. Der Inbegriff von drei verschiedenen Elementen eines und desselben Elementargebildes, von welchen aber das eine zweimal angeschrieben ist, soll ein uneigentlicher Wurf heissen und zwar ein uneigentlicher Wurf I. Art, wenn er die Form abca oder baac hat, ein uneigentlicher Wurf II. Art, wenn er die Form abcb oder babc hat, ein uneigentlicher Wurf III. Art endlich, wenn er die Form abcc oder ccab hat.

Obgleich die Würfe ABCA, baac nicht projektivisch zu einander sind, so sollen doch alle uneigentlichen Würfe, welche nach dem Obigen von einer und derselben Art sind, einander gleich heissen. Ein uneigentlicher Wurf I. Art soll auch durch 0, ein uneigentlicher Wurf II. Art durch 1, ein uneigentlicher Wurf III. Art durch $\infty$ bezeichnet werden. Bemerkt wird noch, dass

jeder uneigentliche Wurf, da seine Elemente einer und derselben Kette angehören, ein neutraler Wurf ist.

257. Wenn man Würfe, welche zu einander projektivisch sind, als gleich, demnach nur als verschiedene Darstellungen eines und desselben Wurfes betrachtet, so kann man zwei Würfe $U, U_1$ schon dann einander conjungirt nennen, wenn sie irgend zwei den Elementen nach einander conjungirten Würfen ABCD, $A_1B_1C_1D_1$ beziehlich gleich sind und also (238) zu jedem Wurfe abcd, welcher dem Wurfe U und daher auch dem Wurfe ABCD gleich ist, einen Wurf $a_1b_1c_1d_1$ es gibt, welcher dem Wurfe $A_1B_1C_1D_1$ und daher auch dem Wurfe $U_1$ gleich ist, so dass überdiess abcd, $a_1b_1c_1d_1$ homologe Würfe von zwei einander conjungirten Elementargebilden sind. Ist $U=ABCD$ ein neutraler Wurf, so ist (239) auch $U=A_1B_1C_1D_1$, daher man von jedem neutralen Wurfe sagen kann, dass er sich selbst conjungirt sei.

Was auch ABCD für ein Wurf sein mag, so ist $ABCD=BADC=CDAB=DCBA$. In der Folge aber ist unter einem Wurfe, wenn nicht ausdrücklich gesagt wird, dass er ein uneigentlicher Wurf III. Art sei, entweder ein eigentlicher Wurf oder ein uneigentlicher Wurf I. oder II. Art zu verstehen.

258. Durch drei Elemente A, B, C eines Elementargebildes und zwei Würfe $U, U_1$ sind drei Elemente D, $D_1$, S des Gebildes bestimmt, so dass nämlich $ABCD=U$, $ABCD_1=U_1$ und in dem involutorischen Gebilde CC. $DD_1$... dem Elemente A das Element S zugeordnet ist. Der Wurf ABCS, welcher auf diese Weise durch die Würfe $U, U_1$ bestimmt ist, soll ihre Summe heissen, und diess durch $ABCS=U+U_1$ ausgedrückt werden. Sucht man in demselben oder in einem andern Elementargebilde zu drei Elementen a, b, c die Elemente d, $d_1$, s, so dass die Würfe abcd, abc$d_1$ den Würfen $U, U_1$ gleich sind und in dem involutorischen Gebilde cc.$dd_1$... dem Elemente a das Element s zugeordnet ist, so ist abcd, s$\nwarrow$ABCD$D_1$S und also abcs=ABCS, woraus eben hervorgeht, dass durch die beiden Würfe $U, U_1$ ein Wurf $U+U_1$ bestimmt ist. Auch ist offenbar, dass $U\,U_1=U_1+U$ ist. Fällt D mit A zusammen, so fällt S mit $D_1$ zusammen. Wenn also der eine von zwei Würfen $=0$ ist, so ist ihre Summe dem andern gleich.



259. Setzt man für irgend zwei von m Würfen $U, U_1, U_2, \ldots U_{m-1}$ ihre Summe, so hat man einen Wurf weniger als vorher. Fährt man so fort, bis man nur noch einen Wurf hat, so ist dieser die Summe der m Würfe.

Ist nämlich irgend einer von den drei Würfen $U, U_1, U_2$ ein uneigentlicher Wurf I. Art, so folgt schon aus dem Vorigen, dass die Summe $(U+U_1)+U_2$ welche auch durch $U+U_1+U_2$ bezeichnet werden kann, dieselbe bleibt, wenn man die Würfe $U_1, U_2$ mit einander vertauscht. Ist $U_1=U_2$, so ist offenbar $U+U_1+U_2=U+U_2+U_1$. Wenn aber keiner von den erwähnten Fällen stattfindet, so suche man in irgend einem Elementargebilde zu drei Elementen A, B, C die Elemente D, $D_1, D_2$, P, Q, so dass $ABCD=U$, $ABCD_1=U_1$, $ABCD_2=U_2$, $ABCP=U+U_1$ und $ABCQ=U+U_2$ ist. Da alsdann CC.$DD_1$. AP, CC.$DD_2$.AQ Involutionen sind, folglich (85, 221) $\overline{CD_1}\,P\nwarrow\overline{CA}\,D$ und eben so $\overline{C}\,D_2\,Q\nwarrow\overline{C}\,A\,D$ ist, so ist auch (223) $\overline{CD_1}\,P\wedge\overline{CD_2}\,Q$ und mithin auch CC.$D_1Q$.$D_2P$ eine Involution. Ist nun R das Element, welches in dem involutorischen Gebilde CC.$D_1Q$.$D_2P$... dem Elemente A zugeordnet ist, so ist $ABCR=ABCP+ABCD_2=U+U_1+U_2$ und eben so $ABCR=ABCQ+ABCD_1=U+U_2+U_1$, folglich auch $U+U_1+U_2=U+U_2+U_1$. Da hiernach auch $(U+U_1\ldots+U_{r-1})+U_r+U_{r+1}=(U+U_1\ldots+U_{r-1})+U_r+U_{r+1}$ ist, so folgt, dass man nicht nur die beiden ersten sondern je zwei auf einander folgende Würfe mit einander vertauschen und daher die m Würfe beliebig unter einander versetzen, mithin auch für irgend zwei, da man sie als die beiden ersten betrachten kann, ihre Summe setzen darf. Dasselbe gilt aber auch von dem m—1 Würfen, aus welchen alsdann die Summe noch besteht.

Sind die Würfe $U, U_1, \ldots U_{m-1}$ den Würfen $V, V_1, \ldots V_{m-1}$ beziehlich gleich, so ist auch die Summe aus den m erstern Würfen der Summe aus dem m letztern gleich.

260. Wenn die Würfe $U, U_1, \ldots U_{m-1}$ den Würfen $V, V_1, \ldots V_{m-1}$ conjungirt sind, so ist auch die Summe der m erstern Würfe der Summe der m letztern conjungirt.

Man suche in einer reellen Curve II. Ordnung zu drei reellen Punkten A, B. C die Punkte D, $D_1$, P, so dass $ABCD=U$, $ABCD_1=U_1$ und $ABCP=U+U_1$ ist. Es seien ferner E, $E_1$, Q die Punkte, welche den Punkten D, $D_1$, P conjungirt sind. Weil

nun der Punkt R, in welchem die Tangente CC von der Geraden $DD_1$ geschnitten wird, in der Geraden AP liegt, so liegt auch der dem Punkte R conjungirte Punkt, in welchem die Geraden CC, $EE_1$ sich schneiden, in der Geraden AQ, woraus hervorgeht, dass der dem Wurfe ABCP conjungirte Wurf ABCQ $= V + V_1$ ist. Da hiernach der Satz für $m=2$ gilt, so gilt er auch für jedes grössere m.

Die Summe von neutralen Würfen ist ein neutraler Wurf. Eben so ist aber auch die Summe von zwei einander conjungirten Würfen ein neutraler Wurf.

261. Durch zwei Würfe U, $U_1$ ist ein Wurf $U_2$ bestimmt, so dass nämlich $U = U_1 + U_2$ ist.

Man suche in einem Elementargebilde zu drei Elementen A, B, C die Elemente S, D, E, so dass die Würfe ABCS, ABCD den Würfen U, $U_1$ gleich sind und in dem involutorischen Gebilde CC.AS .. dem Elemente D das Element E zugeordnet ist. Bezeichnet man nun den Wurf ABCE durch $U_2$, so ist $U = U_1 + U_2$. Und wenn ABCS = ABCD + ABCE sein soll, so muss in dem involutorischen Gebilde CC.AS... dem Elemente D das Element E zugeordnet sein. Sind U, $U_1$ neutrale Würfe, so ist auch $U_2$ ein neutraler Wurf. Ist $U_1 + U_2 = V_1 + V_2$ und $U_1 = V_1$, so ist auch $U_2 = V_2$.

262. Die Summe zweier Würfe ABCD, BACD, von welchen der eine aus dem andern durch Vertauschung der beiden ersten oder auch durch Vertauschung der beiden letzten Elemente entsteht, ist ein uneigentlicher Wurf II. Art.

Es sei in dem involutorischen Gebilde CC.AB... dem Elemente D das Element E zugeordnet, so ist ABCD + ABCE = ABCB = 1 und ABCE = BACD = ABDC, woraus der Satz folgt.

263. Wenn die Involution CC.BD.AE und das Element P einem und demselben Elementargebilde angehören, so ist ABCP + EDCP = ABCE = EDCA und ABCP = DECP + ABCD = DECP + EDCB.

Da nämlich CC.BD.AE eine Involution ist, so ist ABCE = EDCA und ABCD = EDCB. Wenn nun in dem involutorischen Gebilde CC.BD.AE... dem Elemente P das Element Q zugeordnet ist, so ist ABCP + ABCQ = ABCE und ABCQ = EDCP, woraus der erste Theil des Satzes folgt. Es seien ferner N, P homologe



Elemente der zu einander projektivischen Gebilde ABC..., DEC... welche (221) nur das Element C entsprechend gemein haben, so ist ABCN = DECP und in dem involutorischen Gebilde CC.DN.., dem Elemente A das Element P zugeordnet, mithin ABCP = ABCN + ABCD, woraus der letztere Theil des Satzes sich ergiebt.

264. Zwei Würfe, deren Summe = 0 ist, sollen einander entgegengesetzt heissen. Ist CC.BD.AE eine Involution, so ist (263) DECA + EDCB = ABCA = 0 und also der Wurf DECA dem Wurfe EDCB entgegengesetzt.

Sind in einem Elementargebilde, welchem das Element P angehört, die Elemente A, C durch die Elemente B, D harmonisch getrennt, so ist (263) ABCP + ADCP = ABCA und mithin dem Wurfe ABCP der Wurf ADCP entgegengesetzt. Namentlich ist also der harmonische Wurf ABCD dem Wurfe 1 entgegengesetzt.

Wenn den Würfen U, V die Würfe $U_1$, $V_1$ entgegengesetzt sind, und $U + V = W$ ist, so ist $U = U + V + V_1 = W + V_1$ und eben so $V = W + U_1$.

265. Es seien in dem einen von zwei Elementargebilden g, $g_1$ die Elemente E, D, C und im andern die Elemente $A_1$, $B_1$, $C_1$ gegeben. Es sei ferner der Wurf EDCA einem gegebenen Wurfe U gleich und EDCF ein harmonischer Wurf. Nennt man nun zwei Elemente P, $P_1$ der Gebilde g, $g_1$ einander entsprechend, wenn $EDCP + A_1B_1C_1P_1 = U$ und also (264) $A_1B_1C_1P_1 = EFCP + U$ ist, so sind die Gebilde g, $g_1$ projektivisch auf einander bezogen.

Ist nämlich B das Element, welches in dem involutorischen Gebilde CC.AE... dem Elemente D zugeordnet ist, so ist (263) $ABCP + EDCP = U = EDCP + A_1B_1C_1P_1$, mithin $ABCP = A_1B_1C_1P_1$, woraus der Satz folgt.

Wenn die Gebilde g, $g_1$ in einander liegen, aber $C_1$ nicht mit C zusammenfällt, so lässt die Aufgabe: ein Element P zu finden, sodass $EDCP + A_1B_1C_1P = U$ ist, eine oder zwei Auflösungen zu, je nachdem die zu einander projektivischen Gebilde ABC..., $A_1B_1C_1$... ein Element oder zwei Elemente entsprechend gemein haben.

266. Die Summe von m Würfen, deren jeder = U ist, soll das m fache dieses Wurfes heissen und durch mU bezeichnet wer-

den. Es ist hienach $m(U+U_1)=mU+mU_1$, $m(U+U_1+U_2)=mU+mU_1+mU_2$, $(m+n)U=mU+nU$, u. s. w.

Ist $U=U_1$, so ist auch $mU=mU_1$. Ist $U=0$, so ist auch $mU=o$. Sind die Würfe $U, U_1$ einander entgegengesetzt, so sind es auch die Würfe $mU, mU_1$, weil, wenn $U+U_1=0$ ist, auch $mU+mU_1=0$ ist. Sind die Würfe U, V einander conjungirt, so gilt diess auch (260) von den Würfen mU, mV.

267. Durch drei reelle Elemente C, A, $A_1$ eines reellen Elementargebildes g ist in ihm eine Reihe C, A, $A_1$, $A_2$, $A_3$, $A_4$,... von reellen Elementen bestimmt, so dass nämlich $CAA_1A_2$, $CA_1A_2A_3$, $CA_2A_3A_4$, u. s. w. harmonische Würfe sind, daher auch keine zwei auf einander folgende Glieder der Reihe, so weit man dieselbe fortsetzen mag, durch ihre übrigen Glieder in dem Gebilde g getrennt sind. Ist nun B irgend ein von A verschiedenes Element des Gebildes g, so ist $ABCA_m=m(ABCA_1)$. Denn nach 221 haben die zu einander projektivischen Gebilde $CAA_1...$, $CA_1A_2...$ nur das Element C entsprechend gemein. Da ferner den Elementen A, $A_1$, $A_2, A_3...$ des erstern die Elemente $A_1$, $A_2$, $A_3$, $A_4$... des letztern entsprechen, mithin $CCAA_{r+1}.A_1A_r$ eine Involution ist, so ist $ABCA_{r+1}=ABCA+ABCA_1$, woraus, wenn für r nach und nach die Zahlen 1, 2, 3,...m gesetzt werden, der Satz sich ergiebt.

Sucht man in dem Gebilde g das Element B, so dass $ABCA_1=U$ ist, so ist $ABCA_m=mU$. Sind also U, $U_1$ zwei einander nicht gleiche Würfe, so sind auch die Würfe $mU, mU_1$ einander nicht gleich. Sucht man das Element B, so dass $ABCA_m=V$ ist, so ist $ABCA_1$ der Wurf, von welchem der Wurf V das m fache ist. Ist $V=mU=0$, so ist auch $U=0$. Der Wurf $AA_1CA_m$ ist das m fache von 1, der Wurf 1 aber das m fache von $AA_mCA_1$.

---

## §. 20.

### Produkte aus Würfen.

268. Durch drei Elemente M, A, N eines Elementargebildes und zwei Würfe $U, U_1$, von welchen keiner ein uneigentlicher Wurf I. oder III. Art ist, sind zwei Elemente $A_1, A_2$ des Gebildes bestimmt, so dass nämlich $MANA_1=U$ und $MA_1NA_2=U_1$ ist. Der Wurf $MANA_2$, welcher auf diese Weise durch die Würfe $U, U_1$



bestimmt, soll das Produkt aus denselben heissen und durch $UU_1$ bezeichnet werden. Sucht man in demselben oder in einem andern Elementargebilde zu drei Elementen F, B, G die Elemente $B_1$, $B_2$, so dass $FBGB_1=U$ und $FB_1GB_2=U_1$ ist, so ist $FGBB_1B_2\times MNAA_1A_2$, mithin $FBGB_2=MANA_2$, woraus eben hervorgeht, dass durch die Würfe U, $U_1$ ein Wurf $UU_1$ bestimmt ist.

Da den Würfen $U_1$, U auch die Würfe $NA_2MA_1, NA_1MA$ gleich sind, so ist $U_1U=NA_2MA=MANA_2=UU_1$. Ist irgend einer von den beiden Würfen $U, U_1$ ein uneigentlicher Wurf II. Art, so ist ihr Produkt dem andern gleich.

269. Durch drei Elemente M, A, N eines Elementargebildes und m Würfe $U, U_1, U_2...U_{m-1}$, von welchen keiner ein uneigentlicher Wurf I. oder III. Art ist, sind m Elemente $A_1$, $A_2$, $A_3,...A_m$ des Gebildes bestimmt, so dass nämlich die Würfe $MANA_1$, $MA_1NA_2$, $MA_2NA_3,...MA_{m-1}NA_m$ den Würfen $U, U_1, U_2,...U_{m-1}$ gleich sind. Der Wurf $MANA_m$, welcher auf diese Weise durch die gegebenen Würfe bestimmt ist, soll das Produkt aus denselben heissen.

Setzt man für irgend zwei der m Würfe ihr Produkt, so hat man einen Wurf weniger als vorher. Fährt man so fort, bis man nur noch einen Wurf hat, so ist dieser das Produkt aus den sämmtlichen Würfen. Da nämlich $MANA_{r+2}=(MANA_r)(MA_rNA_{r+2})$ ist, so folgt, dass das Produkt aus den gegebenen Würfen dasselbe bleibt, wenn man für irgend zwei auf einander folgende Faktoren ihr Produkt setzt. Da man hiernach auch zwei auf einander folgende Faktoren mit einander vertauschen darf, so kann man mit den Faktoren eines Produkts jede beliebige Versetzung vornehmen und daher auch für irgend zwei derselben ihr Produkt setzen.

Sind die Würfe $U, U_1, U_2,... U_{m-1}$ den Würfen $V, V_1, V_2,...V_{m-1}$ conjungirt, so ist auch das Produkt aus den m ersten Würfen dem Produkte aus den m letztern conjungirt. Wenn man nämlich von drei reellen Elementen M, A, N eines reellen Elementargebildes ausgeht und den Elementen $A_1, A_2, A_3...A_m$ die Elemente $B_1, B_2, B_3,...B_m$ conjungirt sind, so ist auch der Wurf $MANA_m$ dem Wurfe $MANB_m$ conjungirt, welcher das Produkt aus den Würfen $V, V_1, V_2,...V_{m-1}$ ist. Jedes Produkt aus neutralen Würfen ist selbst ein neutraler Wurf.

Uneigentliche Würfe III. Art bleiben auch in der Folge, wenn von Summen oder Produkten die Rede ist, ausgeschlossen. Ist aber irgend ein Faktor eines Produkts ein uneigentlicher Wurf I. Art, so ist unter dem Produkte ebenfalls ein solcher Wurf zu verstehen.

270. Wenn das Produkt aus zwei Würfen $=1$ ist, so soll jeder derselben das Umgekehrte vom andern heissen. Ist der eine von zwei solchen Würfen $=MANA_1$, so ist der andere $=MA_1NA = NAMA_1$. Ist ein Wurf $MANA_1$ dem Wurfe $MA_1NA$, welcher das Umgekehrte von ihm ist, gleich, so ist er entweder ein uneigentlicher Wurf II. Art oder ein harmonischer Wurf, je nachdem nämlich die Elemente A, $A_1$ in einander fallen oder durch die Elemente M, N harmonisch getrennt sind.

Wenn $V_1$ das Umgekehrte von $U_1$, $V_2$ das Umgekehrte von $U_2$ und $U=U_1U_2$ ist, so ist $UV_1=V_1U_1U_2=U_2$ und ebenso $UV_2=U_1$. Vorausgesetzt wird hiebei, dass keiner von den beiden Würfen $U_1$, $U_2$ ein uneigentlicher Wurf I. Art ist, weil das Umgekehrte von einem solchen Wurfe nur ein uneigentlicher Wurf III. Art sein könnte.

271. Durch zwei Würfe U, $U_1$, von welchen der letztere nicht $=0$ ist, ist ein Wurf $U_2$ bestimmt, so dass $U = U_1U_2$ ist.

Es sei $V_1$ das Umgekehrte von $U_1$. Bezeichnet man nun das Produkt $UV_1$ durch $U_2$, so ist $U_1U_2=U$. Und wenn $U_1U_2=U$ sein soll, so muss $U_2=UV_1$ sein. — Ist der Wurf $U_1$, welcher nicht gleich $=0$ ist, dem Wurfe $W_1$ und der Wurf $U_1U_2$ dem Wurfe $W_1W_2$ gleich, so ist auch $U_2=W_2$.

272. Wenn MN.AD.BC eine Involution folglich $MANC=NDMB=MBND$ ist, so ist $(MANB)(MANC)=MAND$. Das m fache $ABCA_m$ eines Wurfes $ABCA_1$ ist (267) nichts anderes als das Produkt aus diesem Wurfe in den Wurf $AA_1CA_m$, welcher das m fache von 1 ist.

273. Das Produkt aus zwei einander conjungirten Würfen U, V ist ein neutraler Wurf.

Wenn nämlich U, V zwei nicht neutrale Würfe sind, so suche man in einer reellen Curve II. Ordnung zu drei reellen Punkten M, A, N die Punkte B, C, so dass $MANB=U$ und $MANC=V$ ist. Da nun die Punkte B, C einander conjungirt sind, mithin die Geraden MN, BC in einem reellen Punkte sich schneiden, so enthält die Curve einen reellen Punkt D, so dass MN.BC.AD eine Involution und also (272) $MAND=UV$ ist. Es geht hieraus zugleich hervor, dass in der Curve die Punkte M, N durch die Punkte A, D nicht getrennt sind.



Ist das Produkt aus zwei einander conjungirten Würfen U, V ein uneigentlicher Wurf II. Art, so gibt es in jedem reellen Elementargebilde zu zwei einander conjungirten imaginären Elementen M, N und einem reellen Elemente A ein reelles Element B, so dass $U=MANB$ und also $V=MBNA$ ist. Es sei nämlich $U=MANF$ und dem Elemente F das Element G conjungirt, so ist $V=NAMG=MGNA$ und also $VU=MGNF$, woraus hervorgeht, dass die Elemente F, G mit einem und demselben reellen Elemente B zusammenfallen.

274 Wenn U, $U_1$, V, $V_1$, $V_2$ beliebige Würfe sind, so ist $U(V+V_1)=UV+UV_1$, also auch $U(V+V_1+V_2)=UV+UV_1+UV_2$, $(U+U_1)(V+V_1)=UV+UV_1+U_1V+U_1V_1$, u. s. w.

Es sei $U=ABCD$, $V=ADCE$ und $V_1=ADCE_1$, so ist $UV=ABCE$ und $UV_1=ABCE_1$. Ist nun in dem involutorischen Gebilde $CC.EE_1\ldots$ dem Elemente A das Element P zugeordnet, so ist $V+V_1=ADCP$ und $UV+UV_1=ABCP=(ABCD)(ADCP)$, woraus der Satz folgt.

275. Es seien in dem einen von zwei Elementargebilden g, $g_1$ die Elemente C, D, A und im andern die Elemente $A_1$, $B_1$, $C_1$ gegeben. Es sei ferner ein Wurf $U=ABCD$ gegeben, welcher nicht $=0$ ist. Nennt man nun zwei Elemente P, $P_1$ der Gebilde g, $g_1$ einander entsprechend, wenn $(CDAP)(A_1B_1C_1P_1)=U$ und also $A_1B_1C_1P_1=U.(ADCP)$ ist, so sind die Gebilde g, $g_1$ projektivisch auf einander bezogen. Da nämlich $U(ADCP)=ABCP$ ist, so ist auch $ABCP=A_1B_1C_1P_1$, woraus der Satz folgt.

Wenn die Gebilde g, $g_1$ in einander liegen, aber weder A mit $A_1$ noch C mit $C_1$ zusammenfällt, so lässt die Aufgabe: ein Element P zu finden, so dass $(CDAP)(A_1B_1C_1P)=U$ ist, eine oder zwei Auflösungen zu, je nachdem die zu einander projektivischen Gebilde ABC..., $A_1B_1C_1\ldots$ ein Element oder zwei Elemente entsprechend gemein haben.

276. Durch zwei Würfe U, V sind zwei Würfe ABCP, ABCQ

bestimmt, deren Summe dem erstern und deren Produkt dem letztern der gegebenen Würfe gleich ist.

Man suche in einer Curve II. Ordnung zu drei Punkten A, B, C die Punkte D, E, so dass ABCD=U und ABCE=V ist. Es werde ferner die Gerade, welche die Curve im Punkte C berührt, von der Geraden AD im Punkte F und die Gerade AC von der Geraden BE im Punkte G geschnitten. Wenn nun die Gerade FG die Curve in den Punkten P, Q schneidet, so ist in dem involutorischen Gebilde CC.AD... dem Punkte P der Punkt Q zugeordnet, folglich (258) ABCP+ABCQ=U. Da ferner die Punkte P, Q auch in dem involutorischen Gebilde AC.BE... einander zugeordnet sind, so ist (272) (ABCP) (ABCQ)=V. Und wenn ABCP+ABCQ=ABCD und zugleich (ABCP) (ABCQ)=ABCE sein soll, so müssen P, Q die Punkte sein, welche sowohl in dem involutorischen Gebilde CC.AD... als auch in dem involutorischen Gebilde AC.BE... einander zugeordnet sind. Wenn die Gerade FG die Curve in einem Punkte P berührt, also Q mit P zusammenfällt, so ist ABCP=ABCQ.

277. Wenn U, V, $U_1$, $V_1$ neutrale Würfe sind, J aber ein nicht neutraler Wurf und $U+VJ=U_1+V_1J$ ist, so ist $U=U_1$ und $V=V_1$.

Es sei $V=V_1+V_2=$, so ist (261) auch $V_2$ ein neutraler Wurf. Da nun $U+V_1J+V_2J=U_1+V_1J$ ist, so ist $U+V_2J=U_1$ und mithin auch $V_2J$ ein neutraler Wurf. Da aber das Produkt MANC aus einem neutralen Wurfe MANB, welcher nicht=0 ist, in einen nicht neutralen Wurf MBNC ebenfalls ein nicht neutraler Wurf ist, so muss $V_2=0$, folglich $U=U_1$ und $V=V_1$ sein.

278. Durch einen nicht neutralen Wurf J und noch einen Wurf W sind zwei neutrale Würfe U, V bestimmt, so dass nämlich W=U+VJ ist.

Ist W ein neutraler Wurf, so ist U=W und V=0. Wenn aber dieser Fall nicht stattfindet, so nehme man in der einen von zwei reellen Geraden, welche in einem Punkte C sich schneiden, noch zwei reelle Punkte A, B, in der andern aber einen imaginären Punkt M an. Man suche ferner in der Ebene ABM die reellen Punkte P, Q, so dass der Wurf M (ABCP) dem Wurfe W und der Wurf M (ABCQ) dem Wurfe J gleich ist. Wenn nun die Gerade AQ von den Geraden CM, CP in den Punkten E, F und die Gerade AC von der Geraden EP im Punkte D geschnitten wird, und man die neutralen Würfe ABCD, AQEF durch U V bezeichnet, so ist W=U+VJ. Da nämlich (G. 222) in dem involutorischen Strahlenbüschel M (CC.AP...) dem Strahle MD der Strahl MF zugeordnet ist, so ist M(ABCP)=M(ABCD)+M(ABCF)=U+M(ABCF) Der Wurf M (ABCF) aber ist das Produkt aus dem Wurfe J in den Wurf M (AQCF), welcher=V ist. Geht die Gerade AQ durch dem Punkt P, so ist F mit P, folglich D mit A identisch und daher U=0.



## §. 21.

### Potenzen von Würfen.

279. Ein Produkt aus m Würfen, deren jeder=U ist, soll die mte Potenz dieses Wurfes heissen und durch $U_m$ bezeichnet werden. Jede Potenz von 0 ist also=0, jede Potenz von 1 ebenfalls=1. Ist U ein neutraler Wurf, so ist auch $U_m$ ein neutraler Wurf.

Gleichnamige Potenzen $U^m$, $V^m$ von gleichen Würfen U, V sind einander gleich. Gleichnamige Potenzen von zwei einander conjungirten Würfen sind (269) einander conjungirt. Ist $U=U_1U_2$, so ist $U_m=U_1{}^mU_2{}^m$. Ist also $U_2$ das Umgekehrte von $U_1$, so ist auch $U_{2m}$ das Umgekehrte von $U_1{}^m$.

280. Wenn MN.AC.BB eine Involution ist, so ist $(MANB)^2=(MANB)(MBNC)=(MBNC)^2=MANC$. Ist MANB ein harmonischer Wurf, so ist $(MANB)^2=1$. Die mte Potenz eines harmonischen Wurfes ist also entweder=1 oder selbst ein harmonischer Wurf, je nachdem nämlich m eine pare oder unpare Zahl ist.

Aus 264 geht hervor, dass das Produkt aus einem harmonischen Wurfe ABCD in einen beliebigen Wurf ABCP das entgegengesetzte von diesem Wurfe ist. Und wenn zwei Würfe U, V einander entgegengesetzt sind, so ist jeder das Produkt aus dem andern in einen harmonischen Wurf. Gleichnamige Potenzen $U^m$, $V^m$ von zwei solchen Würfen sind also einander gleich oder ebenfalls einander entgegengesetzt, je nachdem der Exponent m eine pare oder unpare Zahl ist.

281. Wenn $MANA_1$ ein neutraler eigentlicher Wurf ist, jedoch die Elemente M, N durch die Elemente A, $A_1$ nicht getrennt sind, so ist keine Potenz des Wurfes $MANA_1$ einer andern Potenz desselben gleich und daher auch keine $=1$.

Sucht man nämlich in dem Elementargebilde, in welchem der Wurf $MANA_1$ enthalten ist, die Elemente $A_2$, $A_3$, $A_4$..., so dass Würfe $MANA_1$, $MA_1NA_2$, $MA_2NA_3$, $MA_3NA_4$..., sämmtlich einander gleich sind, so ist $MANA_2=(MANA_1)^2$, $MANA_3=(MANA_1)^3$, $(MANA_4)=(MANA_1)^4$, u. s. w. Da nun, wie man sich leicht überzeugt, die Elemente M, N, A, $A_1$, $A_2$, $A_3$, $A_4$... in der Kette, welcher sie angehören, entweder die angeschriebene Ordnung oder die Ordnung N, M, A, $A_1$, $A_2$, $A_3$, $A_4$... befolgen, so fallen keine zwei derselben in einander, woraus der Satz folgt.

282. Wenn irgend eine Potenz $U^m$ eines Wurfes U dem Wurfe 1 gleich ist, so ist der dem Wurf U conjungirte Wurf V zugleich das Umgekehrte von demselben.

Sind nämlich die Würfe U, V einander conjungirt, so ist das Produkt UV aus denselben einem neutralen Wurfe MAND gleich. Wenn ferner $U^m=1$ ist, so ist (279) auch $V^m=1$ und folglich auch $(MAND)^m=1$. Diess ist aber, da (273) die Elemente M, N durch die Elemente A, D nicht getrennt sind, nach dem vorigen Satze nur möglich, wenn D mit A zusammenfällt und also $UV=1$ ist.

283. Durch jeden Wurf MANC, welcher nicht$=0$ ist, sind zwei einander entgegengesetzte Würfe MANB, $MANB_1$ bestimmt, so dass $(MANB)^2=(MANB_1)^2=MANC$ ist.

Soll nämlich $(MANP)^2=MANC$ sein, so muss (270) $MANP=MPNC=NCMP$ und also in dem involutorischen Gebilde MN.AC... das Element P sich selbst zugeordnet sein. Und wenn B, $B_1$ die Ordnungselemente des erwähnten involutorischen Gebildes sind, so ist $(MANB)^2=(MANB_1)^2=MANC$. Auch ist jeder von den beiden Würfe MANB, $MANB_1$ das Produkt aus dem andern in den harmonischen Wurf $MBNB_1$, folglich dem andern entgegengesetzt.

Ist $MANC=1$, also C mit A identisch, so ist der eine von den beiden Würfen MANB, $MANB_1$ ebenfalls$=1$ und der andere ein harmonischer Wurf. Wenn überhaupt MANC ein neutraler Wurf ist und die Elemente M, N durch die Elemente A, C nicht getrennt sind, so sind auch MANB, $MANB_1$ neutrale Würfe. In



jedem andern Falle aber sind MANB, $MANB_1$ zwei nicht neutrale Würfe.

284. Wenn M, N, A, B, $A_1$, $B_1$ sechs Punkte einer und derselben Curve II. Ordnung sind und S den Schnittpunkt der Geraden $AA_1$, $BB_1$ bezeichnet, so ist der Wurf S (MANB) das Produkt aus den beiden in der Curve enthaltenen Würfen MANB, $MA_1NB_1$.

Es werde die Gerade MN von den Geraden $AA_1$, $BB_1$, $A_1B$ in den Punkten $A_2$, $B_2$, C geschnitten, so ist $S(MANB)=MA_2NB_2=(MA_2NC)(MCNB_2)$. Nun ist aber $MA_2NC=A_1(MANB)=MANB$ und $MCNB_2=B(MA_1NB_1)=MA_1NB_1$, woraus der Satz sich ergibt. Der Satz gilt übrigens auch noch, wenn $A_1$ mit A zusammenfällt und man unter $AA_1$ die der Curve im Punkte A sich anschmiegende Gerade versteht. Lässt man auch noch $B_1$ mit B zusammenfallen, so folgt:

Wenn eine Curve, welche durch die vier Punkte M, N, A, B geht, in den Punkten A, B die Geraden SA, SB berührt, so ist $S(MANB)=(MANB)^2$ und also auch $S(AMBN)=(AMBN)^2$. Eben so ist, wenn die durch die Punkte M, N gehenden Tangenten der Curve im Punkte $S_1$ sich schneiden, $S_1(MANB)=(MANB)^2$, daher auch $S(MANB)=S_1(MANB)$.

285. In jedem Elementargebilde sind durch drei Elemente A, C, E drei harmonische Würfe ABCE, CDEA, EFAC bestimmt. Nach G. 220 sind alsdann auch BCDF, DEFB, FABD harmonische Würfe, während AD.BE.CF eine Involution ist. Sind nun M, N die Ordnungselemente des involutorischen Gebildes AD.BE.CF..., so ist $MNABCDEF\overline{\wedge}MNBCDEFA$.

Da nämlich in dem involutorischen Gebilde BB.EE.. dem Elemente M das Element N und dem Elemente A das Element C zugeordnet ist, so ist $MNAB=NMCB=MNBC$. Eben so wird bewiesen, dass $MNBC=MNCD=MNDE=MNEF=MNFA$ ist, woraus der Satz folgt.— Wenn das Gebilde und die Elemente A, C, E, mithin auch die Elemente B, D, F reell sind, so sind M, N zwei einander conjungirte imaginäre Elemente, woraus hervorgeht, dass der Wurf MANB, welcher von dem Wurfe $MBNA=NAMB$ das Umgekehrte ist, diesem zugleich conjungirt ist. Dasselbe gilt von den Würfen MANC, MCNA.

286. Durch drei Elemente A, C, E eines Elementargebildes sind zwei Elemente M, N desselben bestimmt, so dass nämlich MNACE$\pi$MNCEA ist.

Es seien wieder ABCE, CDEA, EFAC harmonische Würfe. Soll nun MNAC$\pi$MNCE$\pi$NMEC, mithin MN.AE.CC eine Involution sein, so müssen die Elemente M, N durch die Elemente C, F und eben so, da auch MNCE$\pi$MNEA sein soll, auch durch die Elemente E, B harmonisch getrennt sein. Und wenn M, N die Ordnungselemente des involutorischen Gebildes AD.BE.CF... sind, so ist MNAC=MNCE=MNEA und also MNACE$\pi$MNCEA.

287. Durch drei Elemente M, N, A eines Elementargebildes sind zwei Elemente C, E desselben bestimmt, so dass nämlich MNACE$\bar{\pi}$MNCEA ist.

Man suche die Elemente D, C, E, so dass MAND ein harmonischer Wurf und (286) CEMND$\pi$CENDM ist. Da alsdann die Elemente A, D auch durch die Elemente C, E harmonisch getrennt sind, so ist (85) AD.MC.NE eine Involution, folglich MNAC=CEDM. Da ferner MNCE=CEMN ist, so ist MNACE$\bar{\pi}$CEDMN$\pi$CEMND $\pi$ MNCEA. Und wenn MNACE$\bar{\pi}$MNCEA sein soll, so muss auch CEMND$\pi$CENDM sein, woraus hervorgeht, dass die Elemente C, E durch die gegebenen bestimmt sind.

Wenn das Gebilde eine Curve II. Ordnung ist und diese in den Punkten M, N die Geraden SM, SN berührt, so ist (248) S(MNAC)=MNAE=NMEA=NMAC und eben so S(MNAE)=NMAE, folglich S (MNACE)$\pi$NMACE$\pi$D(NMACE), woraus man schliessen kann, dass die Punkte C, E in der Geraden liegen, welche den Schnittpunkt von SM und DN mit dem Schnittpunkte von SN und DM verbindet, folglich von der Geraden MN durch die Punkte S, D harmonisch getrennt ist und durch den in der Geraden MN befindlichen Pol der Geraden SD geht. Wenn also die Curve und die Punkte M, N, A reell sind, so sind C, E zwei einander conjungirte imaginäre Punkte. Sind die Curve und der Punkt A reell, hingegen M, N zwei einander conjungirte imaginäre Punkte, so sind C, E reelle Punkte.

288. Wenn die Elemente M, N, A, C, E einem und demselben Elementargebilde angehören und MNACE $\pi$ MNCEA ist, so ist $(MANC)^2$=(MANC) (MCNE)=MANE=MCNA, folglich





$(MANC)^3$=1 und eben so $(MCNA)^3$=1. Und wenn die dritte Potenz eines Wurfes W dem Wurfe 1 gleich ist, ohne dass W selbst=1 ist, so ist der Wurf W einem von den beiden Würfen MANC, MCNA gleich. Es sei nämlich W=MANP=MPNQ, so ist $W^2$=MANQ Da nun W das Umgekehrte von $W^2$ ist, so ist auch MANP=MQNA, folglich MNAPQ$\pi$MNPQA, woraus man (287) schliessen kann, dass entweder P mit C und Q mit E oder P mit E und Q mit C zusammenfällt, also entweder W=MANC oder W=MANE=MCNA ist.

Sind die dritten Potenzen von zwei Würfen U, $U_1$ einander gleich, ohne dass U=$U_1$ ist, so ist der Wurf U entweder das Produkt aus dem Wurfe $U_1$ in den Wurf MANC oder das Produkt aus dem Wurfe $U_1$ in den Wurf MCNA. Ist nämlich U=$U_1 U_2$, so ist auch $U_1^3 U_2^3 = U_1^3$, mithin $U_2^3$=1, ohne dass $U_2$=1 ist, woraus der Satz folgt.

Bezeichnet B wieder das Element, welches vom Elemente E durch die Elemente M, N harmonisch getrennt ist, so ist MANB das Produkt aus MCNA=MANE in den harmonischen Wurf MENB und eben so der Wurf MBNA dem Wurfe MANC entgegengesetzt, daher $(MANB)^3$, $(MBNA)^3$ harmonische Würfe sind. Und wenn die dritte Potenz eines Wurfes V ein harmonischer Wurf ist, so ist der Wurf V entweder selbst ein harmonischer Wurf oder einem von den beiden Würfen MANB, MBNA gleich.

289. Durch jeden neutralen Wurf U, welcher nicht=0 ist, sind ein neutraler Wurf V und zwei einander conjungirte nicht neutrale Würfe $V_1$, $V_2$ bestimmt, so dass $V^3 = V_1^3 = V_2^3$=U ist.

Die Fälle, in welchen $U^2$=1 ist, sind bereits in der vorigen Nummer betrachtet worden. Wenn aber U weder=1 noch ein harmonischer Wurf ist, so suche man in einer reellen Curve K II. Ordnung zu drei reellen Punkten M, A, N den Punkt D, so dass MAND=U ist. Es sei ferner S der Schnittpunkt der der Curve in den Punkten M, N sich anschmiegenden Geraden, so erzeugen die zu einander projektivischen Strahlenbüschel S(MAN...), A(NDM...) eine andere reelle Curve II. Ordnung, welche die erstere im Punkte A und daher in noch einem reellen Punkte P schneidet. Da nun (284) $(MANP)^2$=S(MANP) A(NDMP)=NDMP=MPND ist, so ist $(MANP)^3$=MAND=U. Die

Curve K enthält ferner (287) zwei einander conjungirte imaginäre Punkte $P_1$, $P_2$, so dass $MNPP_1P_2 \pi MNP_1P_2P$ ist. Da alsdann $(MPNP_1)^3 = (MPNP_2)^3 = 1$ ist, so ist $(MANP_1)^3 = (MANP_2)^3 = (MANP)^3 = U$. Zugleich geht hieraus hervor, dass die beiden Curven auch in den Punkten $P_1$, $P_2$ sich schneiden.

290. Durch jeden nicht neutralen Wurf U, welcher von dem ihm conjungirten Würfe das Umgekehrte ist, sind drei andere solche Würfe V, $V_1$, $V_2$ bestimmt, so dass $V^3 = V_1^3 = V_2^3 = U$ ist.

Man suche in einer reellen Curve K II. Ordnung zu zwei einander conjungirten imaginären Punkten M, N und einem reellen Punkte A den Punkt D, so dass MAND = U ist. Da alsdann auch der Pol S der Geraden MN und (273) der Punkt D reelle Punkte sind, so erzeugen, wie aus 150 leicht hervorgeht, die zu einander projektivischen Strahlenbüschel S(MAN...), A(NDM...) eine andere reelle Curve II. Ordnung, welche die erstere im Punkte A und daher wenigstens in noch einem reellen Punkte P schneidet. Ferner enthält (287) die Curve K zwei reelle Punkte $P_1$, $P_2$, so dass $MNPP_1P_2 \pi MNP_1P_2P$ ist, und nun lässt sich wie in der vorigen Nummer beweisen, dass $(MANP_1)^3 = (MANP_2)^3 = (MANP)^3 = U$ ist. Die beiden Curven schneiden sich daher auch in den Punkten $P_1$, $P_2$.

291. Durch jeden Wurf U, welcher nicht = 0 ist, sind drei Würfe V, $V_1$, $V_2$ bestimmt, so dass $V^3 = V_1^3 = V_2^3 = U$ ist.

Man nehme in der einen von zwei reellen Geraden, welche in einem Punkte C sich schneiden, noch zwei reelle Punkte S, B, in der andern aber zwei einander conjungirte imaginäre Punkte M, N an und suche alsdann in der Ebene SMN den reellen Punkt D, so dass der Wurf M(SBCD) dem gegebenen Wurfe U gleich ist. Es sei ferner A irgend ein Schnittpunkt der Geraden SC und der reellen Curve, welche in den Punkten M, N die Geraden SM, SN berührt und durch den Punkt D geht, so giebt es (289) in der Geraden SC einen reellen Punkt R und (290) in der erwähnten Curve drei reelle Punkte P, $P_1$, $P_2$, so dass $(SRCA)^3 = SBCA$ und $(MANP)^3 = (MANP_1)^3 = (MANP_2)^3 = MAND$ ist. Bezeichnet man nun den Wurf M(SRCP), welcher das Produkt aus dem Wurfe M(SRCA) = SRCA in den Wurf M(SACP) = MANP ist, durch V, so ist $V^3 = (SRCA)^3 (MANP)^3 = (SBCA)(MAND) =$





M(SBCD) = U. Eben so wird bewiesen, dass, wenn man die Würfe M(SRCP), M(SRCP$_2$) durch $V_1$, $V_2$ bezeichnet, $V_1^3 = V_2^3 = U$ ist.

---

## §. 22.

### Gemeinschaftliche Punkte und Tangenten zweier Curven II. Ordnung.

292. Wenn zwei in einerlei Ebene liegende Curven II. Ordnung in einem gemeinschaftlichen Punkte S eine und dieselbe Gerade berühren, so findet irgend einer von den in 187 betrachteten Fällen statt. Dass die Curven und der Punkt S reell sind, musste in der erwähnten Nummer nur desshalb vorausgesetzt werden, weil vorläufig nur, was reell-projektivisch heisse, erklärt und daher auch noch nicht nachgewiesen war, dass je zwei Elementargebilde projektivisch so auf einander bezogen werden können, dass drei gegebenen Elementen des einen drei gegebene Elemente des andern entsprechen. Haben zwei Curven II. Ordnung drei Punkte A, B, C mit einander gemein, ohne in einem dieser Punkte eine und dieselbe Gerade zu berühren, so schneiden sie sich noch in einem vierten Punkte, der (11) schon durch die drei Punkte A, B, C und die Gerade bestimmt ist, welche in Hinsicht auf beide Curven der Geraden AB conjugirt ist.

293. Wenn man die eine K von zwei Curven K, $K_1$ II. Ordnung, welche in einerlei Ebene liegen und die Punkte A, B mit einander gemein haben, aus jedem dieser Punkte auf die andere projicirt, so hat man zwei zu einander projektivische Punktgebilde, welche beide in der Curve $K_1$ liegen. Dem Punkte $P_1$ des einen dieser Gebilde entspricht der Punkt $P_2$ des andern, wenn die Strahlen $AP_1$, $BP_2$ einen und denselben Punkt P der Curve K projiciren. Haben nun die zu einander projektivischen Gebilde zwei ausserhalb der Geraden AB liegende Punkte C, D entsprechend gemein, so schneiden sich die Curven K, $K_1$ in den vier Punkten A, B, C, D. Wenn aber der Punkt D mit einem der drei Punkte A, B, C zusammenfällt, so berühren sich die Curven in diesem Punkte, während sie in den beiden übrigen sich schneiden. Haben die zu einander projektivischen Gebilde die Punkte A, B entsprechend gemein, so berühren sich die Curven K, $K_1$ in diesen

## A.5 The real field with 3D models as its elements

# Field of Cows:
# ... construction of the projective scale

see also /attachments/fieldofcows/fieldofcows.cdf
CDF Player download and installation https://www.wolfram.com/cdf-player/

Points of the cow are uploaded from the “Cow” model. We use homogenous projective coordinates, therefore the last coordinate 1 is appended.

In[1]:=
```
data = ExampleData[{"Geometry3D", "Cow"}, "GraphicsComplex"];
fig0 = Append[#, 1] & /@ data[[1]];
```

In[3]:=
```
fig0[1]
```

Out[3]=
```
{{0.107994, 0.0846667, -0.218938, 1}, {0.11579, 0.0886993, -0.228472, 1}, {0.107198, 0.0955648, -0.232789, 1},
  {0.12329, 0.0940333, -0.241455, 1}, <<2895>>, {-0.264289, -0.0454424, -0.129848, 1},
  {-0.259885, -0.043327, -0.131191, 1}, {-0.222232, -0.0354683, -0.135835, 1}, {-0.265862, -0.0401731, -0.136741, 1}}[1]
```

large output | show less | show more | show all | set size limit...

In[4]:=
```
(*points drops the last coordinate of points in the figure fig; used for unhomogenization*)
points[fig_] := Take[# / Last[#] & /@ fig, All, 3]
```

Let us find the highest point of the cow - the horn point and the labeling point to give the cow a name.

Printed by Wolfram Mathematica Student Edition



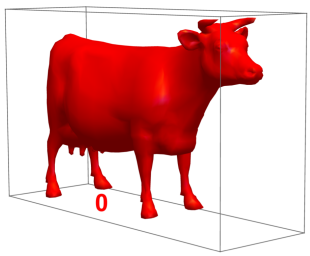
```
maxpoint0 = Append[First[MaximalBy[points[fig0], Last]], 1];
(*label finds the labeling point of the figure fig*)
label[fig_] := First[MinimalBy[points[fig], Last]];
label0 = label[fig0];

(*graphcomplex adds directives to GraphicComplex of the figure fig*)
graphcomplex[fig_] := {Specularity[White, 100], EdgeForm[], ReplacePart[data, {1} → points[fig]]}
(*graphicscol adds specific color col to the graphcomplex gc*)
graphicscol[gc_, col_] := Prepend[gc, col]
```

The first cow, let us call it 0-cow:

Printed by Wolfram Mathematica Student Edition

```
graph0 = Join[graphicscol[graphcomplex[fig0], Hue[0]], {Text[Style[0, Large, Bold, Red], label0, {0, 1}]}];
Graphics3D[graph0]
```




Let us have a collineation matrix P.

```
P = {{2, 0, 0, 5}, {0, 2, 0, 5}, {0, 0, 2, 0}, {1, 0, 0, 1}};
MatrixForm[P]
```

MatrixForm=

$$\begin{pmatrix} 2 & 0 & 0 & 5 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Printed by Wolfram Mathematica Student Edition



```
(*homofig normalize the last coordinate to 1*)
homofig[fig_] := # / Last[#] & /@ fig
```

The image of the 0-cow after collineation is called the ∞-cow.

```
fig∞ = homofig[fig0.Transpose[P]];
maxpoint∞ = (# / Last[#]) &@ (maxpoint0.Transpose[P]);
label∞ = label[fig∞];
graph∞ = Join[graphicscol[graphcomplex[fig∞], Hue[7 / 10]], {Text[Style[∞, Large, Bold, Red], label∞, {0, 1}]}];
maxline = InfiniteLine[{Take[maxpoint0, 3], Take[maxpoint∞, 3]}];
Show[Graphics3D[{graph0, graph∞, maxline}, PlotRange → {{-1, 7}, {-2, 10}, {-1, .5}}],
 Lighting → Automatic, ViewPoint → {.75, -1, .2}, Boxed → False, ImageSize → {800, 300}]
```




Printed by Wolfram Mathematica Student Edition

```
fig∞[1]
```

```
{{4.7076, 4.66549, -0.395197, 1.}, {4.68868, 4.64012, -0.409525, 1.}, {4.70954, 4.68853, -0.420501, 1.},
  {4.67073, 4.61864, -0.429907, 1.}, ··· 2895 ···, {6.07769, 6.67261, -0.352986, 1.},
  {6.05342, 6.63863, -0.354515, 1.}, {5.85719, 6.33745, -0.349294, 1.}, {6.08643, 6.70127, -0.372521, 1.}}[1]
```

large output | show less | show more | show all | set size limit...

Any point $X$ on the line through the horn points $H_0$ of the 0-cow and $H_\infty$ of the ∞-cow is a linear combination of these two points: $X = k_1 \cdot H_\infty + k_0 \cdot H_0$. Coordinates $(k_1, k_0)$; $k_1, k_0 \in \mathbb{R} - \{(0, 0)\}$ are the homogenous projective coordinates of the point $X(k_1, k_0)$ with respect to $H_0(0, 1)$; $H_\infty(1, 0)$, up to a nonzero factor. If $k_0 \neq 0$ we have $x(k, 1)$ for $k = k_1 / k_0$.
Hence, all points of the $k$-cow have projective coordinates $(k, 1)$ on the lines through points of the 0-cow and their images on the ∞-cow.
To complete the projective coordinate system, we choose a 1-cow with the coordinates $(1/3, 1)$.

Printed by Wolfram Mathematica Student Edition



```
j = 1 / 3;
fig1 = homofig[j * fig∞ + fig0];
maxpoint1 = (# / Last[#]) &@(j * maxpoint∞ + maxpoint0);
graph1 = Join[graphicscol[graphcomplex[fig1], Hue[7 / (10 Sqrt[2])]], {Text[Style[1, Large, Bold, Red], label[fig1], {0, 1}]}];
Show[Graphics3D[{graph0, graph∞, graph1, maxline}, PlotRange → {{-1, 7}, {-2, 10}, {-1, .5}}],
 Lighting → Automatic, ViewPoint → {.75, -1, .2}, Boxed → False, ImageSize → {800, 300}]
```

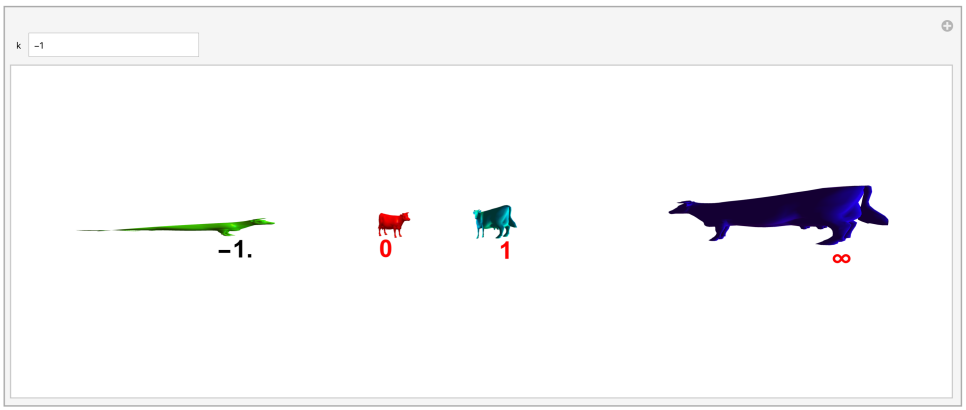



Given the projective coordinate system, all cows can be found. The cow with the number $(-1/j)$ disappears in our affine computer screen.

```
figx[k_] := homofig[k * j * fig∞ + fig0]
koefcol[k_] := (k) / Sqrt[(k)^2 + 1]
graph[k_] := If[k ≠ -1 / j, Join[graphicscol[graphcomplex[figx[k]], Hue[koefcol[k]]],
    {Text[Style[Round[k, .01], Large, Bold, Black], label[figx[k]], {0, 1}]}],
   {Text[Style["You have found the infinite cow ", Large, Bold, Red], label0, {0, -5}]}];
```

Printed by Wolfram Mathematica Student Edition

```
Manipulate[Graphics3D[{graph[k], graph0, graph1, graph∞}, PlotRange → {{-5, 7}, {-7, 10}, {-1, .5}},
  ImageSize → {900, 300}, Boxed → False, ViewPoint → {.75, -1, .2}], {k, -Infinity, Infinity, ControlType → InputField}]
```

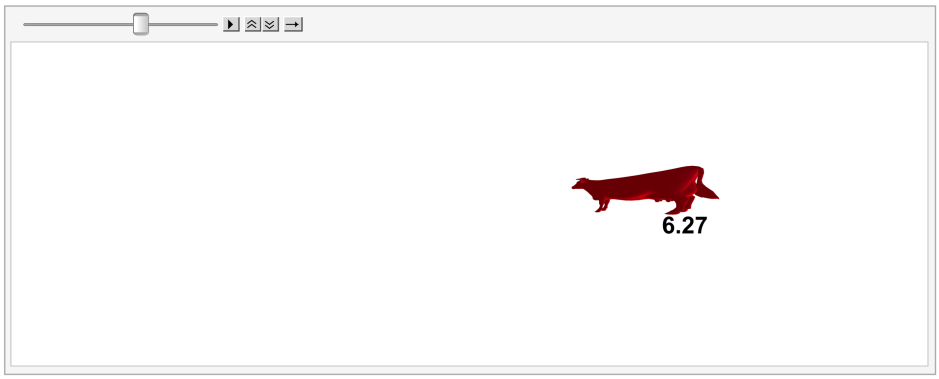



See the animation of the projective scale of cows.

Printed by Wolfram Mathematica Student Edition



```
figs = Table[Graphics3D[graph[i^7], PlotRange → {{-5, 7}, {-7, 10}, {-1, .5}},
    ImageSize → {900, 300}, Boxed → False, ViewPoint → {.75, -1, .2}], {i, -3 / 2, 3, 1 / 20}];
ListAnimate[
 figs,
 10]
```




To any of our cows a real number or ∞ could be assigned. The set of our cows is therefore isomorphic to the real projective line $\mathbb{RP}_1$. On the other hand, the structure of real numbers could be constructed synthetically. So, let us see how addition and subtraction of cows work. The following constructions are variations of von Staudt's constructions.

Let us have 0-cow, ∞-cow and A-cow and B-cow.

Printed by Wolfram Mathematica Student Edition

```
a = 0.5;
b = 1.5;
figA = figx[a];
figB = figx[b];
(*maxpoint finds the image of the horn point*)
maxpoint[x_] := (# / Last[#]) &@(maxpoint0 + x * j * maxpoint∞);
maxpointA = maxpoint[a];
labelA = label[figA];
maxpointB = maxpoint[b];
labelB = label[figB];
graphA =
  Join[graphicscol[graphcomplex[figA], Hue[7 / 10 * koefcol[a]]], {Text[Style[N[a], Large, Bold, Blue], label[figA], {0, 1}]}];
graphB = Join[graphicscol[graphcomplex[figB], Hue[7 / 10 * koefcol[b]]],
   {Text[Style[N[b], Large, Bold, Blue], label[figB], {0, 1}]}];
maxpoints = Point[#] & /@ points[{maxpoint0, maxpoint∞, maxpointA, maxpointB}];
```

Printed by Wolfram Mathematica Student Edition



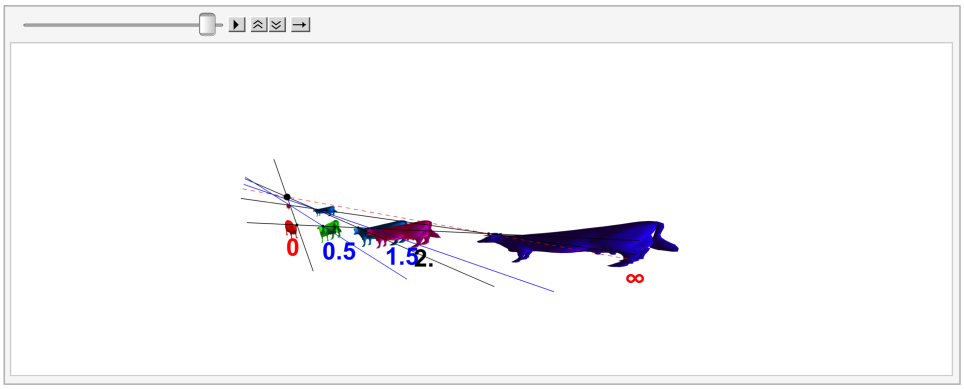
```
(*ADDITION - von Staudt's construction - see the step-by-step construction below*)
point1 = {0, 0, 1};
line01 = InfiniteLine[{point1, Take[maxpoint0, 3]}];
scaleon0 = 3 / 10;
pointon01 = Take[maxpoint0, 3] + (1 - scaleon0) * (point1 - Take[maxpoint0, 3]);
linek1 = InfiniteLine[{point1, Take[maxpointB, 3]}];
lineInf01 = InfiniteLine[{pointon01, Take[maxpoint∞, 3]}];
line01l = InfiniteLine[{pointon01, Take[maxpointA, 3]}];
lineInf1 = InfiniteLine[{point1, Take[maxpoint∞, 3]}];
precisionplane = InfinitePlane[{pointon01, Take[maxpointA, 3], {-1, -1, 0}}];
pointonInf = {x, y, z} //. Solve[{x, y, z} ∈ precisionplane ∧ {x, y, z} ∈ lineInf1, {x, y, z}];
pointonk1 = {x, y, z} //. Solve[{x, y, z} ∈ lineInf01 ∧ {x, y, z} ∈ linek1, {x, y, z}];
scaleonk1 = 1 - Norm[(pointonk1[[1]] - Take[maxpointB, 3])] / (Norm[(point1 - Take[maxpointB, 3])]);
linek1Inf = InfiniteLine[{pointonInf[[1]], pointonk1[[1]]}];
precisionplane2 = InfinitePlane[{pointonInf[[1]], pointonk1[[1]], {-1, -1, 0}}];
pointCplusD = {x, y, z} //. Solve[{x, y, z} ∈ maxline ∧ {x, y, z} ∈ precisionplane2, {x, y, z}];

(*scaled cows for visualization*)
graph0 = Drop[ReplacePart[graph0, {4} → Scale[graph0[[4]], scaleon0, point1]], -1];
graphB = Drop[ReplacePart[graphB, {4} → Scale[graphB[[4]], scaleonk1, point1]], -1];

(*list of graphics*)
list1 = {{graph0, graph∞}, {graphA, graphB}, {maxpoints, maxline}, {PointSize[Large], Point[point1]}, {Red, Dashed, lineInf1},
   line01, graph0, linek1, lineInf01, graphB, {Blue, line01l}, {Blue, linek1Inf}, graph[a + b]};
animlist1 = Table[Take[list1, i], {i, 1, Length[list1]}];
graphanimlist1 = Graphics3D[#, PlotRange → {{-1, 7}, {-1, 10}, {-1, 2}},
     ImageSize → {900, 300}, Boxed → False, ViewPoint → {1, -.75, .5}] & /@ animlist1;
```

Printed by Wolfram Mathematica Student Edition

```
(*A+B=0.5+1.5=2 step-by-step construction*)
ListAnimate[graphanimlist1, 1]
```



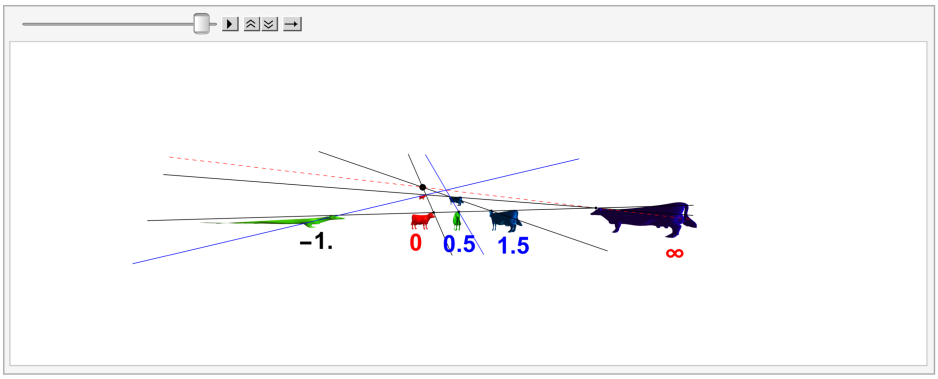

```
(*SUBTRACTION - von Staudt's construction - see the step-by-step construction below*)
linek1l = InfiniteLine[{pointonk1[[1]], Take[maxpoint_A, 3]}];
precisionplane3 = InfinitePlane[{pointonk1[[1]], Take[maxpoint_A, 3], {-1, -1, 0}}];
pointonInfk = {x, y, z} //. Solve[{x, y, z} ∈ precisionplane3 ∧ {x, y, z} ∈ lineInf1, {x, y, z}];
line01Inf = InfiniteLine[{pointonInfk[[1]], pointon01}];
precisionplane4 = InfinitePlane[{pointonInfk[[1]], point1, {-1, -1, 0}}];
pointCminusD = {x, y, z} //. Solve[{x, y, z} ∈ maxline ∧ {x, y, z} ∈ precisionplane4, {x, y, z}];
```

Printed by Wolfram Mathematica Student Edition



```
(*list of graphics*)
list2 = {{graph_0, graph_∞}, {graph_A, graph_B}, {maxpoints, maxline}, {PointSize[Large], Point[point1]}, {Red, Dashed, lineInf1},
   line01, graph0, linek1, lineInf01, graphB, {Blue, linek1l}, {Blue, line01Inf}, graph[a - b]};
animlist2 = Table[Take[list2, i], {i, 1, Length[list2]}];
graphanimlist2 =
  Graphics3D[#, PlotRange → {{-5, 7}, {-8, 10}, {-1, 2}}, ImageSize → {900, 300}, Boxed → False, ViewPoint → {.4, -.8, 0}] & /@
   animlist2;
```

```
(*A-B=0.5-1.5=-1 step-by-step construction*)
ListAnimate[graphanimlist2, 1]
```




```
(*MULTIPLICATION - von Staudt's construction - see the step-by-step construction below*)
```

Printed by Wolfram Mathematica Student Edition

```
line101 = InfiniteLine[{pointon01, Take[maxpoint1, 3]}];
precisionplane5 = InfinitePlane[{pointon01, Take[maxpoint1, 3], {-1, -1, 0}}];
pointonInf01k = {x, y, z} //. Solve[{x, y, z} ∈ precisionplane5 ∧ {x, y, z} ∈ lineInf1, {x, y, z}];
linek01Inf = InfiniteLine[{pointonInf01k[[1]], Take[maxpointB, 3]}];
precisionplane6 = InfinitePlane[{pointonInf01k[[1]], Take[maxpointB, 3], {-1, -1, 0}}];
pointon01Inf = {x, y, z} //. Solve[{x, y, z} ∈ precisionplane6 ∧ {x, y, z} ∈ line01, {x, y, z}];
scaleonk01 = 1 - Norm[(pointon01Inf[[1]] - Take[maxpointB, 3])] / (Norm[(pointonInf01k[[1]] - Take[maxpointB, 3])]);
precisionplane7 = InfinitePlane[{pointon01Inf[[1]], Take[maxpointA, 3], {-1, -1, 0}}];
pointonkInf = {x, y, z} //. Solve[{x, y, z} ∈ precisionplane7 ∧ {x, y, z} ∈ lineInf1, {x, y, z}];
linekInf = InfiniteLine[{pointon01, Take[maxpointA, 3]}];
precisionplane8 = InfinitePlane[{pointon01, Take[maxpointA, 3], {-1, -1, 0}}];
pointonlInf = {x, y, z} //. Solve[{x, y, z} ∈ precisionplane8 ∧ {x, y, z} ∈ lineInf1, {x, y, z}];
linelInf = InfiniteLine[{pointonlInf[[1]], pointon01Inf[[1]]}];

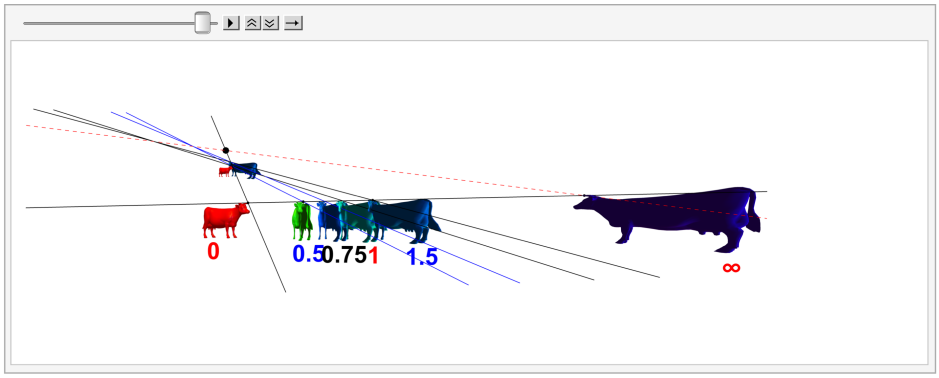
graphB2 = Drop[ReplacePart[graphB, {4} → Scale[graphB[[4]], scaleonk01, pointonInf01k[[1]]]], -1];

(*list of graphics*)
list3 = {{graph0, graph∞, graph1}, {graphA, graphB}, {maxpoints, maxline}, {PointSize[Large], Point[point1]},
   {Red, Dashed, lineInf1}, line01, graph0, line101, linek01Inf, graphB2, {Blue, linekInf}, {Blue, linelInf}, graph[a * b]};
animlist3 = Table[Take[list3, i], {i, 1, Length[list3]}];
graphanimlist3 = Graphics3D[#, PlotRange → {{-2, 7}, {-2, 8}, {-1, 1.5}},
     ImageSize → {900, 300}, Boxed → False, ViewPoint → {.5, -1.2, 0}] & /@ animlist3;
```

Printed by Wolfram Mathematica Student Edition



```
(*A*B=0.5*1.5=0.75*)
ListAnimate[graphanimlist3, 1]
```

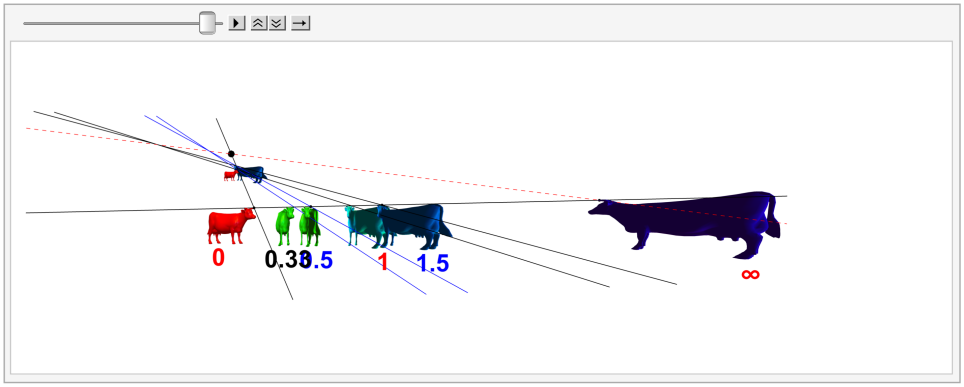



```
(*DIVISION - von Staudt's construction - see the step-by-step construction below*)
linek01l = InfiniteLine[{pointon01Inf[[1]], Take[maxpointA, 3]}];
precisionplane9 = InfinitePlane[{pointon01Inf[[1]], Take[maxpointA, 3], {-1, -1, 0}}];
pointonInfl1 = {x, y, z} //. Solve[{x, y, z} ∈ precisionplane9 ∧ {x, y, z} ∈ lineInf1, {x, y, z}];
lineInfl1 = InfiniteLine[{pointonInfl1[[1]], pointon01}];
```

Printed by Wolfram Mathematica Student Edition

```
(*list of graphics*)
list4 = {{graph0, graph∞, graph1}, {graphA, graphB}, {maxpoints, maxline}, {PointSize[Large], Point[point1]},
   {Red, Dashed, lineInf1}, line01, graph0, line101, linek01Inf, graphB2, {Blue, linek01l}, {Blue, lineInfl1}, graph[a / b]};
animlist4 = Table[Take[list4, i], {i, 1, Length[list4]}];
graphanimlist4 = Graphics3D[#, PlotRange → {{-2, 7}, {-2, 8}, {-1, 1.5}},
     ImageSize → {900, 300}, Boxed → False, ViewPoint → {.5, -1.2, 0}] & /@ animlist4;

(*A/B=0.5/1.5=1/3*)
ListAnimate[graphanimlist4, 1]
```




The whole field (ℝ,+,-,·,/,0,1) of cows.

Printed by Wolfram Mathematica Student Edition